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Superposition and Thinning of Counting Processes in Non-life Insurance

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Title: Superposition and Thinning of Counting Processes in Non-life Insurance

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Abstract: The thesis examines a model for representing the number of claims after merging or splitting different lines of business of an insurance company. The model is based on counting processes, the Poisson and the renewal processes are considered in particular. The operations of superposition and thinning are the proposed solution to this problem. We present the well-known results that the Poisson processes are closed under superposition and several types of thinning and explore the necessary conditions for this statement to also hold for renewal processes. Specifically, the previous work on the superposition of renewal processes is studied and further clarified, and an original result is derived for two types of thinning of a renewal process. The theoretical results are then used to analyze real insurance data in a model situation when an insurance company wants to estimate the future number of claims after merging two of its lines of business.

Keywords: counting processes, superposition, thinning, non-life insurance, Poisson process, renewal process

Contents

Introduction

An insurance company may occasionally encounter the requirement to merge or split some of its lines of business. Among other reasons, this may be a result of change in legislation or an internal business decision. Our interest lies in the mathematical background of these operations, especially from the perspective of claim numbers models. We consider counting processes as a general model for the number of insurance claims, the merging and splitting of the lines of business then translates to the operations of superposition and thinning. The main goal is to describe several classes of counting processes which are closed under superposition and two elementary types of thinning, as some of the properties of the initial models entering these operations are preserved in the resulting claim numbers models for the merged or split lines of business.

The concept of a counting process is introduced in the first chapter and the specific types of counting processes considered in the next parts of the thesis are described. In particular, a Poisson process is introduced along with two of its numerous possible generalizations, a nonhomogeneous Poisson process and a renewal process. Several important properties of these types of processes are also derived in this chapter.

The second chapter is concerned with the operations of superposition and thinning. The definitions of superposition and two elementary types of thinning are stated in the general setting of counting processes and several results about superposition and thinning of the previously introduced types of counting processes are presented. The well-known results that the Poisson processes are closed under these operations are stated and the necessary conditions for this assertion to also hold true for renewal processes are explored. The previous work on the superposition of renewal processes is studied and clarified in further detail, and an original result for thinning of a renewal process is derived.

The third chapter covers the necessary theory for application of the previously stated results and the analysis of insurance data provided by the Czech Insurers' Bureau. Specifically, an approach to estimate the parameters of one particular type of intensity function is presented and the superposition of nonhomogeneous Poisson processes is then used to estimate the future number of reported claims in a model situation when an insurance company merges two of its lines of business.

1. Counting Processes

This chapter is concerned with a particular class of stochastic processes often used in claim numbers modeling, the so-called counting processes. We formally define a counting process and introduce some specific types of counting processes, namely a Poisson process and two of its generalizations, a non-homogeneous Poisson process and a renewal process. The definitions throughout this chapter are inspired by Ross [2014].

1.1 Preliminaries

Let us begin with definitions of several concepts that will be used throughout this text. We first introduce the fundamental idea of a stochastic process.

Definition 1.1. *A collection of random variables* $\{X(t), t \in T\}$*,* T *being an arbitrary index set, is called a* stochastic process*.*

We will interpret index t as time and call $X(t)$ the *state* of the process at time t. Stochastic process $\{X(t), t \in T\}$ is called a *discrete-time* stochastic process if the set T is at most countable or a *continuous-time* stochastic process if T is uncountable. A realization of a stochastic process is called a *sample path*.

A counting process is a specific type of continuous-time stochastic process whose state at time t represents the number of certain events occurring in the time interval $(0, t]$. This idea gives several formal conditions for a stochastic process to be called a counting process.

Definition 1.2. *A stochastic process* $\{N(t), t \geq 0\}$ *is said to be a* counting process *if the following conditions are satisfied:*

(1) $N(t) \geq 0$*,*

(2) $N(t)$ *is integer valued,*

(3) if $s < t$*, then* $N(s) \leq N(t)$*.*

The sample path of a counting process is always a nondecreasing step function, the points of discontinuity represent times when the observed events occur. We will denote nth such time by T_n and call it the *nth arrival time*. Another characterization is possible in terms of waiting times between two consecutive events. We will denote the elapsed time between the $(n-1)$st and the nth event by X_n and call it the *nth interarrival time*.

The arrival and the interarrival times are naturally interconnected. If we formally let $T_0 = 0$, the nth interarrival time can be expressed as $X_n = T_n - T_{n-1}$, and the converse expression for T_n is obtained as the sum of the first n interarrival times, $T_n = \sum_{i=1}^{n} X_i$.

The figure below shows the sample path of a counting process with four events occurring at times 1, 2, 4 and 7 (denoted by $T_1, \ldots, T_4$) and the corresponding interarrival times (denoted by $X_1, \ldots, X_4$).

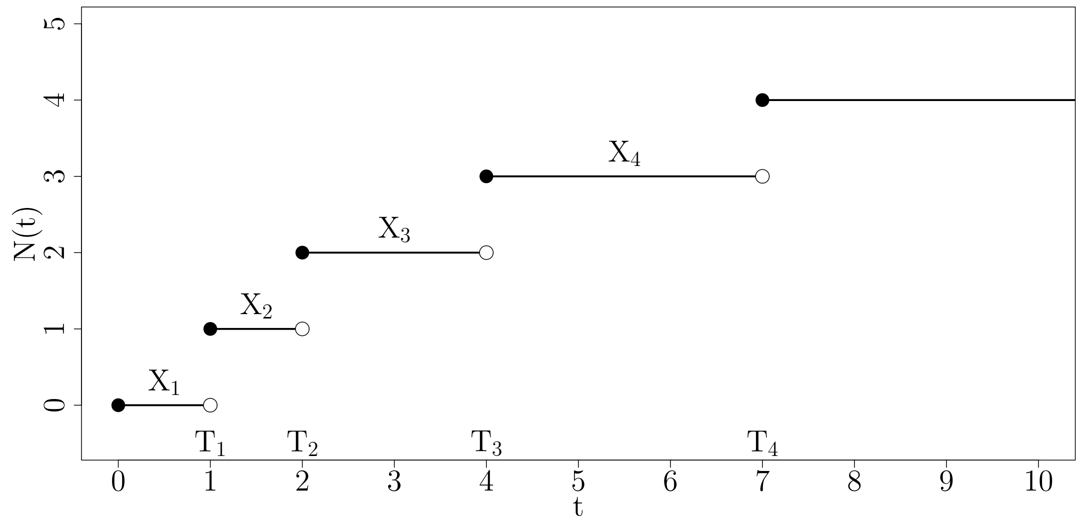

Figure 1.1: *The sample path of an arbitrary counting process $N(t)$*

We continue by defining two possible properties of a counting process, the independent and stationary increments.

Definition 1.3. *A counting process $\{N(t), t \geq 0\}$ is said to possess* independent increments *if $N(t_1) - N(t_0), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1})$ are independent random variables for all $n \in \mathbb{N}$ and for all $0 \leq t_0 < t_1 < \cdots < t_n$. The process is said to possess* stationary increments *if for any $t \geq 0$ and $s > 0$, the probability distribution of variables $N(t+s) - N(t)$ only depends on s.*

To conclude this introductory section, let us define an integral transform that will be used in various parts of this text, the Laplace-Stieltjes transform.

Definition 1.4. *Let $g : [0, \infty) \to [0, \infty)$ be a non-decreasing function. The* Laplace-Stieltjes transform *of g, denoted by g^*, is defined as*

$$g^*(t) = \int_0^\infty e^{-tx} dg(x)$$

for every t where the right-hand side integral converges.

Remark. There is a convenient connection between the Laplace-Stieltjes transform and the moment generating functions of random variables. If we let X be a nonnegative random variable with distribution function F, it is easy to verify that $F^*(t) = \mathbb{E}\left[e^{-tX}\right]$, which is just the moment generating function of X evaluated in $-t$. It can be therefore concluded that the Laplace-Stieltjes transform uniquely determines the probability distribution of X.

1.2 Poisson Process

The first type of counting process that we introduce is a Poisson process. The process is in its simplest form determined by a single constant parameter called rate or intensity. A Poisson process with constant rate is also called homogeneous.

Three different definitions of a homogeneous Poisson process will be given in this section. The first describes the probability distribution of increments in an interval of a fixed length. This definition will be referred to as *axiomatic*.

Definition 1.5. *A counting process* $\{N(t), t \geq 0\}$ *is said to be a* Poisson process *with rate* $\lambda > 0$ *if the following conditions are satisfied:*

(A1) $N(0) = 0$,

(A2) the process has independent increments,

(A3) for all $t, s \geq 0$, $\mathbb{P}[N(t+s) - N(t) = n] = e^{-\lambda s}\frac{(\lambda s)^n}{n!}$, $n = 0, 1, \dots$

Remark. It follows from the property (A3) that this process also has stationary increments, as for a given constant rate λ, the probability distribution of increments is identical for every interval of a fixed length s.

The name of the process of course also follows from the probability distribution of increments, which is Poisson. Using the well-known properties of this distribution, the expectation of a Poisson process at time t can be easily derived as $\mathbb{E}[N(t)] = \lambda t$, which also explains why λ is called rate.

The second definition is concerned with number of increments in an interval as its length h tends to zero. This definition will thus be referred to as *infinitesimal*.

Definition 1.6. *A counting process* $\{N(t), t \geq 0\}$ *is said to be a* Poisson process *with rate* $\lambda > 0$ *if the following conditions are satisfied:*

(I1) $N(0) = 0$,

(I2) the process has independent and stationary increments,

(I3) $\mathbb{P}[N(h) = 1] = \lambda h + o(h)$,

(I4) $\mathbb{P}[N(h) \geq 2] = o(h)$,

where $o(h)$ *denotes any function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

Remark. Properties (I3) and (I4) also determine the probability of zero events occurring in an interval of length h as $\mathbb{P}[N(h) = 0] = 1 - \lambda h + o(h)$.

The infinitesimal definition states that in a Poisson process, as the length h of an interval tends to zero, the probability of one event occurring is approximately proportional to this length, while the probability of more events occurring is negligible. This property is useful for modeling events which do not occur multiple times in a quick succession.

We now show that the first two definitions truly describe the same process.

Claim 1.1. *The definitions 1.5 and 1.6 of a Poisson process are equivalent.*

Proof. The proof of 1.6 $\Rightarrow$ 1.5 makes use of the Laplace-Stieltjes transform of $N(t)$ and the differential equations derived from the definition 1.6. It can be found in full in Ross [2014, p. 299–300, Theorem 5.1].

We will add the proof of the converse implication 1.5 $\Rightarrow$ 1.6. The first property is identical in both definitions, stationary and independent increments follow from the property (A2) and the remark below the axiomatic definition. We will use the property (A3) to show that (I3) and (I4) are also satisfied.

$$\begin{aligned}
(I3)\quad \mathbb{P}[N(h)=1] &= \lambda h e^{-\lambda h} = \lambda h(1-1+e^{-\lambda h}) \\
&= \lambda h + \lambda h(e^{-\lambda h}-1) = \lambda h + o(h), \\
(I4)\quad \mathbb{P}[N(h)\geq 2] &= 1-(\mathbb{P}[N(h)=1]+\mathbb{P}[N(h)=0]) \\
&= 1-(\lambda h e^{-\lambda h}+e^{-\lambda h}) \\
&= 1-(\lambda h e^{-\lambda h}+1-\lambda h+o(h)) = o(h).
\end{aligned}$$

□

The third definition utilizes sequences of arrival and interarrival times described in the previous section and can be thus thought of as a more constructive one. We first notice that the probability distribution of the interarrival times can be derived directly from the axiomatic definition.

Claim 1.2. *The sequence of interarrival times $\{X_n, n \in \mathbb{N}\}$ of a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ consists of independent and identically distributed random variables, he common distribution is exponential with parameter λ.*

Remark. The exponential distribution with parameter λ is considered such that its distribution function is in form $F(x) = 1 - e^{-\lambda x}, x \geq 0$. The fact that X is exponentially distributed with parameter λ will be denoted by $X \sim Exp(\lambda)$.

Proof. We first determine the distribution of waiting time for the first event, X_1. If $N(t) = 0$ at time t, the first event has not occurred yet, X_1 must be therefore greater than t. This observation and the axiomatic definition together give

$$\mathbb{P}[X_1 > t] = \mathbb{P}[N(t) = 0] = e^{-\lambda t},$$

hence, by the properties of distribution functions, $X_1 \sim Exp(\lambda)$. We continue by conditioning X_2 on X_1 and obtaining the distribution

$$\begin{aligned}
\mathbb{P}[X_2 > t | X_1 = s] &= \mathbb{P}[N(t+s) - N(s) = 0 | N(s) = 1] \\
&= \mathbb{P}[N(t+s) - N(s) = 0] \quad \textit{(by independent increments)} \\
&= e^{-\lambda t} \quad \textit{(by property A3)}.
\end{aligned}$$

The conditional distribution does not depend on s, variables X_1 and X_2 are thus independent. This immediately implies that also $\mathbb{P}[X_2 > t] = e^{-\lambda t}$, hence, $X_2 \sim Exp(\lambda)$. Repeating this argument gives independence and exponential distribution for all X_n. □

These properties of interarrival times provide necessary framework for the third definition of a Poisson process. We will later see that the general idea of identifying a counting process with its arrival and interarrival times can be also used to define different types of counting processes.

Definition 1.7. *Let* $\{X_n, n \in \mathbb{N}\}$ *be a sequence of independent, identically distributed random variables having an exponential distribution with parameter* $\lambda > 0$*. Let* $\{T_n, n \in \mathbb{N}\}$ *be a sequence of random variables such that* $T_n = \sum_{i=1}^{n} X_i$*. A counting process* $\{N(t), t \geq 0\}$ *given by the sum*

$$N(t) = \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(T_n), \quad t \geq 0,$$

is said to be a Poisson process *with rate* λ*,* $\mathbb{I}_{(0,t]}(x)$ *denotes the indicator function of interval* $(0, t]$*.*

Remark. The independence and exponential distribution of interarrival times X_n along with their relationship to the arrival times T_n all together imply that the nth arrival time T_n has gamma distribution with parameters n and λ, we consider the gamma distribution with density function in form $f(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}$, $x > 0$.

We see that the formal statement of the last definition can be fully derived from the axiomatic one. The converse is also true, the Poisson process defined as in 1.7 has all the properties described in the axiomatic definition, the proof of this claim can be found in Durrett [2019, p. 152–153]. All three definitions of a Poisson process are thus equivalent and can be used interchangeably in the following parts of this text.

1.3 Nonhomogeneous Poisson Process

The applications of a Poisson process can be sometimes limited by the requirement of a constant arrival rate.

Consider for instance a counting process which tracks the occurrence of car accidents in time. We have an information that the number of accidents tends to increase during summer months or weekends and we would like to capture these trends in our model.

If we let the arrival rate of a Poisson process vary over time, we obtain a generalization called a nonhomogeneous Poisson process.

Two definitions of a nonhomogeneous Poisson process will be given. Similarly to the axiomatic definition in the previous section, the first one describes the distribution of increments in an interval of a fixed length.

Definition 1.8. *A counting process* $\{N(t), t \geq 0\}$ *is said to be a* nonhomogeneous Poisson process *with intensity function* $\lambda(t), t \geq 0$ *if the following conditions are satisfied:*

(NA1) $N(0) = 0$*,*

(NA2) the process has independent increments,

(NA3) for all $t, s \geq 0$*, the number of increments* $N(t + s) - N(t)$ *is Poisson distributed with mean* $\int_t^{t+s} \lambda(x)dx$*.*

Remark. In general, a nonhomogeneous Poisson process does not possess stationary increments. This property is preserved only if $\lambda(t) = \lambda$ for some constant λ, the resulting Poisson process is of course the homogeneous one.

The lack of stationary increments must be of course also considered in the second definition, which again describes the behavior of increments in an interval whose length h tends to zero.

Definition 1.9. *A counting process* $\{N(t), t \geq 0\}$ *is said to be a* nonhomogeneous Poisson process *with rate function* $\lambda(t), t \geq 0$ *if the following conditions are satisfied:*

(NI1) $N(0) = 0$,

(NI2) the process has independent increments,

(NI3) $\mathbb{P}[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$,

(NI4) $\mathbb{P}[N(t+h) - N(t) \geq 2] = o(h)$.

Just like in the homogeneous case, we can show that these two definitions describe the same process.

Claim 1.3. *The definitions 1.8 and 1.9 of a nonhomogeneous Poisson process are equivalent.*

Proof. The proof of 1.9 ⇒ 1.8 follows in a similar manner to the homogeneous case and in full can be found in Ross [2014, p. 322–323, Theorem 5.3].

For the implication 1.8 ⇒ 1.9, it must be shown that (NI3) and (NI4) are satisfied. Fix $t \geq 0$ and let $m(h) = \int_t^{t+h} \lambda(x)dx$. The Taylor expansion of $m(h)$ at 0 is

$$m(h) = m(0) + m'(0)h + o(h) = \lambda(t)h + o(h)$$

Using the Taylor expansions of $m(h)$ and the exponential function, the property (NA3) and the fact that

$$\mathbb{P}[N(t+h) - N(t) \geq 2] = 1 - (\mathbb{P}[N(t+h) - N(t) = 1] + \mathbb{P}[N(t+h) - N(t) = 0]),$$

we obtain

$$\begin{aligned}
(NI3)\ \mathbb{P}[N(t+h) - N(t) = 1] &= e^{-m(h)}m(h) = [1 - m(h) + o(h)]m(h) \\
&= [1 - \lambda(t)h + o(h)][\lambda(t)h + o(h)] \\
&= \lambda(t)h - [\lambda(t)h]^2 + o(h) \\
&= \lambda(t)h + o(h), \\
(NI4)\ \mathbb{P}[N(t+h) - N(t) \geq 2] &= 1 - [e^{-m(h)}m(h) + e^{-m(h)}] \\
&= 1 - [[1 - m(h) + o(h)]m(h) + 1 - m(h) + o(h)] \\
&= 1 - [1 - [m(h)]^2 + o(h)] = [m(h)]^2 + o(h) \\
&= [\lambda(t)h + o(h)]^2 + o(h) = [\lambda(t)h]^2 + o(h) \\
&= o(h).
\end{aligned}$$

□

1.4 Renewal Process

Another generalization of a Poisson process is obtained by allowing for other than exponentially distributed interarrival times.

Let us recall that in a homogeneous Poisson process with with rate λ, the sequence of interarrival times $\{X_n, n \in \mathbb{N}\}$ consists of independent, indentically distributed random variables having an exponential distribution with parameter λ and the nth arrival time T_n is gamma distributed with parameters n and λ.

If we preserve the requirements of independence and identical distribution of interarrival times but allow for any nonnegative distribution that is not degenerate at 0, we obtain a new type of counting process, the so-called renewal process.

Definition 1.10. *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of independent, identically distributed nonnegative random variables with a common distribution function F, such that $F(0) = \mathbb{P}[X_n = 0] < 1$. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of random variables where $T_n = \sum_{i=1}^{n} X_i$. A counting process $\{N(t), t \geq 0\}$ obtained as*

$$N(t) = \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(T_n), \quad t \geq 0,$$

is said to be a renewal process.

It is well known that the distribution function of the sum of n independent, identically distributed random variables can be obtained as the n-fold convolution of their common distribution function F with itself. We will denote such n-fold convolution by F_n and additionally define $F_0 = 1$.

It is easy to see that F_n determines the distribution of the nth arrival time T_n. If the distribution of T_n is known, we can also determine the distribution of the state of the process at time t.

Claim 1.4. *Let $\{N(t), t \geq 0\}$ be a renewal process and let F be the distribution function of its interarrival times. The distribution of the state of the process $N(t)$ at time t can be determined as $\mathbb{P}[N(t) = n] = F_n(t) - F_{n+1}(t), n = 0, 1, \ldots$*

Proof. We first notice that if $N(t) \geq n$ at time t, the nth event must have occurred at the latest at time t, hence $T_n \leq t$. We then obtain

$$\begin{aligned}
\mathbb{P}[N(t) = n] &= \mathbb{P}[N(t) \geq n] - \mathbb{P}[N(t) \geq n+1] \\
&= \mathbb{P}[T_n \leq t] - \mathbb{P}[T_{n+1} \leq t] \\
&= F_n(t) - F_{n+1}(t).
\end{aligned}$$

□

The mean function $m(t) = \mathbb{E}\left[N(t)\right]$ of a renewal process $N(t)$ is another object of interest. The function will be also called the *renewal function*.

It is rather straightforward to demonstrate that the renewal function is uniquely determined by the distribution function F of the interarrival times, the n-fold convolutions of F again play an important role.

Claim 1.5. *Let $\{N(t), t \geq 0\}$ be a renewal process and let F be the distribution function of its interarrival times. The renewal function $m(t)$ of the process $N(t)$ can be expressed as*

$$m(t) = \sum_{n=1}^{\infty} F_n(t).$$

Proof.

$$m(t) = \mathbb{E}\,[N(t)] = \mathbb{E}\,[\sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(T_n)] = \sum_{n=1}^{\infty} \mathbb{E}\,[\mathbb{I}_{(0,t]}(T_n)] = \sum_{n=1}^{\infty} \mathbb{P}[T_n \le t] = \sum_{n=1}^{\infty} F_n(t)$$

□

Perhaps more interesting is the opposite statement which will conclude this section and chapter.

Theorem 1.6. *The renewal function $m(t)$ uniquely determines the distribution of the interarrival times and thus the whole renewal process $\{N(t), t \ge 0\}$.*

Proof. We begin by noticing two particular properties of the Laplace-Stieltjes transform. First we show that the Laplace-Stieltjes transform translates convolution to multiplication. Let F_n be the distribution function of the sum $\sum_{i=1}^{n} X_i$ where the variables X_i are independent with a common distribution function F. It then holds true that

$$F_n^*(t) = \mathbb{E}\,[e^{-t\sum_{i=1}^{n} X_i}] = \prod_{i=1}^{n} \mathbb{E}\,[e^{-tX_i}] = [F^*(t)]^n.$$

Second, the fact that $F \le 1$ implies that for a nonnegative random variable X which is not degenerate at 0, the Laplace-Stieltjes transform $F^*(t) < 1$ for all $t > 0$ and $F^*(0) = 1$.

Using these two observation and the claim 1.5 we obtain for all $t > 0$

$$\begin{aligned} m^*(t) &= \int_0^{\infty} e^{-tx} dm(x) = \int_0^{\infty} e^{-tx} d\left(\sum_{n=1}^{\infty} F_n(x)\right) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-tx} dF_n(x) \\ &= \sum_{n=1}^{\infty} F_n^*(t) = \sum_{n=1}^{\infty} [F^*(t)]^n = \frac{F^*(t)}{1 - F^*(t)}. \end{aligned} \tag{1}$$

The proof is completed by rearranging the equation as

$$F^*(t) = \frac{m^*(t)}{1 + m^*(t)}, \tag{2}$$

noticing that the fraction on the right-hand side tends to 1 for $t \to 0$, so we can formally let $F^*(0) = 1$ and by the fact that the Laplace-Stieltjes transform uniquely determines the probability distribution of a random variable. □

2. Superposition and Thinning

Now that the general concept of a counting process has been introduced, we can describe two particular operations on counting processes, the superposition and thinning. We begin by stating the general idea of these operations and continue with several results that emerge when superposing and thinning the specific types of counting processes defined in the previous chapter.

2.1 General Idea

The superposition of two counting processes results in a new counting process whose events occur every time that an event has occurred in either of the two initial processes.

Imagine we have two counting processes tracking the number of claims in two lines of business of an insurance company, say that these are material damage claims and bodily injury claims. One way to represent the combined number of claims in these lines of business, for instance when merging them into one, is to superpose the two initial processes.

Definition 2.1. *Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two counting processes with sequences of arrival times $R = \{R_n, n \in \mathbb{N}\}$ and $S = \{S_n, n \in \mathbb{N}\}$ respectively. A counting process $\{N(t), t \geq 0\}$ whose sequence of arrival times is the union $T = R \cup S$ ordered increasingly, is said to be a* superposition *of $N_1(t)$ and $N_2(t)$.*

Remark. The state of the superposition process $N(t)$ at time t is once again obtained as $N(t) = \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(T_n), T_n \in T$. It clearly does not make any difference whether the occurrences of events until time t are counted jointly in the union T or separately in the sequences R and S. One can thus write that

$$N(t) = \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(T_n) = \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(R_n) + \sum_{n=1}^{\infty} \mathbb{I}_{(0,t]}(S_n) = N_1(t) + N_2(t),$$

hence the superposition $N(t)$ can be also obtained as the sum of the two initial processes at every time t.

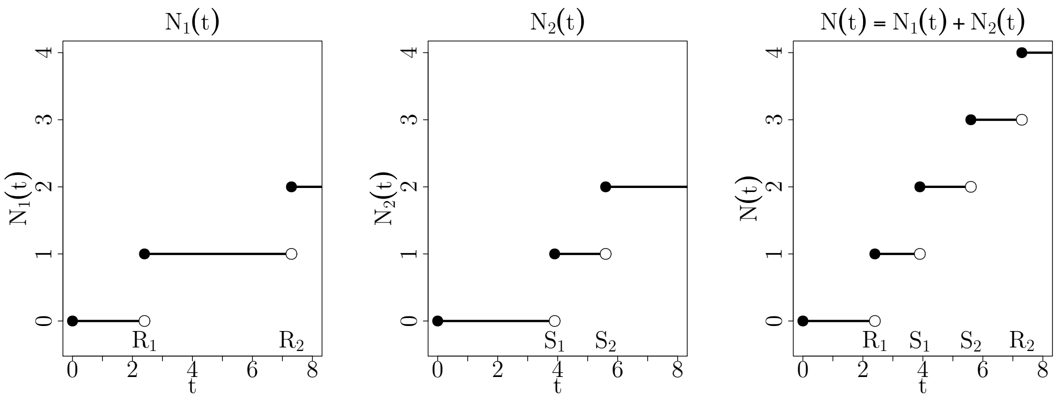

Figure 2.1: *The superposition of two arbitrary counting processes*

The aim of thinning is to classify each event of a counting process into one of different categories based on some given rule.

Consider again a counting process tracking the number of claims in a certain line of business. If an insurance company wanted to split this line of business in two or more new ones, the processes resulting from thinning of the initial counting process would provide a good representation of claim numbers for these new lines of business.

Although various distinct thinning rules can be thought of, we will only consider thinning based on the Bernoulli distribution which splits the initial process in two processes and thinning based on the multinomial distribution which splits the process in k different processes.

Definition 2.2. *Let* $\{N(t), t \geq 0\}$ *be a counting process with sequence of arrival times* $\{T_n, n \in \mathbb{N}\}$ *and let* $\{Y_n, n \in \mathbb{N}\}$ *be a sequence of independent Bernoulli random variables where* $\mathbb{P}[Y_n = j] = p^j(1-p)^{1-j}$, $j \in \{0,1\}, p \in (0,1)$. *Furthermore, let every* Y_n *be independent of every* T_n. *The processes*

$$N_j(t) = \sum_{n=1}^{\infty} \mathbb{I}_{\{j\}}(Y_n)\mathbb{I}_{(0,t]}(T_n), \quad t \geq 0,\ j = 0,1$$

are said to be thinned counting processes*, the thinning rule will be referred to as* Bernoulli thinning.

Definition 2.3. *Let* $\{N(t), t \geq 0\}$ *be a counting process with sequence of arrival times* $\{T_n, n \in \mathbb{N}\}$*, let* $\{Y_n, n \in \mathbb{N}\}$ *be a sequence of independent random variables such that* $\mathbb{P}[Y_n = j] = p_j$, $j \in \{1, 2, \ldots, k\}$, $p_j \in (0,1)$ *for all* j *and* $\sum_{j=1}^{k} p_j = 1$*, and again let every* Y_n *be independent of every* T_n. *The processes*

$$N_j(t) = \sum_{n=1}^{\infty} \mathbb{I}_{\{j\}}(Y_n)\mathbb{I}_{(0,t]}(T_n), \quad t \geq 0,\ j = 1, 2, \ldots, k$$

are said to be thinned counting processes*, the thinning rule will be referred to as* multinomial thinning.

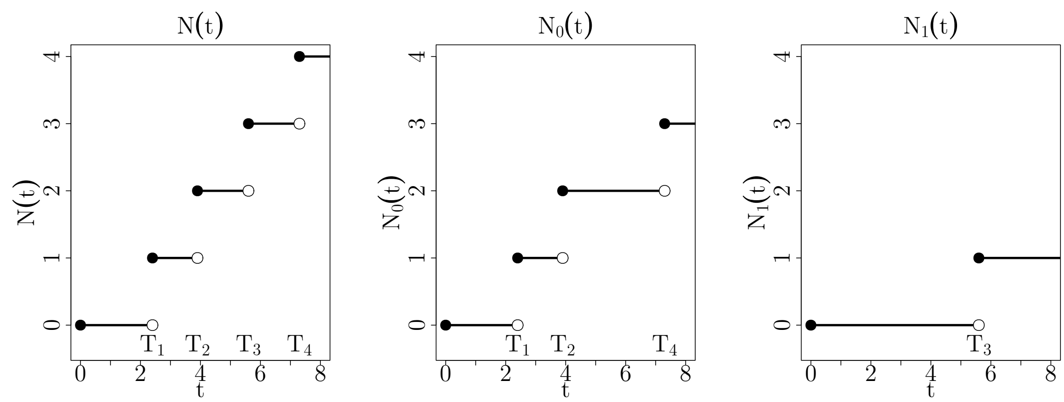

Figure 2.2: *Bernoulli thinning of an arbitrary counting process*

2.2 Poisson Processes

One of the reasons why Poisson processes are vastly used in applications is the fact that with the assumption of independence, they are closed under superposition, Bernoulli and multinomial thinning. The resulting processes are again Poisson and their intensities can be determined in a rather simple manner.

We start with the theorem about superposition and for generality assume the initial Poisson processes to be nonhomogeneous. Throughout the proofs, $m(t) = \int_0^t \lambda(x)dx$ will denote the integral of the intensity function $\lambda(t)$.

Theorem 2.1. *Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent nonhomogeneous Poisson processes with intensity functions $\lambda_1(t)$ and $\lambda_2(t)$ respectively. The superposition $\{N(t), t \geq 0\}$ of $N_1(t)$ and $N_2(t)$ is again a nonhomogeneous Poisson process whose intensity function is $\lambda(t) = \lambda_1(t) + \lambda_2(t)$.*

Proof. We will use the fact that the superposition $N(t)$ can be obtained as the sum $N(t) = N_1(t) + N_2(t)$ at every time t.

As both $N_1(0) = 0$ and $N_2(0) = 0$, also $N(0) = 0$. The property of independent increments of the superposition follows from the independence of the initial processes and the fact that they themselves possess independent increments.

We now show that the number of increments in a time interval $(0, t]$ is Poisson distributed. The proof would proceed similarly for an arbitrary interval $(t, t+s]$.

$$\mathbb{P}[N(t) = n] = \mathbb{P}[N_1(t) + N_2(t) = n] = \sum_{k=0}^{n} \mathbb{P}[N_1(t) = k, N_2(t) = n-k]$$
$$= \sum_{k=0}^{n} \mathbb{P}[N_1(t) = k]\,\mathbb{P}[N_2(t) = n-k] = \sum_{k=0}^{n} e^{-m_1(t)} \frac{[m_1(t)]^k}{k!} e^{-m_2(t)} \frac{[m_2(t)]^{n-k}}{(n-k)!}$$
$$= e^{-[m_1(t)+m_2(t)]} \frac{[m_1(t)+m_2(t)]^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{m_1(t)}{m_1(t)+m_2(t)}\right]^k \left[\frac{m_2(t)}{m_1(t)+m_2(t)}\right]^{n-k}.$$

We notice that the last sum can be thought of as the sum of probabilities of the binomial distribution with parameters n and $\frac{m_1(t)}{m_1(t)+m_2(t)}$, hence it must equal 1. The proof is then finished by the linearity property of integrals, as $m_1(t) + m_2(t) = \int_0^t \lambda_1(x) + \lambda_2(x)dx$. □

Corollary. The theorem naturally holds true also for homogeneous Poisson processes, if we let $\lambda_1(t) = \lambda_1$ and $\lambda_2(t) = \lambda_2$ for some constants $\lambda_1, \lambda_2 > 0$, the superposition is a Poisson process with rate $\lambda_1 + \lambda_2$.

Corollary. The result can be easily generalized for the superposition of any finite number k of independent Poisson processes. The resulting process is Poisson with rate $\sum_{j=1}^{k} \lambda_j$ or intensity function $\sum_{j=1}^{k} \lambda_j(t)$ if the initial processes are generally nonhomogeneous.

The theorem about thinning will be again proved for nonhomogeneous Poisson process and for the multinomial thinning rule, as Bernoulli thinning and homogeneous process variations are once more only special cases of this theorem.

Theorem 2.2. *Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson process with intensity function $\lambda(t)$. The thinned processes $\{N_j(t), t \geq 0\}, j \in \{1, 2, \ldots, k\}$*

resulting from the multinomial thinning of $N(t)$ in the sense of definition 2.3 are independent nonhomogeneous Poisson processes with intensity functions $p_j\lambda(t)$.

Proof. The properties of the initial process again imply that $N_j(0) = 0$ for every j and that the thinned processes possess independent increments.

The independence of thinned processes and the Poisson distribution of increments will be again shown for a time interval $(0, t]$.

We first notice that if n_j events occurred until time t in the jth thinned process, then $n = \sum_{j=1}^{k} n_j$ events must have occurred until time t in the initial process $N(t)$. The probability that these n events are classified such that there are n_j events in the jth thinned process is given by the multinomial distribution and since the classification is independent of the initial process, we can write

$$\mathbb{P}[N_1(t) = n_1, \ldots, N_k(t) = n_k] = e^{-m(t)} \frac{[m(t)]^n}{n!} \frac{n!}{n_1! n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$$

$$= e^{-\sum_{j=1}^{k} p_j m(t)} \frac{[m(t)]^{\sum_{j=1}^{k} n_j}}{n_1! n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} = \prod_{j=1}^{k} e^{-p_j m(t)} \frac{[p_j m(t)]^{n_j}}{n_j!}.$$

The proof is once again finished by the linearity property of integrals, as $p_j m(t) = \int_0^t p_j \lambda(x) dx$ □

Corollary. The result for Bernoulli thinning is a special case of the theorem if we choose the number of categories $k = 2$, if we choose $\lambda(t) = \lambda$ for some constant $\lambda > 0$, we obtain the result for homogeneous Poisson processes.

2.3 Renewal Processes

Consider now that the initial processes are renewal. We are once again interested in conditions that must be satisfied in order for the renewal processes to be closed under superposition and the two thinning rules mentioned earlier.

Just like in the previous section, we begin with results about superposition. Let us first divide the renewal processes in two groups based on the distribution of their interarrival times. We say that the renewal process is *ordinary*, if its interarrival times are strictly positive random variables. If there is a positive probability that any interarrival time is zero, the renewal process is said to have *multiple renewals*.

It is shown in Samuels [1974] that if the superposition of two ordinary renewal processes is again an ordinary renewal process, all processes must be Poisson. We will now state an extension of this theorem which also considers processes with multiple renewals. The statement and the proof of the theorem comes from Ferreira [2000], we will provide further clarification of several steps of the proof and correct some minor mistakes appearing in the original paper.

Theorem 2.3. *Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent renewal processes with sequences of interarrival times $\{X_n, n \in \mathbb{N}\}$ and $\{Y_n, n \in \mathbb{N}\}$, and let F and G be the common distribution functions of these interarrival times. The superposition $\{N(t), t \geq 0\}$ of $N_1(t)$ and $N_2(t)$ is a renewal process if and only if*

(1) all processes are ordinary, in which case F and G are exponential, hence $N_1(t), N_2(t)$ and $N(t)$ are all Poisson,

(2) one and only one of the two processes has multiple renewals, in which case F and G are concentrated on a set $\{0, \delta, 2\delta, \dots\}$ for some $\delta > 0$ and belong to one of these families of distributions:

- $F(x) = 1 - p^{[x/\delta]+1}, \quad x \geq 0, 0 < p < 1,$
  $G(x)$ *degenerate at* δ,
- $F(x) = 1 - p^{[x/\delta]+1}, \quad x \geq 0, 0 < p < 1,$
  $G(x) = 1 - q^{[x/\delta]}, \quad x \geq 0, 0 < q < 1,$

where $[x]$ denotes the integer part of x. The distributions of F and G can be also interchanged.

Proof. The first alternative corresponds to the result proved in Samuels [1974] and to the results about Poisson processes proved in the previous section.

Throughout the proof of the second alternative, $\overline{F} = 1 - F$ will denote the survival function of any distribution F, $\operatorname{supp}(F)$ will denote the support of the distribution and x_F^+ will denote the right endpoint of the distribution, formally $x_F^+ = \sup\{x : F(x) < 1\}$. Furthermore, let us denote the sequence of interarrival times of the superposition $N(t)$ by $\{Z_n, n \in \mathbb{N}\}$.

To start with, let us look at the first two interarrival times of the superposition. A necessary condition for the superposition to be a renewal process is that Z_1 and Z_2 are independent and identically distributed. This already enforces several conditions for the interarrival distributions of the initial processes.

It is easy to determine that $Z_1 = \min\{X_1, Y_1\}$, Z_2 then depends on whichever of X_1 and Y_1 is smaller and is either the distance between X_1 and Y_1 or the next interarrival time, so $Z_2 = \min\{|X_1 - Y_1|, X_2\mathbb{I}(X_1 < Y_1) + Y_2\mathbb{I}(Y_1 < X_1)\}$.

Let H be the distribution of Z_1. Since the initial processes are independent, $\mathbb{P}[\min\{X_1, Y_1\} > u] = \mathbb{P}[X_1 > u]\,\mathbb{P}[Y_1 > u]$, hence H is determined by F and G as $\overline{H} = \overline{F}\,\overline{G}$. The independence and identical distribution of Z_1 and Z_2 is now satisfied if the following equation holds

$$\mathbb{P}[Z_1 > u, Z_2 > v] = \overline{F}(u)\overline{G}(u)\overline{F}(v)\overline{G}(v), \quad u, v \in \mathbb{R}.$$

For $v \geq 0$, the left-hand side can be also expressed as

$$\begin{aligned}\mathbb{P}[Z_1 > u, Z_2 > v] &= \mathbb{P}[Y_1 > u, X_1 - Y_1 > v, Y_2 > v] \\ &\quad + \mathbb{P}[X_1 > u, Y_1 - X_1 > v, X_2 > v] \\ &= \int_u^\infty \overline{F}(x+v)dG(x)\overline{G}(v) + \int_u^\infty \overline{G}(x+v)dF(x)\overline{F}(v),\ u \in \mathbb{R},\end{aligned}$$

which is just the split into cases when $X_1 < Y_1$ and when $Y_1 < X_1$, the final equality expresses the probabilities as general Lebesgue-Stieltjes integrals, since F and G can be arbitrary distribution functions, multiplied by the respective survival functions, since X_2 and Y_2 are independent of the previous interarrival times.

For $v < 0$, the right-hand side simplifies to $\overline{F}(u)\overline{G}(u), u \in \mathbb{R}$ and so the formula for integration by parts of right continuous functions in Lebesgue-Stieltjes integral must hold. We have thus derived the two following formulas

$$\int_u^\infty \overline{F}(x+v)dG(x)\overline{G}(v) + \int_u^\infty \overline{G}(x+v)dF(x)\overline{F}(v) = \overline{F}(u)\overline{G}(u)\overline{F}(v)\overline{G}(v)$$

for $u \in \mathbb{R}, v \geq 0$ and

$$\int_u^\infty \overline{F}(x)dG(x) + \int_u^\infty \overline{G}(x)dF(x) + \int_u^\infty [G(x) - G(x_-)]dF(x) = \overline{F}(u)\overline{G}(u)$$

for $u \in \mathbb{R}$, where x_- denotes the left-hand limit. By the additivity property of integrals, the formulas can be also rewritten as

$$\begin{aligned} &\int_{u-h}^{u+h} \overline{F}(x+v)dG(x)\overline{G}(v) + \int_{u-h}^{u+h} \overline{G}(x+v)dF(x)\overline{F}(v) \\ &= \overline{F}(v)\overline{G}(v)[\overline{F}(u-h)\overline{G}(u-h) - \overline{F}(u+h)\overline{G}(u+h)], \quad u \in \mathbb{R}, v \geq 0, h > 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\int_{u-h}^{u+h} \overline{F}(x)dG(x) + \int_{u-h}^{u+h} \overline{G}(x)dF(x) + \int_{u-h}^{u+h} [G(x) - G(x_-)]dF(x) \\ &= \overline{F}(u-h)\overline{G}(u-h) - \overline{F}(u+h)\overline{G}(u+h), \quad u \in \mathbb{R}, h > 0. \end{aligned} \tag{4}$$

Let us now assume that one of the initial processes has multiple renewals, say $\overline{F}(0) = p < 1$ and let $\overline{G}(0) = q$. By putting $u = 0$ and letting $h \searrow 0$ in (3), we obtain

$$\overline{F}(v)\overline{G}(v)(1-q) + \overline{F}(v)\overline{G}(v)(1-p) = \overline{F}(v)\overline{G}(v)(1-pq), \quad v \geq 0,$$

which only holds if $(1-p)(1-q) = 0$ and since $p < 1$, q must equal 1, meaning that $\overline{G}(0) = 1$. Hence, for the superposition to be a renewal process, at most one of the initial processes may possess multiple renewals.

We continue with the assumption that $\overline{F}(0) = p < 1$ and thus $\overline{G}(0) = 1$. For $v = 0$, (3) can be simplified as

$$\int_{u-h}^{u+h} \overline{F}(x)dG(x)p^{-1} + \int_{u-h}^{u+h} \overline{G}(x)dF(x) = \overline{F}(u-h)\overline{G}(u-h) - \overline{F}(u+h)\overline{G}(u+h), \tag{5}$$

$u \in \mathbb{R}, h > 0$. Now we combine (4) with (5). Since $p^{-1} > 1$, the combined equation holds only if

$$\int_{u-h}^{u+h} [G(x) - G(x_-)]dF(x) = \frac{1-p}{p} \int_{u-h}^{u+h} \overline{F}(x)dG(x),$$

which can only be true if every $u \leq x_F^+$ such that $u \in \text{supp}(G)$ is also a point of increase of F. Hence, if $u \in [0, x_F^+] \cap \text{supp}(G)$, u must be also in $\text{supp}(F)$ and so the sets $[0, x_F^+] \cap \text{supp}(G)$ and $\text{supp}(F) \cap \text{supp}(G)$ must be identical.

Since (4) is a general formula for integration by parts, it must also hold if we interchange the two distributions. The combined equation of (4) and (5) can be then simplified to

$$\int_{u-h}^{u+h} \overline{F}(x)dG(x) = \frac{p}{1-p} \int_{u-h}^{u+h} [\overline{F}(x_-) - \overline{F}(x)]dG(x),$$

letting $h \searrow 0$, we obtain

$$\overline{F}(u) = p\overline{F}(u_-), \quad u \in \mathrm{supp}(F) \cap \mathrm{supp}(G). \tag{6}$$

We will now derive several additional properties of F and G.

(a) *The set* $\mathrm{supp}(F) \cap \mathrm{supp}(G) = [0, x_F^+] \cap \mathrm{supp}(G)$ *is not empty.*

If $[0, x_F^+] \cap \mathrm{supp}(G)$ was empty, then $x \in \mathrm{supp}(G)$ would imply $x > x_F^+$, the interarrival distribution of the superposition would thus be $H = F$ and the superposition $N(t)$ would be a copy of the first initial process $N_1(t)$. Since the equation $m_1(t) + m_2(t) = m(t)$ must hold for the renewal functions of the respective processes and for all $t \geq 0$, $m_2(t)$ would have to equal 0 for all $t \geq 0$. This is only possible if G is concentrated at ∞, an option that we exclude. This point then implies that $\inf\{\mathrm{supp}(F) \cap \mathrm{supp}(G)\} < \infty$.

(b) *The set* $\mathrm{supp}(F) \cap \mathrm{supp}(G) = [0, x_F^+] \cap \mathrm{supp}(G)$ *is not dense in any right neighbourhood of* 0.

If it was, we could choose $0 < \varepsilon < x_F^+$ and an infinite sequence $\{x_n\}$ in the intersection of the two supports such that $0 < x_1 < x_2 < \cdots < x_n < \varepsilon$ for every n. If we repeatedly apply (6), we obtain

$$\overline{F}(\varepsilon) \leq p\overline{F}(x_n) = p\overline{F}(x_{n^-}) \leq p\overline{F}(x_{n-1}) \leq p^2\overline{F}(x_{n-1^-}) \leq \cdots \leq p^{n-1}\overline{F}(x_1) \leq p^n$$

for all n, hence $F(\varepsilon) = 1$, which is a contradiction with the definition of x_F^+. It follows that $\inf\{\mathrm{supp}(F) \cap \mathrm{supp}(G)\} = \min\{\mathrm{supp}(F) \cap \mathrm{supp}(G)\} > 0$. Now let $u_1 = \min\{\mathrm{supp}(F) \cap \mathrm{supp}(G)\}$. It can be then shown that u_1 must be the first support point of G. For if it was not, there would be a $u \in \mathrm{supp}(G), u < u_1$ and since $u < x_F^+, u \in \mathrm{supp}(F)$, which contradicts the definition of u_1.

(c) *The interval* $(0, u_1)$ contains no point of $\mathrm{supp}(F)$.

If there was a u, $0 < u < u_1$ in $\mathrm{supp}(F)$, we could take $v = u_1 - (u - h)$ for a sufficiently small h, $0 < h < u$ and since u_1 is the first support point of G, (3) reduces to

$$\int_{u-h}^{u+h} \overline{G}(x)dF(x) = [F(u+h) - F(u-h)].$$

But $\overline{G}(x+v) = \overline{G}(x + u_1 - (u-h)) < 1$ for every $x > u - h$, hence

$$\int_{u-h}^{u+h} \overline{G}(x)dF(x) < [F(u+h) - F(u-h)],$$

which contradicts the previous equation and u_1 is thus the first positive point of increase of F. To sum up, we now know that $\overline{F}(0) = p, \overline{F}(u_1) = p\overline{F}(0) = p^2$ and $\overline{G}(u_1) = q$. Let us now set $u_1 = 1$ for simplicity, although the next steps would proceed similarly for any $u_1 = \delta > 0$.

(d) *F and G are concentrated on* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If we let $x \in \mathrm{supp}(F)$ be a non-integer, it can be easily derived that the two following events have positive probabilites.

$$\mathbb{P}[Y_1 + Y_2 + \cdots + Y_{[x]} = [x]] \geq \prod_{i=1}^{[x]} \mathbb{P}[Y_1 = 1] = (1-q)^{[x]} > 0$$

and

$$\mathbb{P}[X_1 \in (x-h, x+h)] > 0 \text{ for every } h > 0.$$

For any $0 < h < \min\{x-[x], 1-x+[x]\}$, the two events imply that $Y_1 + Y_2 + \cdots + Y_{[x]} = [x] < X_1$ and since $Y_{[x]+1} \geq 1$, $Z_{[x]+1} = X_1 - [x]$. But since all previous events have positive probabilities, it must also be true that

$$\mathbb{P}[Z_{[x]+1} \in (x-[x]-h, x-[x]+h)] > 0$$

and for the specified choice of h this also means that

$$\mathbb{P}[Z_{[x]+1} \in (0,1)] \geq \mathbb{P}[Z_{[x]+1} \in (x-[x]-h, x-[x]+h)] > 0,$$

which is a contradiction, because since F nor G has a point of increase in $(0,1)$, neither can the interarrival distribution H of the superposition. The argument also holds for F and G interchanged, which finishes the proof of this point.

The exact distributions of F and G can now be derived. We start with the first family of distributions appearing in the statement of this theorem, we let G be degenerate at $u_1 = 1$. Since $F(0) = \mathbb{P}[X_n = 0] = 1-p, \mathbb{P}[Y_n = 1] = 1$ and F is concentrated on $\mathbb{N}_0$, the interarrival distribution H is detemined as

$$H(x) = \begin{cases} 0 & x < 0, \\ 1-p & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

The Laplace-Stieltjes transforms $m^*(t)$ and $m_2^*(t)$ of the renewal functions $m(t)$ and $m_2(t)$ can be derived from (1) if we realize that the distribution function G increases by 1 in one and H increases by $1-p$ in zero and by p in one. It follows that

$$G^*(t) = \int_0^\infty e^{-tx} dG(x) = e^{-t}, \quad m_2^*(t) = \frac{e^{-t}}{1-e^{-t}}, \quad t \geq 0$$

and

$$H^*(t) = \int_0^\infty e^{-tx} dH(x) = 1-p+pe^{-t}, \quad m^*(t) = \frac{1-p(1-e^{-t})}{p(1-e^{-t})}, \quad t \geq 0,$$

the functions are defined by the right-hand limit in $t = 0$ if necessary. The Laplace-Stieltjes transform $m_1^*(t)$ of the renewal function $m_1(t)$ can now be derived, and using (2), so can be the Laplace-Stieltjes transform $F^*(t)$ of the distribution function F

$$m_1^*(t) = m^*(t) - m_2^*(t) = \frac{1-p}{p(1-e^{-t})}, \quad F^*(t) = \frac{1-p}{1-pe^{-t}}, \quad t \geq 0.$$

It can be easily shown that $F^*(t)$ is the Laplace-Stieltjes transform of the distribution function $F(x) = 1 - p^{[x]+1}, x \geq 0$ if we notice that the function increases by $p^n(1-p)$ in every $n \in \mathbb{N}_0$.

$$F^*(t) = \int_0^\infty e^{-tx} dF(x) = \sum_{n=0}^\infty e^{-tn} p^n (1-p) = (1-p) \sum_{n=0}^\infty (pe^{-t})^n = \frac{1-p}{1-pe^{-t}}, t \geq 0.$$

Now consider that G is not degenerate at $u_1 = 1$ and so $\overline{G}(u_1) = q > 0$. We also know that $\overline{F}(0) = p$ and $\overline{F}(u_1) = p^2$. It follows that both distribution must have at least one more support point in $\{2, 3, \dots\}$. Let us assume that F and G are in form

$$\overline{F}(k) = p^{k+1},\ k = 0, 1, \dots, r, \qquad \overline{G}(k) = q^k,\ k = 1, \dots, r. \tag{7}$$

The statement obviously holds for $r = 1$, we will now use the proof by induction. Suppose the statement holds for an arbitrary r, we want to show that it also holds for $r + 1$. The next equations for interarrival times of the superposition again follow as a split into cases of different possible interarrivals of the initial processes. We have

$$\begin{aligned}
&\mathbb{P}[Z_1 = 1, Z_2 > r] \\
&= \mathbb{P}[X_1 = 1, X_2 > r, Y_1 > r + 1] + \mathbb{P}[Y_1 = 1, Y_2 > r, X_1 > r + 1] \\
&= \mathbb{P}[X_1 = 1]\,\mathbb{P}[X_2 > r]\,\mathbb{P}[Y_1 > r + 1] + \mathbb{P}[Y_1 = 1]\,\mathbb{P}[Y_2 > r]\,\mathbb{P}[X_1 > r + 1] \\
&= p(1 - p)p^{r+1}\overline{G}(r + 1) + (1 - q)q^r\overline{F}(r + 1)
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\mathbb{P}[Z_1 = 1, Z_2 = 0, Z_3 > r] \\
&= \mathbb{P}[X_1 = 1, Y_1 = 1, Z_3 > r] + \mathbb{P}[X_1 = 1, X_2 = 0, Y_1 > 1, Z_3 > r] \\
&= \mathbb{P}[X_1 = 1]\,\mathbb{P}[Y_1 = 1]\,\mathbb{P}[X_2 > r]\,\mathbb{P}[Y_2 > r] \\
&+ \mathbb{P}[X_1 = 1]\,\mathbb{P}[X_2 = 0]\,\mathbb{P}[X_3 > r]\,\mathbb{P}[Y_1 > r + 1] \\
&= p(1 - p)(1 - q)p^{r+1}q^r + p(1 - p)^2p^{r+1}\overline{G}(r + 1)
\end{aligned} \tag{9}$$

Since $\overline{H} = \overline{F}\,\overline{G}$ and by independence of the interrarival times of the superposition, it must also be true that

$$\begin{aligned}
&\mathbb{P}[Z_1 = 1, Z_2 > r] = \mathbb{P}[Z_1 = 1]\,\mathbb{P}[Z_2 > r] \\
&= p^2(1 - pq)(pq)^r
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\mathbb{P}[Z_1 = 1, Z_2 = 0, Z_3 > r] = \mathbb{P}[Z_1 = 1]\,\mathbb{P}[Z_2 > r]\,\mathbb{P}[Z_3 > r] \\
&= p^2(1 - pq)(1 - p)(pq)^r
\end{aligned} \tag{11}$$

Setting (9) and (11) equal yields $\overline{G}(r + 1) = q^{r+1}$, if we plug this in (8) and set it equal with (10) we also obtain that $\overline{F}(r + 1) = p^{r+2}$. Hence, if (7) holds for r, it also holds for $r + 1$ and the two distributions are precisely the ones presented in the second family of distributions in the statement of this theorem.

The proof is finally completed if we can show that the superposition of the initial processes, whose interarrival times have the derived distributions F and G, is in fact a renewal process, in other words that

$$\mathbb{P}[Z_1 = i_1, \dots, Z_{k+1} = i_{k+1}] = \mathbb{P}[Z_1 = i_1] \dots \mathbb{P}[Z_{k+1} = i_{k+1}] = \prod_{j=1}^{k+1} \mathbb{P}[Z = i_j]. \tag{12}$$

Without loss of generality, let again $\delta = 1$ and so the equation must hold for $k = 1, 2, \dots$ and $i_1, \dots, i_{k+1} \in \mathbb{N}_0$. The variable Z in the last expression is not indexed in order to show that the interarrival times must be also identically distributed.

We begin with the second family of distributions and obtain

$$\begin{aligned}&\mathbb{P}[Z_1 = i_1, \dots, Z_{k+1} = i_{k+1}]\\&= \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1}, X_1 = i_1, Y_1 > i_1]\\&+ \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1}, X_1 \geq i_1, Y_1 = i_1]\\&= \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 = i_1, Y_1 > i_1]\, \mathbb{P}[X_1 = i_1]\, \mathbb{P}[Y_1 > i_1]\\&+ \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 \geq i_1, Y_1 = i_1]\, \mathbb{P}[X_1 \geq i_1]\, \mathbb{P}[Y_1 = i_1]. \end{aligned} \tag{13}$$

The interarrival times $Z_j, j = 2, \dots, k+1$ in the first conditional probability are functions of $X_2, X_3, \dots, Y_2, Y_3, \dots$ and $Y_1 - i_1$ and thus only depend on the latter of the two conditioning events. Likewise, the interarrival times in the second conditional probability only depend on the variable X_1 through $X_1 - i_1$. We now notice that the derived distributions F and G are memoryless, in other words

$$\mathbb{P}[Y_1 - i_1 > x | Y_1 > i_1] = \frac{\mathbb{P}[Y_1 > x + i_1, Y_1 > i_1]}{\mathbb{P}[Y_1 > i_1]} = \frac{q^{[x+i_1]}}{q^{[i_1]}} = q^{[x]} = \mathbb{P}[Y_1 > x]$$

and similarly

$$\mathbb{P}[X_1 - i_1 > x | X_1 \geq i_1] = \frac{p^{[x+i_1]+1}}{p^{[i_1]+1} + p^{i_1}(1-p)} = p^{[x]+1} = \mathbb{P}[X_1 > x].$$

Hence, if $Z_1 = i_1$, the next interarrival times $Z_j, j = 2, \dots, k+1$ behave as if the process was restarted and so

$$\begin{aligned}&\mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 = i_1, Y_1 > i_1]\\&= \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 \geq i_1, Y_1 = i_1] = \mathbb{P}[Z_1 = i_2, \dots, Z_k = i_{k+1}].\end{aligned}$$

The equation (13) can thus be rewritten as

$$\mathbb{P}[Z_1 = i_1, \dots, Z_{k+1} = i_{k+1}] = \mathbb{P}[Z_1 = i_2, \dots, Z_k = i_{k+1}]\, \mathbb{P}[Z = i_1],$$

$\mathbb{P}[Z = i_1]$ of course being $\mathbb{P}[X_1 = i_1]\, \mathbb{P}[Y_1 > i_1] + \mathbb{P}[X_1 \geq i_1]\, \mathbb{P}[Y_1 = i_1]$. The induction on k would prove that (12) truly holds.

For the first family of distributions, $i_1, \dots, i_{k+1}$ in (12) can only be either 0 or 1. For $i_1 = 0$, the equation reads

$$\begin{aligned}&\mathbb{P}[Z_1 = 0, \dots, Z_{k+1} = i_{k+1}] = \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1}, X_1 = 0]\\&= \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 = 0]\, \mathbb{P}[X_1 = 0]\\&= \mathbb{P}[Z_1 = i_2, \dots, Z_k = i_{k+1}]\, \mathbb{P}[Z = 0],\end{aligned}$$

the last equality follows from the fact that $Z_j, j = 2, \dots, k+1$ are determined by $X_2, X_3, \dots, Y_1, Y_2, \dots$ upon conditioning on $X_1 = 0$. For $i_1 = 1$, the equation reads

$$\begin{aligned}&\mathbb{P}[Z_1 = 1, \dots, Z_{k+1} = i_{k+1}] = \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1}, X_1 \geq 0]\\&= \mathbb{P}[Z_2 = i_2, \dots, Z_{k+1} = i_{k+1} | X_1 \geq 1]\, \mathbb{P}[X_1 \geq 1]\\&= \mathbb{P}[Z_1 = i_2, \dots, Z_k = i_{k+1}]\, \mathbb{P}[Z = 1],\end{aligned}$$

memorylessness of F and the dependence of $Z_j, j = 2, \dots, k+1$ on X_1 only through $X_1 - 1$ are once more used to obtain the final equality, the proof is again finished by the induction on k. □

The result for multinomial and Bernoulli thinning of a renewal process is much simpler to demonstrate. It turns out that the renewal processes are closed under these two thinning rules and that the interarrival distributions of the thinned processes can be determined with the help of moment generating functions. It is again sufficient to prove the result for the multinomial thinning rule, as the Bernoulli thinning is a special case for $k = 2$.

Theorem 2.4. *Let $\{N(t), t \geq 0\}$ be a renewal process, let $\{X_n, n \in \mathbb{N}\}$ be a sequence of its interarrival times and let $M_X(s)$ be the moment generating function of any X_n. Let $\{N_j(t), t \geq 0\}, j \in \{1, 2, \ldots, k\}$ be the counting processes resulting from multinomial thinning of $N(t)$ in the sense of definition 2.3. Then the following statements are true.*

(1) Every thinned process $N_j(t)$ is a renewal process.

(2) The probability distribution of interarrival times $\{Z_n^j, n \in \mathbb{N}\}$ of any process $N_j(t)$ is determined by the moment generating function $M_{Zj}(s)$ of any Z_n^j as

$$M_{Zj}(s) = \frac{p_j M_X(s)}{1 - (1 - p_j) M_X(s)}.$$

Proof. Fix $j \in \{1, 2, \ldots, k\}$ and for a better clarity, let the interarrival times of $N_j(t)$ be denoted by $\{Z_n, n \in \mathbb{N}\}$

(1) We notice that the first interarrival time Z_1 can be expressed as a random sum, $Z_1 = \sum_{i=1}^{Y} X_i$, where Y is a geometric random variable with parameter p_j, the distribution is considered such that $\mathbb{P}[Y = n] = p_j(1 - p_j)^{n-1}, n = 1, 2, \ldots$ It follows from the definition of multinomial thinning that Y is also independent of every X_n.

Let y_1 denote the index of the first arrival time classified into jth thinned process. The second interarrival time Z_2 can then be expressed as $Z_2 = \sum_{i=y_1+1}^{Y+y_1} X_i$, where Y again has a geometric distribution with parameter p_j.

Since every X_n is nonnegative, so are Z_1 and Z_2. Since the sequence of interarrival times $\{X_n, n \in \mathbb{N}\}$ consists of independent variables and no X_n is repeated in both random sums determining Z_1 and Z_2, they are also independent, and since the random sum determining Z_2 is the same as for Z_1, only shifted by a fixed number y_1, this together with identical distribution of variables X_n implies that Z_1 and Z_2 are also identically distributed.

The same argument can be used to show that these properties hold for all Z_n. The sequence $\{Z_n, n \in \mathbb{N}\}$ thus consists of nonnegative, independent and identically distributed random variables, hence $N_j(t)$ is by definition a renewal process.

(2) To determine the probability distribution of any Z_n, it is convenient to use moment generating functions. Let $M_Z(s)$ denote the moment generating function

of any Z_n. It can be then written that

$$\begin{aligned}
M_Z(s) = \mathbb{E}\left[e^{sZ}\right] &= \mathbb{E}\left[\mathbb{E}\left(e^{sZ}|Y\right)\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{sZ}|Y = n\right] \mathbb{P}[Y = n] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{s\sum_{i=1}^{n} X_i}\right] \mathbb{P}[Y = n] \qquad (\textit{by independence of } Y \textit{ and } \{X_n\}) \\
&= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \mathbb{E}\left[e^{sX_i}\right] \mathbb{P}[Y = n] \qquad (\textit{by independence of } \{X_n\}) \\
&= \sum_{n=0}^{\infty} \mathbb{P}[Y = n][M_X(s)]^n \qquad (\textit{by identical distribution of } \{X_n\})
\end{aligned}$$

which is just the probability generating function of Y evaluated at $M_X(s)$. Since Y is geometric with parameter p_j, the proof is completed by plugging $M_X(s)$ into the probability generating function of this distribution, we obtain

$$M_Z(s) = \frac{p_j M_X(s)}{1 - (1 - p_j)M_X(s)}.$$

□

3. Application

To demonstrate the practical application of the theoretical results discussed in the previous chapters, we will consider the following situation. An insurance company is required to merge two of its lines of business. The company possesses the data of reported claims until the day of merging and wants to estimate the number of claims in the merged line of business for the following month.

We will model this situation by superposing the counting processes representing the numbers of claims in the initial lines of business. Although the necessary conditions for the renewal processes to be closed under superposition have been stated, they are a bit limiting due to the requirement of geometric-like distributions of interarrival times of the initial processes. The distribution of interarrival times of the superposition process is also not that trivial to determine. However, if we assume that the claims in the initial lines of business come from nonhomogeneous Poisson processes, the superposition will again be a nonhomogeneous Poisson process whose intensity function can be easily determined as the sum of the intensity functions of the two initial processes. The number of claims in the following month will then be Poisson distributed, the parameter being the integral of the intensity function of the superposition process taken over the desired month. Thus, from now on, we will only work with this type of counting process.

The chapter covers the necessary theory for parameter estimation of a particular type of intensity function used as a model for the intensities of the initial processes and an application to insurance data provided by the Czech Insurers' Bureau with a short discussion of different possible estimates for the future number of claims. Version 4.3.0 of the programming language R is used as a software solution to perform the calculations.

3.1 Parameter estimation

The first step after choosing an appropriate model to represent the data is to estimate its parameters. In this case, we will utilize the likelihood function to estimate a particular type of intensity function of a nonhomogeneous Poisson process. Let us first begin with a more general setting of this approach.

Suppose we have observed the times of events $(t_1, t_2, \ldots, t_n)^T$ in a certain time period $(0, T], T > 0$. As stated before, we assume that this observation is a realization of a nonhomogeneous Poisson process. Furthermore, let us assume that the intensity function of this process can be written as a function of parameter $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_m)^T$ from a certain set $\Theta \subset \mathbb{R}^{m+1}$. We will denote this intensity function by $\lambda(t|\boldsymbol{\theta})$. According to Streit [2010, p.17, Equation 2.12], the likelihood function in this setting can be written as

$$L_n(\boldsymbol{\theta}) = \left(\prod_{i=1}^{n} \lambda(t_i|\boldsymbol{\theta})\right) e^{-m(T)}, \quad m(T) = \int_0^T \lambda(x|\boldsymbol{\theta}) dx,$$

the log likelihood function is then

$$\ell_n(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log(\lambda(t_i|\boldsymbol{\theta})) - \int_0^T \lambda(x|\boldsymbol{\theta}) dx.$$

The maximum likelihood estimate of the parameter $\boldsymbol{\theta}$ is of course obtained as

$$\widehat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}\in\Theta} \ell_n(\boldsymbol{\theta}).$$

Various different types of intensity functions could be thought of, for instance the polynomial intensities in form $\sum_{j=0}^{m} \theta_j t^j$ or intensities containing trigonometric terms to model the potential seasonality of events.

Despite that, we will only consider one specific type of the intensity function proposed in Cox and Lewis [1966], the so-called *log linear* intensity function which is in form $\lambda(t) = e^{\theta_0+\theta_1 t}$. The most notable advantage of this type of intensity function compared to the polynomial models is that the log linear function is always positive and so is its integral taken over any bounded interval. The latter is a necessary condition for the Poisson distribution in the definition 1.8 to always make sense.

Monotonically increasing or decreasing trends in the number of events can be modeled with the log linear intensity for the choices of $\theta_1 > 0$ and $\theta_1 < 0$ respectively, the trend is locally close to linear for the values of θ_1 near zero. If $\theta_1 = 0$, there is no trend in the number of events, the Poisson process is thus homogeneous with a constant parameter $\lambda = e^{\theta_0}$.

The log likelihood function for the log linear intensity is in form

$$\ell_n(\boldsymbol{\theta}) = n\theta_0 + \theta_1 \sum_{i=1}^{n} t_i - \frac{e^{\theta_0}(e^{\theta_1 T} - 1)}{\theta_1}. \tag{14}$$

Cox and Lewis [1966, p.46] however suggest that since the observations enter the log likelihood function only through n and $\sum t_i$ and for a given θ_1, the sufficient statistic to determine θ_0 is the number of events n, the conditional distribution of $\sum t_i$ given n can be used for inference about θ_1. The conditional log likelihood function in form

$$\ell_n^*(\theta_1) = n\log\theta_1 - n\log(e^{\theta_1 T} - 1) + \theta_1 \sum_{i=1}^{n} t_i + \log n!$$

is proposed to be used in order to determine the estimate of θ_1. The score and the Fisher information are then derived as

$$\frac{\partial \ell_n^*(\theta_1)}{\partial \theta_1} = \begin{cases} \frac{n}{\theta_1} - \frac{nT}{1-e^{-\theta_1 T}} + \sum_{i=1}^{n} t_i, & \theta_1 \neq 0, \\ \\ -\frac{1}{2}nT + \sum_{i=1}^{n} t_i, & \theta_1 = 0 \end{cases} \tag{15}$$

and

$$I(\theta_1) = \mathbb{E}\left[-\frac{\partial^2 \ell_n^*(\theta_1)}{\partial \theta_1^2}\right] = \begin{cases} n\left(\frac{1}{\theta_1^2} - \frac{T^2 e^{-\theta_1 T}}{(1-e^{-\theta_1 T})^2}\right), & \theta_1 \neq 0, \\ \\ \frac{nT^2}{12}, & \theta_1 = 0. \end{cases}$$

The maximum likelihood estimate $\widehat{\theta}_1$ of θ_1 must of course be the root of (15). It can be found numerically, we will use the function `uniroot` built in the programming langauge R for this purpose. The positivity of $I(\widehat{\theta}_1)$ should also be checked to confirm that the log likelihood function is concave at $\widehat{\theta}_1$.

Once the estimate $\widehat{\theta}_1$ is obtained, it can be used for the estimation of parameter θ_0. Differentiating (14) with respect to θ_0 gives

$$\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \theta_0} = n - \frac{e^{\theta_0}(e^{\theta_1 T} - 1)}{\theta_1},$$

plugging in the estimate $\widehat{\theta}_1$ and setting the derivative equal to zero then gives the maximum likelihood estimate of θ_0 as

$$\widehat{\theta}_0 = \log\left(\frac{n\widehat{\theta}_1}{e^{\widehat{\theta}_1 T} - 1}\right).$$

A formal test for homogeneity of the process is also stated in Cox and Lewis [1966, p. 47], in other words, the null hypothesis

$$H_0 : \theta_1 = 0$$

is tested against the alternative

$$H_1 : \theta_1 \neq 0.$$

The test utilizes the fact that in the homogeneous Poisson process, the arrival times T_i are independent variables with uniform distributon on $(0, T]$ when conditioned on the total number n of events occuring in $(0, T]$. With the help of the central limit theorem, it can be then determined that the variable

$$Z = \frac{\sum_{i=1}^{n} T_i - \frac{1}{2} nT}{T\sqrt{\frac{n}{12}}}$$

has an asymptotic standard normal distribution under the null hypothesis. The test statistic is then

$$z = \frac{\frac{\sum_{i=1}^{n} t_i}{n} - \frac{1}{2}T}{T\sqrt{\frac{1}{12n}}}, \tag{16}$$

the null hypothesis is rejected at a significance level α if $|z| \geq u_{1-\alpha/2}$, the latter denoting the $(1 - \alpha/2)$-quantile of the standard normal distribution.

Once the intensity functions of the initial nonhomogeneous Poisson processes have been estimated, we can also estimate the parameter of the Poisson distribution representing the number of events in the desired following month. The estimate will be denoted by $\widehat{\mu}$, according to definition 1.8 and theorem 2.1 it can be obtained as

$$\widehat{\mu} = \int_T^{T+d} \lambda_1(x|\widehat{\boldsymbol{\theta}}_1) + \lambda_2(x|\widehat{\boldsymbol{\theta}}_2)dx,$$

where d denotes the length of the month in the corresponding unit of time. Letting $\widehat{\boldsymbol{\theta}}_1 = (\widehat{\theta}_0, \widehat{\theta}_1)^T$ and $\widehat{\boldsymbol{\theta}}_2 = (\widehat{\theta}_2, \widehat{\theta}_3)^T$, for the log linear intesities of the initial processes, the desired parameter can be determined as

$$\widehat{\mu} = \int_T^{T+d} e^{\widehat{\theta}_0 + \widehat{\theta}_1 x} + e^{\widehat{\theta}_2 + \widehat{\theta}_3 x} dx = \frac{e^{\widehat{\theta}_0}\left(e^{\widehat{\theta}_1(T+d)} - e^{\widehat{\theta}_1 T}\right)}{\widehat{\theta}_1} + \frac{e^{\widehat{\theta}_2}\left(e^{\widehat{\theta}_3(T+d)} - e^{\widehat{\theta}_3 T}\right)}{\widehat{\theta}_3}. \tag{17}$$

3.2 Application to Insurance Data

The theoretical approach summarized in this chapter can now be applied to real insurance data. The data set analyzed in this part was provided by the Czech Insurers' Bureau, it contains the reported insurance claims from car accidents in the time span between 1 January 2010 and 31 December 2016. In total, there are 16141 reported insurance claims, 14078 of which are labeled as material damage claims, the remaining 2063 are labeled as bodily injury claims. The reporting time is measured in days, 1 January 2010 being day 0 and 31 December 2016 being day 2557. The value itself is a decimal number, specifying also the time of the day when the claim was reported and as such can be considered continuous.

The model situation is that the material damage and the bodily injury lines of business are to be merged as of 1 December 2016, we are interested in an estimate for the number of claims in this merged line of business for December 2016. The estimate will then be compared to the total number of material damage and bodily injury claims actually reported during this month.

We first analyze the numbers of claims in the initial lines of business separately, only the claims before 1 December 2016 are considered. To examine the potential trends in the numbers of claims, the claims are grouped by month and visualized. The descriptive statistics of the numbers of claims per month are also calculated.

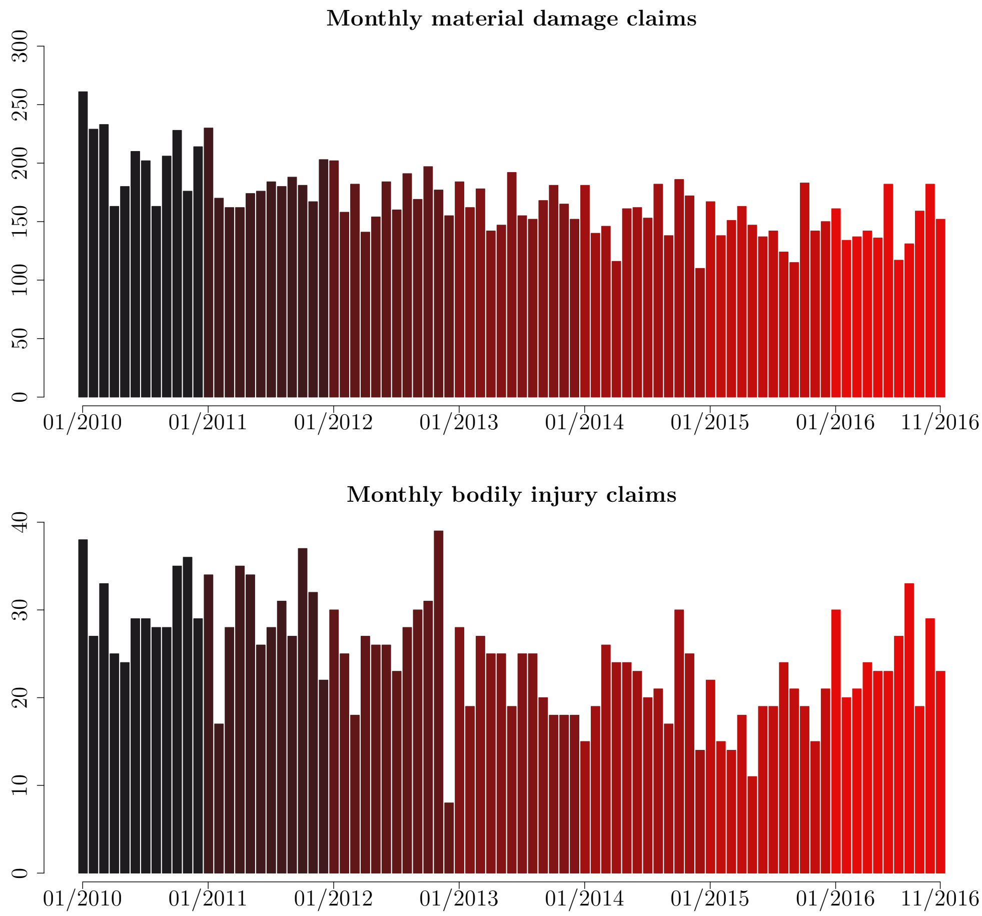

Figure 3.1: *Numbers of claims per month between 01/2010 – 11/2016*

Table 3.1: *The descriptive statistics of numbers of claims per month*

Claim type	Min	25% q.	Median	Mean	75% q.	Max
MD	110.00	148.50	163.00	167.80	182.00	261.00
BI	8.00	19.50	25.00	24.55	28.50	39.00

It is obvious that the bodily injury claims are less frequent than the material damage claims. When looking at the figure, a certain periodic trend in the number of claims could be noticed within every calendar year, which indicates that an intensity function with trigonometric components might be a more appropriate model for these particular data. But since the parameter estimates have only been stated for the log linear model, we ignore the periodic trend and only focus on a long term trend in the numbers of claims. The numbers of claims seem to be decreasing in the long term in both lines of business, we would thus expect the trend parameters of the log linear models to be negative, although very close to zero, because, thanks to the reporting time being measured in days, the observation period is quite long.

We formally test the hypotheses whether the trend parameters of the initial processes are zero and the processes are thus homogeneous. The test statistic stated in (16) is used, the time T entering the test statistics is $T = 2526$ which corresponds to the midnight between 30 Novemeber 2016 and 1 December 2016. The significance level α is considered $\alpha = 0.05$.

Table 3.2: *The results of tests for homogeneity of the initial processes*

Claim type	z	p-value
MD	−13.23	< 0.0001
BI	−5.74	< 0.0001

The hypothesis of the trend component equaling zero and the homogeneity of the process is rejected in both cases, the observation that the trend parameters might be negative is therefore sensible.

We proceed to the estimation of parameters of the log linear intensity functions as described in the first section of this chapter. Once again, let $\widehat{\boldsymbol{\theta}}_1 = (\widehat{\theta}_0, \widehat{\theta}_1)^T$ denote the parameters of the intensity of the first process and $\widehat{\boldsymbol{\theta}}_2 = (\widehat{\theta}_2, \widehat{\theta}_3)^T$ the parameters of the intensity of the second process. The obtained estimates are summarized in the table below, the values are rounded to five decimal places for a better readability.

Table 3.3: *Estimates of intensity functions parameters of the initial processes*

MD		BI	
$\widehat{\theta}_0$	1.90093	$\widehat{\theta}_2$	−0.02901
$\widehat{\theta}_1$	−0.00016	$\widehat{\theta}_3$	−0.00015

The estimated intensity functions are thus in form $\lambda_1(t) = e^{1.90093-0.00016t}$ and $\lambda_2(t) = e^{-0.02901-0.00015t}$. The functions are visualized, on the given scale, they almost seem to be linearly decreasing.

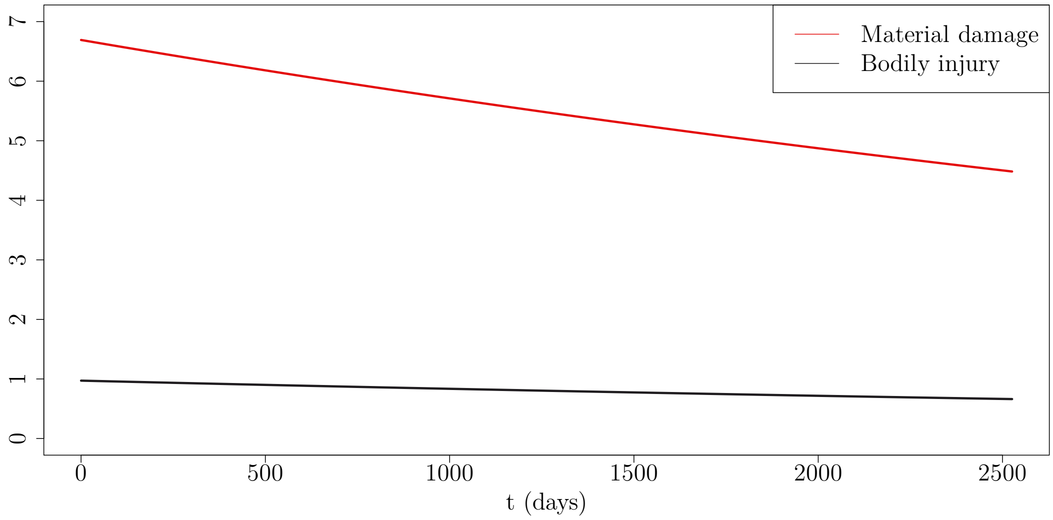

Figure 3.2: *Estimated intensities of material damage and bodily injury claims*

The estimate for the parameter $\widehat{\mu}$ of the Poisson distribution representing the number of claims occurring in the merged line of business in December 2016 is then obtained by plugging the estimated parameters $\widehat{\theta}_0, \ldots, \widehat{\theta}_3$ into (17), the endpoints of the time interval representing December 2016 are $T = 2526$ and $T + 31 = 2557$. The value of the estimate is

$$\widehat{\mu} = 159.1184.$$

Once the distribution of the number of claims in December 2016 is determined to be Poisson with parameter $\widehat{\mu}$, various estimates for the precise number of claims can be thought of. The insurance company might prefer some more conservative estimates, such as the 95%-quantile or even the 99.9%-quantile of the distribution. The estimates obtained in this way are compared with the actual number of claims reported in December 2016, we see that both estimates are above the actual number of reported claims.

Table 3.4: *The estimates and the actual number of claims in 12/2016*

MD + BI	
Actual	174
95%-quantile	180
99.9%-quantile	199

As already mentioned, numerous more complex approaches could be used to model this situation, such as different types of intensity functions containing for instance the trigonometric terms to model the seasonality of events or even different than Poisson processes, if the superposition of such processes turns out to have reasonable properties. Additionally, the insurance company could also be interested in the precise times when the future claims are likely to be reported. The approaches for simulating realizations of a nonhomogeneous Poisson process should be studied in that case.

Conclusion

The purpose of this thesis was to describe several classes of counting processes which are closed under the operations of superposition and thinning. Modeling the number of claims with these types of processes is especially useful when an insurance company faces the requirement to merge or split some of its lines of business. The resulting models retain the basic properties of the models entering the operations and in some cases, the parameters of the resulting models can be directly determined from the parameters of the initial processes.

The term counting process was introduced in the first chapter and some specific types of counting processes were presented. Several definitions of a Poisson and a nonhomogeneous Poisson process were stated and their equivalence was demonstrated in both cases. A different generalization of a Poisson process, the renewal process, was also introduced and some of its properties were derived.

The second chapter introduced the general idea of superposition and thinning of the counting processes, Bernoulli and multinomial thinning rules were considered in particular. The classic results about the superposition and thinning of the Poisson processes were proved for the more general, nonhomogeneous version of this process. An interesting result about the necessary conditions for the renewal processes to be closed under superposition was studied and explained in further detail. An original result was derived, stating that the renewal processes are also closed under the two mentioned thinning rules.

The third chapter covered the necessary theory to apply the previously stated results to a model situation when an insurance company is required to merge its lines business. The described approach was then used to estimate the future number of claims in the merged line of business, real insurance data provided by the Czech Insurers' Bureau were used for a practical demonstration. In particular, the maximum likelihood approach for the estimation of a log linear intensity function of a nonhomogeneous Poisson process was described, the estimate of a parameter of the Poisson distribution representing the future number of claims in the merged line of business was derived and the quantiles of this distribution were proposed as conservative estimates of the number of reported claims.

The future work related to this topic could examine closure under different types of thinning, which could be for instance time-dependent. The interaction of superposition and thinning with a generalized version of the renewal process, the so-called renewal reward process, could be also investigated, as this process provides a more complex approach for modeling both the claim numbers and claim amounts. From the application point of view, different types of intensity functions could be used to model the trends in number of claims and the algorithms for simulating the realizations of a nonhomogeneous Poisson process could be studied.

Bibliography

D. R. Cox and P. A. W. Lewis. *The Statistical Analysis of Series of Events*. First Edition. Methuen, London, 1966. ISBN 978-94-011-7801-3.

R. Durrett. *Probability: Theory and Examples*. Fifth Edition. Cambridge University Press, Cambridge, 2019. ISBN 978-1-108-47368-2.

J. A. Ferreira. Pairs of renewal processes whose superposition is a renewal process. *Stochastic Processes and their Applications*, 86:217–230, 2000.

S. M. Ross. *Introduction to Probability Models*. Eleventh Edition. Academic Press, San Diego, 2014. ISBN 978-0-12-407948-9.

S. M. Samuels. A Characterization of the Poisson Process. *Journal of Applied Probability*, 11(1):72–85, 1974.

R. L. Streit. *Poisson Point Processes: Imaging, Tracking and Sensing*. First Edition. Springer, New York, 2010. ISBN 978-1-4419-6922-4.