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Inverse limits in module categories

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Abstract: For a class of modules $\mathcal{C}$, we study the class $\varprojlim \mathcal{C}$ of modules that can be obtained as inverse limits of modules from $\mathcal{C}$. In particular, we investigate how additional properties of the class $\mathcal{C}$ are reflected by properties of the class $\varprojlim \mathcal{C}$. We also address the question of whether for a given module M, every inverse limit of products of M is an inverse limit of finite products of M. We provide examples of modules for which the answer is positive, negative, and for which there is a reason to believe that it depends on additional set-theoretic assumptions.

Keywords: Inverse limit, Slender module, Self-slender module, Measurable cardinal

Contents

Introduction

The inverse limit is a categorical construction, more specifically, it is a type of limit, for which the shape of the diagram is an inversely directed poset. The question we will be asking is as follows: given a class of modules over some ring, which modules can we get by inverse limits? For the dual construction called the direct limit, new results were published in the article [8]. In this thesis, we will see whether they (and some results from other sources) can be dualized in order to better understand the structure of inverse limits.

In the first chapter, we will provide the definitions needed for the rest of the thesis, as well as references for a reader who isn't familiar with category theory. In the second chapter, we will study the general properties of the class $\varprojlim \mathcal{C}$ consisting of inverse limits constructed from modules from some given class $\mathcal{C}$. Some of the results work in any category and are stated as such (therefore for example generalizing the original statements about direct limits). In the third chapter, we will focus on a specific problem, whether any inverse limit of products of some module M is also an inverse limit of finite products of M. Compared to the case of direct limits and direct sums, it is pretty easy to find a counterexample, but as we will see in the fourth chapter, the answer to the question may depend on the model of the set theory in which we are working. At the end, we will provide a list of open problems related to the topic of the thesis.

1. Basic definitions and notation

In this section, we introduce the basic definitions from category theory, because the notation there isn't as standardized as in other areas of mathematics we are going to use. However, we assume the reader is familiar with the basic notions of it (functors, limits, colimits, natural transformations, ...), as well as the notions from module theory (direct sums and products, functors *Hom* and ⊗, ...) and set theory (ordinals, cardinals). Our basic references for category theory are [7] and [2], for module theory [2] and for set theory [6].

Definition 1.1. *A category $\mathcal{C}$ consists of*

- *class $Ob(\mathcal{C})$ of objects*
- *for each two objects $c_1, c_2 \in Ob(\mathcal{C})$ a class $Hom_{\mathcal{C}}(c_1, c_2)$ of morphism (also denoted $Hom(c_1, c_2)$ if there is no ambiguity). The disjoint union of those classes is denoted by $Hom(\mathcal{C})$.*
- *for every $c \in Ob(\mathcal{C})$ a morphism $id_c \in Hom(c, c)$ (the identity morphism)*
- *for each three $c_1, c_2, c_3 \in Ob(\mathcal{C})$ a function $Hom(c_2, c_3) \times Hom(c_1, c_2) \to Hom(c_1, c_3), (f, g) \to f \circ g$ (composition of morphisms)*

and has to satisfy the following:

- *the composition of morphisms is associative, i.e. for every $f \in Hom(c_4, c_3)$, $g \in Hom(c_3, c_2)$ and $h \in Hom(c_2, c_1))$ it holds that $f \circ (g \circ h) = (f \circ g) \circ h$*
- *for every $c, c' \in Ob(\mathcal{C})$, for every $f \in Hom(c, c')$ it holds that $f \circ id_c = f$ and for every $g \in Hom(c', c)$ it holds that $id_c \circ g = g$*

A category is called locally small if for every $c_1, c_2 \in Ob(\mathcal{C})$ the class $Hom(c_1, c_2)$ is a set. A category is called small if it is locally small and the class of objects is a set.

Example 1.2. *Some of the categories we will be using are:*

- *The category Set: $Ob(Set)$ is the class of all sets, $Hom_{Set}(U, V) = \{f : U \to V\}$, (composition of two morphisms is their composition as functions)*
- *For a ring R the categories Mod-R and R-Mod:*
  - *$Ob(Mod\text{-}R)$ is the class of all right R-modules, $Hom_{Mod\text{-}R}(M_1, M_2)$ is the set of all right R-module homomorphisms $M_1 \to M_2$*
  - *$Ob(R\text{-}Mod)$ is the class of all left R-modules, $Hom_{R\text{-}Mod}(M_1, M_2)$ is the set of all left R-module homomorphisms $M_1 \to M_2$*
- *for a cardinal κ the discrete category of size κ with κ-many objects and no morphisms except the identity morphisms*
- *For a poset $P = (|P|, \leq)$ a category also denoted P, whose objects are elements of the poset and there is one morphism $p_1 \to p_2$ if $p_1 \leq p_2$ and no morphism otherwise*

In the first three examples, the composition of the morphisms is their composition as functions. In the last two examples, there is only one way how to define the composition, as there is at most one morphism between any two objects.

Definition 1.3. *Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of*

- *a map $Ob(\mathcal{C}) \to Ob(\mathcal{D})$, $c \mapsto F(c)$*
- *for every $c_1, c_2 \in \mathcal{C}$ a map $Hom_{\mathcal{C}}(c_1, c_2) \to Hom_{\mathcal{D}}(F(c_1), F(c_2))$, $f \mapsto F(f)$*

and has to satisfy the following:

- *for every $c \in Ob(\mathcal{C})$ it holds that $F(id_c) = id_{F(c)}$*
- *for every $f \in Hom(c_3, c_2)$ and $g \in Hom(c_2, c_1))$ it holds that $F(f \circ g) = F(f) \circ F(g)$*

Definition 1.4. *Let $\mathcal{C}, \mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ functors. A natural transformation $\varphi : F \to G$ is a class of morphisms $(\varphi_c)_{c \in Ob(\mathcal{C})}$, such that $\varphi_c \in Hom(F(c), G(c))$ and for every $f \in Hom_{\mathcal{C}}(c_1, c_2)$ it holds that $G(f) \circ \varphi_{c_1} = \varphi_{c_2} \circ F(f)$.*

Definition 1.5. *Let $\mathcal{C}$ be a small category, $\mathcal{D}$ a category, $F : \mathcal{C} \to \mathcal{D}$ a functor.*

- *A cone over F is an object $d \in Ob(\mathcal{D})$ together with a set of morphisms $(\varphi_c)_{c \in Ob(\mathcal{C})}$, $\varphi_c \in Hom_{\mathcal{D}}(d, F(c))$, such that for every $f \in Hom_{\mathcal{C}}(c_1, c_2)$ it holds that $F(f) \circ \varphi_{c_1} = \varphi_{c_2}$. A limit of F is a cone $(d, (\varphi_c)_{c \in Ob(\mathcal{C})})$, such that for every cone $(d', (\varphi'_c)_{c \in Ob\mathcal{C}})$ over F there is a unique morphism $g \in Hom_{\mathcal{D}}(d', d)$ satisfying $\varphi'_c = \varphi_c \circ g$ for all $c \in Ob(\mathcal{C})$.*
- *A cocone under F is an object $d \in Ob(\mathcal{D})$ together with a set of morphisms $(\varphi_c)_{c \in Ob(\mathcal{C})}$, $\varphi_c \in Hom_{\mathcal{D}}(F(c), d)$, such that for every $f \in Hom_{\mathcal{C}}(c_1, c_2)$ it holds that $\varphi_{c_1} = \varphi_{c_2} \circ F(f)$. A colimit of F is a cocone $(d, (\varphi_c)_{c \in Ob(\mathcal{C})})$, such that for every cocone $(d', (\varphi'_c)_{c \in Ob\mathcal{C}})$ under F there is a unique morphism $g \in Hom_{\mathcal{D}}(d, d')$ satisfying $\varphi'_c = g \circ \varphi_c$ for all $c \in Ob(\mathcal{C})$.*

We will often omit the morphisms and use the word limit/colimit just for the object d.

Definition 1.6. *A directed poset P is a poset, where for every two elements $p_1, p_2 \in P$ there exists an upper bound $q \in P$, $q \geq p_1, p_2$. An inversely directed poset P is a poset, where for every two elements $p_1, p_2 \in P$ there exists a lower bound $q \in P$, $q \leq p_1, p_2$.*

Definition 1.7. *Let $\mathcal{D}$ be a complete category. A direct limit $\varinjlim F$ is a colimit of a functor $F : P \to \mathcal{D}$, where P is a directed poset considered as a category. An inverse limit $\varprojlim F$ is a limit of a functor $F : P \to \mathcal{D}$, where P is an inversely directed poset. For a class $\mathcal{C} \subseteq Ob(\mathcal{D})$ we define classes*

$$\varinjlim \mathcal{C} := \{\varinjlim F;\ F : P \to \mathcal{C} \text{ a functor, } P \text{ a directed poset}\}$$

$$\varprojlim \mathcal{C} := \{\varprojlim F;\ F : P \to \mathcal{C} \text{ a functor, } P \text{ an inversely directed poset}\}.$$

(By a functor $P \to \mathcal{C}$ we mean a functor, which maps every object of P to an object in $\mathcal{C}$. The limit/colimit of a such functor is computed in category $\mathcal{D}$.)

Definition 1.8. *Let $\mathcal{C}$ be a category. An opposite category $\mathcal{C}^{op}$ is a category with $Ob(\mathcal{C}^{op}) := Ob(\mathcal{C})$, $Hom_{\mathcal{C}^{op}}(c_1, c_2) := Hom_{\mathcal{C}}(c_2, c_1)$ and $f \circ_{\mathcal{C}^{op}} g := g \circ_{\mathcal{C}} f$.*

Definition 1.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. An opposite functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is defined by $F^{op}(c) := F(c)$ for any $c \in Ob(\mathcal{C})$ and $F^{op}(f) = F(f)$ for any $f \in Hom(\mathcal{C})$. Since this functor is determined by the same data as the functor F, they are sometimes interchanged, but it is important to distinguish them for example when we are talking about their limits/colimits because the limit of one is the colimit of the other.*

Theorem 1.10. ***(Maranda's theorem)*** *[7, Theorem V.2.2] Let D be a small category, $\mathcal{C}$ be a category, $F : D \to \mathcal{C}$ be a functor. Then the limit $\lim F$ is isomorphic to the equalizer of $f, g : \prod_{d \in Ob(D)} F(D) \to \prod_{h : d_1 \to d_2 \in Hom(D)} F(d_2)$, where in the component $h' : d_1' \to d_2'$*

- *f is the projection $\pi_{d_2} : \prod_{d \in Ob(D)} F(D) \to F(d_2)$*
- *g is the composition $h' \circ \pi_{d_1}$*

if those products and this equalizer exist.

Theorem 1.11. ***(Yoneda lemma)*** *[7, Section III.2] Let D be a locally small category, $F : D \to Set$ be a functor, $D \in Ob(D)$. Then there is a bijection between the natural transformations $Hom_D(d, -) \to F$ and $F(d)$, which sends a natural transformation φ to $\varphi_d(id_d) \in F(d)$.*

Corollary 1.12. *[7, Section III.2] Let D be a locally small category, $d_1, d_2 \in Ob(D)$. Then the following are equivalent:*

1. *the objects d_1 and d_2 are isomorphic*
2. *the functors $Hom_D(d_1, -)$ and $Hom_D(d_2, -)$ are naturally isomorphic*
3. *the functors $Hom_D(-, d_1)$ and $Hom_D(-, d_2)$ are naturally isomorphic*

Note. The book [7] only shows the equivalence 1. $\iff$ 2., the equivalence 1. $\iff$ 3. can be obtained by applying the same argument to the category D^{op}.

Theorem 1.13. ***(Commutativity of limits)*** *[7, Corollary in IX.8] Let D_1, D_2 be small categories, $\mathcal{C}$ be a complete category, $F : D_1 \times D_2 \to \mathcal{C}$ be a functor. We can also view F as a functor from D_1 to the category of functors $D_2 \to \mathcal{C}$ (or vice versa). Then it holds that*

$$\lim_{(d_1, d_2) \in D_1 \times D_2} F(d_1, d_2) \cong \lim_{d_2 \in D_2} \lim_{d_1 \in d_1} F(d_1, d_2) \cong \lim_{d_1 \in D_1} \lim_{d_2 \in d_2} F(d_1, d_2).$$

2. General properties of inverse limits

In this section, we will examine a few results from the article [8] and the book [5] about direct limits and present their dualization for the inverse limits. Some of them hold in any category, not just in categories of modules, therefore the proofs are also alternative proofs for the theorems about direct limits (as direct limits in category $\mathcal{C}$ are the inverse limits in the category $\mathcal{C}^{op}$). Some of the results also hold for any limits without the requirement on the shape of the diagram. We will start by proving a lemma, which will allow us to write a limit as an inverse limit of smaller limits:

Lemma 2.1. *Let $\mathcal{D}$ be a complete category, D be a small category, $F : D \to \mathcal{D}$ a functor. Let P be the set of all subcategories of D ordered by opposite inclusion ($p \geq p' \iff p \subseteq p'$) and Q be some inversely directed subposet of P (considered as a category), such that $\bigcup Q = D$. Define a functor $G : Q \to \mathcal{D}$ as follows:*

- *For $q \in Ob(Q)$, let $G(q) = \lim_{i \in q} F(i)$.*
- *For $q_1 \subseteq q_2$, the limit $G(q_2)$ as a cone over q_2 is also a cone over q_1, therefore the universal property of $G(q_1) = \lim_{i \in q_1} F(i)$ uniquely defines a morphism $G(q_2) \to G(q_1)$. Define $G(q_2 \to q_1)$ to be this morphism.*

Then $\varprojlim_{q \in Q} G(q) \cong \lim_{i \in D} F(i)$.

Proof. We will show this using the Corollary of the Yoneda lemma. From the universal property of the limit, the functor $Hom(-, \varprojlim_{q \in Q} G(q))$ is (naturally) isomorphic to $\{(f_q)_{q \in Q} \in \prod_{q \in Q} Hom(-, G(q)); f_{q_1} = G(q_2 \to q_1) \circ f_{q_2}\}$. Each $G(q)$ is by itself a limit, so we can decompose $Hom(-, G(q)) \cong \{(f_i)_{i \in q} \in \prod_{i \in q} Hom(-, F(i)); \ f_{i_1} = F(f) \circ f_{i_2} \forall f \in Hom_q(i_2, i_1)\}$. Because the morphisms $G(q_2 \to q_1)$ are defined so that

$$\begin{array}{ccc} G(q_2) & \longrightarrow & G(q_1) \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ F(i) & \xrightarrow[id_{F(i)}]{} & F(i) \end{array}$$

commute for every $i \in q_1$, it holds that $Hom(-, \varprojlim_{q \in Q} G(q))$ is isomorphic to the subset of $\{(f_{i,q})_{i \in q \in Q} \in \prod_{i \in q \in Q} Hom(-, F(i))\}$ given by the following conditions:

- $\forall q \in Q, \forall i_1, i_2 \in q, \forall f \in Hom_q(i_2, i_1) : f_{i_1} = F(f) \circ f_{i_2}$
- $\forall i \in q_1 \subseteq q_2 \in Q : f_{i,q_1} = f_{i,q_2}$

Since the poset Q is inversely directed, from the second condition it follows that $f_{i,q_1} = f_{i,q_2}$ holds for all q_1, q_2 containing i. Therefore $Hom(-, \lim_{q \in Q} G(q))$ is isomorphic to

$$\{(f_i) \in \prod_{i \in \bigcup_{q \in Q} Ob(q)} Hom(-, F(i)); f_{i_2} = f_{i_1} \circ F(f) \forall f \in \bigcup_{q \in Q} Hom_q(i_2, i_1)\}$$

which under the assumption $\bigcup Q = D$ is isomorphic to $Hom(-, \lim_{i\in D} F(I))$.

□

Lemma 2.2. *[8, Lemma 1.1] Let R be a ring, $\mathcal{C}$ be a class of R-modules, $\mathcal{C}'$ its closure under direct summands. Then $\varinjlim \mathcal{C} = \varinjlim \mathcal{C}'$.*

Lemma 2.3. *Let R be a ring, $\mathcal{C}$ be a class of R-modules, $\mathcal{C}'$ its closure under direct summands. Then $\varprojlim \mathcal{C} = \varprojlim \mathcal{C}'$.*

Proof. Let P be an inversely directed poset, $F : P \to \mathcal{C}'$ a functor. For an object $p \in P$ fix a module M_p such that $F(p) \oplus M_p \in \mathcal{C}$. We will distinguish two cases:

- If P has a minimal element m, then $\varprojlim F = F(m)$. In that case, we can get it as an inverse limit of

$$\dots \xrightarrow{\pi_1} F(m) \oplus M_m \xrightarrow{\pi_1} F(m) \oplus M_m \xrightarrow{\pi_1} F(m) \oplus M_m$$

- If P does not have a minimal element, define a functor $G : P \to \mathcal{C}'$, such that $G(p) = M_p$ for any object $p \in Ob(P)$ and $G(f) = 0$ for any morphism $f \in P$ except identities. Then $\varprojlim G = 0$ and therefore $\varprojlim F \cong \varprojlim F \oplus \varprojlim G \cong \varprojlim (F \oplus G)$.

In both cases, we got $\varprojlim F$ as an inverse limit of some functor to $\mathcal{C}$.

□

Since most of the interesting classes of modules are closed under finite direct sums, it is important to know what this property implies for the classes $\varinjlim \mathcal{C}$ and $\varprojlim \mathcal{C}$.

Theorem 2.4. *[8, Proposition 2.2] Let R be a ring, $\mathcal{C}$ be a class of R-modules. If $\mathcal{C}$ is closed under finite direct sums, then $\varinjlim \mathcal{C}$ is closed under arbitrary direct sums.*

Theorem 2.5. *Let $\mathcal{D}$ be a complete category, $\mathcal{C}$ be a class of objects in $\mathcal{D}$ closed under finite products. Then $\varprojlim \mathcal{C}$ is closed under arbitrary products.*

Proof. For $i \in I$, let D_i be an inversely directed poset considered as a category and $F_i : D_i \to \mathcal{C}$ be a functor, such that $\varprojlim F_i = C_i$. Define another poset Q with the underlying set $\{q \subseteq I; |q| < \omega\}$ and the order defined by opposite inclusion (this poset is clearly inversely directed, since $q \cup q' \leq q, q'$ for any q, q'). On objects, define a functor $F : \prod_{i\in I} D_i \times Q \to \mathcal{C}$ as $F((d_i)_{i\in I}, q) = \prod_{i\in q} F_i(d_i)$ for $f_q : q \to q'$. Define it on morphism by $F((f_i)_{i\in I}, f_q) = \prod_{i\in q_2} F_i(f_i)$ if $f_q : q_1 \to q_2$ Now using Theorem 1.13 about commutativity of limits

$$\varprojlim_{(d_i)\in\prod_{i\in I} D_i} \left(\varprojlim_{q\in Q} F \right) \cong \varprojlim_{q\in Q} \left(\varprojlim_{(d_i)\in\prod_{i\in I} D_i} F \right)$$

The right-hand side is an inverse limit of finite products of elements of $\mathcal{C}$. From Lemma 2.1 it follows that $\varprojlim_{q\in Q} F((d_i), q) \cong \prod_{i\in I} F_i(d_i)$, because product over

I is a limit from a discrete category on I and $F((d_i), q)$ are limits over the finite subcategories. Since products commute with limits, $\varprojlim_{(d_i)\in\prod_{i\in I} D_i} \prod_{i\in I} F_i(d_i) \cong \prod_{i\in I} \varprojlim_{(d_i)\in\prod_{i\in I} D_i} F_i(D_i) \cong \prod_{i\in I} C_i$.

□

For the next theorem, the dual statement doesn't hold. If we replace direct sums with products, homomorphic images with submodules and direct limits with inverse limits, the class of finitely dimensional vector spaces will work as a counterexample, as we will see in Corollary 3.0.4.. Therefore we present a statement to the dual, which works in any category and the requirements are a bit weaker - closure under finite limits in the category Mod-R is equivalent to closure under finite products and those submodules, for which the quotient is a submodule of some module from $\mathcal{C}$.

Theorem 2.6. *[8, Lemma 2.5] Let R be a ring, $\mathcal{C}$ be a class of R-modules. If $\mathcal{C}$ is closed under finite direct sums and homomorphic images, then $\varinjlim \mathcal{C}$ coincides with the class of homomorphic images of direct sums of modules from $\mathcal{C}$.*

Theorem 2.7. *Let $\mathcal{D}$ be a complete category, $\mathcal{C}$ be a class of objects in $\mathcal{D}$ closed under finite limits. Then $\varprojlim \mathcal{C}$ coincides with the class of all limits of $\mathcal{C}$.*

Proof. Let D be a small category, $F : D \to \mathcal{C}$ a functor. When we chose P in Lemma 2.1 to be the set of all finitely generated subcategories of D, it follows that $\lim F \in \varprojlim \mathcal{C}$.

□

Definition 2.8. *Let R be a ring, $\mathcal{C}$ be a class of R-modules, M an R-module. We say that M is*

- *$\mathcal{C}$-filtered if there exists an ordinal α and na increasing chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha = M$, such that*
  - *$M_{\beta+1}/M_\beta \in \mathcal{C}$.*
  - *If $\beta \leq \alpha$ is a limit ordinal, then $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$*
- *$\mathcal{C}$-cofiltered if there exists an ordinal α and a sequence $M = M_\alpha \to \cdots \to M_1 \to M_0 = 0$, such that*
  - *the morphisms $M_{\beta+1} \to M_\beta$ are epimorphisms with kernel in $\mathcal{C}$*
  - *If $\beta \leq \alpha$ is a limit ordinal, then $M_\beta = \varprojlim_{\gamma<\beta} M_\gamma$*

Theorem 2.9. *[8, Lemma 2.5] Let R be a ring, $\mathcal{C}$ be a class of R-modules closed under finite direct sums and homomorphic images. Then $\varinjlim \mathcal{C}$ consists of $\mathcal{C}$-filtered modules.*

Theorem 2.10. *Let R be a ring, $\mathcal{C}$ be a class of objects in Mod-R closed under submodules, D a small category, $F : D \to \mathcal{C}$ a functor. Then $\lim F$ is a $\mathcal{C}$-cofiltered module.*

Proof. Let $\kappa := |Ob(D)|$, fix a bijection $\kappa \to Ob(D), \alpha \mapsto d_\alpha$. Let M_α be the image of the map $\lim F \to \prod_{\beta<\alpha} F(d_\beta)$, whose components are the limit morphisms. The maps $M_\alpha \to M_\beta$ for $\beta < \alpha$ are the projections, therefore epimorphisms. The kernel of the map $M_{\beta+1} \to M_\beta$ contains only elements, which have all components except the β component zero, therefore it is isomorphic to a submodule of $F(d_\beta) \in \mathcal{C}$. Obviously $M_0 = 0$ and from the Maranda's theorem it follows that $M_\kappa \cong M$ and for a limit ordinal α it holds that $M_\alpha = \varprojlim_{\beta<\alpha} M_\beta$.

□

Theorem 2.11. *[5, Lemma 2.14] Let R be a ring and $\mathcal{C}$ a class of modules closed under direct limits of well-ordered chains. Then $\mathcal{C}$ is closed under direct limits.*

Theorem 2.12. *Let $\mathcal{D}$ be a complete category, $\mathcal{C}$ a class of objects in $\mathcal{D}$. Suppose that $\mathcal{C}$ is closed under well-ordered inverse limits (inverse limits of functors $D \to \mathcal{C}$ where D^{op} is well-ordered). Then it is closed under arbitrary inverse limits.*

Proof. Let D be an inversely directed poset of size $\kappa \geq \omega$ (finite inversely directed posets are not important, since they have a minimal element and the limit of a functor from them is the value of the functor on the minimal element), $G : D \to \mathcal{C}$ be a functor with a limit C. We will prove the theorem by induction on κ. Fix a bijection $\kappa \to D$, $\alpha \mapsto d_\alpha$. For $\alpha \in \kappa$, define inductively an inversely directed subposet $D_\alpha \subseteq D$:

- If $\alpha = 0$, $D_\alpha = \emptyset$.
- If α is limit, $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$
- If $\alpha = \gamma + 1$ is succesor cardinal, define $\tilde{D}_\alpha = D_\gamma \cup \{d_\gamma\}$. This poset doesn't have to be inversely directed, therefore we will add elements to it until it is. If α is finite, we only need to add one element $e \in D$, which is smaller than all elements of $\tilde{D}_\alpha$. Then $D_\alpha = \tilde{D}_\alpha \cup \{e\}$ is inversely directed. If α is infinite, let $P(\tilde{D}_\alpha)$ be the poset $\tilde{D}_\alpha \cup \{e_{u,v}; u, v \in \tilde{D}_\alpha\}$, where $e_{u,v} \in D$ is some lower bound of u and v. The poset $P(\tilde{D}_\alpha)$ doesn't have to be inversely directed, since there doesn't have to be a common lower bound for the elements of the form $e_{u,v}$. However, we can repeat this construction to get $P(P(\tilde{D}_\alpha))$ and so on and then $D_\alpha = \bigcup_{n\in\omega} P^n(\tilde{D}_\alpha)$ is inversely directed.

Note that if $\alpha \in \kappa$ is finite, D_α is also finite, and if α is infinite, $|D_\alpha| = |\alpha|$. Let G_α be the restriction of G to D_α. From Lemma 2.1 it follows that $\varprojlim G \cong \varprojlim_{\alpha\in\kappa^{op}} \varprojlim G_\alpha$. By the induction hypothesis, $\varprojlim G_\alpha \in \mathcal{C}$ and therefore $C \in \mathcal{C}$.

□

3. $\varprojlim\ prod(M)$ and $\varprojlim Prod(M)$

For the direct limits, one can ask a question, whether for a given module M the class of the direct limits of finite direct sums of M is the same as the class of the direct limits of all direct sums of M. The question is easy for so-called self-small modules (in particular for all finitely generated modules), where equality holds. In general, there are descriptions of both of the classes, but they both use the endomorphism ring of M, which is usually difficult to describe. More about that in [8].

We can dualize this question and ask, whether for any given module M the class of the inverse limits of finite products of M is the same as the class of the inverse limits of all products of M. As in the case of the direct limit, we can describe the classes using the endomorphism ring of M and there is a dual property of modules (being strongly self-slender) which makes it easy to show equality. However, finding such modules is tricky - whether a module has this property can depend on the underlying set theory. And it turns out that we can take even the simplest possible module - a field - as a counterexample.

Definition 3.1. *Let R be a ring and M be an R-module. We define classes of modules*

$$add(M) = \{\textit{direct summands of finite direct sums of } M\}$$

$$Add(M) = \{\textit{direct summands of any direct sums of } M\}$$

$$prod(M) = \{\textit{direct summands of finite direct products of } M\}$$

$$Prod(M) = \{\textit{direct summands of any direct products of } M\}$$

Since we will study the classes of direct/inverse limit of those classes, by Lemma 2.2 and Lemma 2.3 it doesn't matter whether we include the direct summands or only the sums/products themselves. Also, since the finite direct sums are the same as finite direct products, the first and the third class are the same, but it makes more sense to consider them as direct sums in the case of direct limits and as direct products in the case of inverse limits.

Theorem 3.2. *[8, Theorem 3.3]*
Let R be a ring and M be a right R-module. Then $\varinjlim add(M) = \{F \otimes_S M; F$ is a flat right S-module$\}$, where S is the endomorphism ring of M.

Theorem 3.3. *Let R be a ring and M be a right R-module. Then $\varprojlim prod(M) = \{Hom_S(F, M); F$ is a flat left S-module$\}$, where S is the endomorphism ring of M.*

Proof. Let D be an inversely directed poset, $G : D \to prod(M)$ a functor. We will construct a contravariant functor $H : D \to prod(S)^{op}$, such that there is a natural isomorphism $G \cong Hom_S(-, M) \circ H$. For $d \in D$, if $G(d) = M^{n_d}$, let $H(d) = S^{n_d}$. For a morphism $f \in Hom_D(d, d')$, if $G(f)$ is represented by a matrix $A \in S^{n_{d'} \times n_d}$, let $H(f)$ be represented by the matrix $A^T \in S^{n_d \times n_{d'}}$. The isomorphism $\varphi_d : G(d) \to Hom_S(H(d), M)$ is defined by $(m_1, \dots, m_{n_d}) \mapsto (f : e_i \mapsto m_i)$. On one hand, for $f \in Hom_D(d, d')$ it holds that

$$\varphi_{d'} \circ G(f)(m_1, \dots, m_{n_d}) = \varphi_d(A \cdot (m_1, \dots, m_{n_d})^T) =$$

$$= h : e_j \mapsto \sum_{i=1}^{n_d} A_{ji} m_i.$$

On the other hand,

$$Hom_S(H(f), M) \circ \varphi_d(m_1, \dots, m_{n_d}) =$$

$$= Hom_S(H(f), M)(g : e_j \mapsto m_j) =$$

$$= (g : e_j \mapsto m_j) \circ H(f) = h : e_j \mapsto \sum_{i=1}^{n_d} A_{ji} m_i.$$

Therefore the isomorphism is natural. Therefore

$$Hom_S(\varinjlim_{d \in D^{op}} H^{op}(d), M) \cong \varprojlim_{d \in D}(Hom_S(H(d), M)) \cong \varprojlim_{d \in D} G.$$

According to Lazard's theorem, direct limits of finitely generated free modules are precisely the flat modules, therefore one inclusion holds. For any functor H we can construct back the functor $G := Hom_S(-, M) \circ H$, therefore the theorem is proved.

□

Corollary 3.4. *Let F be a field considered as a module over itself. Then*

$$\varprojlim prod(F) \neq \varprojlim Prod(F)$$

Proof. According to previous Theorem, $F^{(\omega)} \notin \varprojlim prod(F)$, because $End_F(F) = F$ and for any F-vector space V, $Hom(V, F)$ has finite dimension iff V has finite dimension and uncountable dimension otherwise. On the other hand, $F^{(\omega)}$ is a direct summand of F^ω, therefore $F^{(\omega)} \in \varprojlim Prod(F)$.

□

On the other hand, for a vector space with a countable dimension (and thus for any infinite-dimensional vector space), the equality $\varprojlim prod(M) = \varprojlim Prod(M)$ holds.

Theorem 3.5. *[1, Proposition 4.2] Let F be a field. Then $\varprojlim F^{(\omega)}$ is the class of all F-vector spaces.*

Definition 3.6. *Let R be a ring and M be an R-module. We say that M is:*

- *strongly slender if for any cardinal κ and any collection of R-modules M_α, $\alpha \in \kappa$ the natural homomorphism*

$$\bigoplus_{\alpha \in \kappa} Hom(M_\alpha, M) \to Hom(\prod_{\alpha \in \kappa} M_\alpha, M)$$

$$(f_\alpha)_{\alpha \in \kappa} \mapsto \left((m_\alpha)_{\alpha \in \kappa} \mapsto \sum_{\alpha \in \kappa} f_\alpha(m_\alpha)\right)$$

  of abelian groups is an isomorphism (it is always injective, not necessarily surjective).

- *slender if the above holds for* $\kappa = \omega$.
- *strongly self-slender if the above holds (for any cardinal* κ*) in the case when* $\forall \alpha \in \kappa : \ M_\alpha = M$.
- *self-slender the above holds in the case when all* M_α *are isomorphic to* M *and* $\kappa = \omega$.
- *self-small if the natural homomorphism*

$$Hom(M,M)^{(\kappa)} \to Hom(M, M^{(\kappa)})$$

$$(f_\alpha)_{\alpha\in\kappa} \mapsto (g : m \mapsto (f_\alpha(m))_{\alpha\in\kappa})$$

is an isomorphism for any cardinal κ.

Theorem 3.7. *[8, Corollary 5.3.] Let* R *be a ring and* M *be a self-small* R*-module. Then* $\varinjlim add(M) = \varinjlim Add(M)$.

Theorem 3.8. *Let* R *be a ring and* M *be a strongly self-slender right* R*-module. Then* $\varprojlim prod(M) = \varprojlim Prod(M)$.

Proof. One inclusion is obvious. For the second one, suppose an inversely directed poset P and a functor $G : P \to Prod(M)$. Since P is a set and not a proper class, the values of G on objects have to lie in some set of the form $\{M^\alpha; \alpha < \kappa\}$ for some cardinal κ. Since elements of this set are direct summands of M^κ, it holds that $\varprojlim G \in \varprojlim\{M^\kappa\} = \varprojlim prod(M^\kappa)$. Denote $S = End_R(M)$ and $S_\kappa = End_R(M^\kappa)$. From the previous theorem $\varprojlim G \cong Hom_{S_\kappa}(F, M^\kappa)$ for some flat left S_κ-module F. There is an isomorphism of right R-modules $M^\kappa \cong Hom_S(S^{(\kappa)}, M)$. Since M is strongly self-slender, $S^{(\kappa)}$ also has the right S_κ-module structure of $Hom(M, M^\kappa)$ and this isomorphism $M^\kappa \cong Hom_S(S^{(\kappa)}, M)$ is also an S_κ-module isomorphism. Using the Hom-tensor adjunction it then follows that

$$\varprojlim G \cong Hom_{S_\kappa}(F, Hom_S(S^{(\kappa)}, M)) \cong Hom_S(S^{(\kappa)} \otimes_{S_\kappa} F, M).$$

From the fact that $S^{(\kappa)}$ is a flat left S-module follows that $S^{(\kappa)} \otimes_{S_\kappa} F$ is flat left S-module and therefore $Hom_S(S^{(\kappa)} \otimes_{S_\kappa} F, M) \in \varprojlim prod(M)$ by Theorem 3.3. □

4. Strongly self-slender modules and measurable cardinals

As we will see, the existence of strongly self-slender modules heavily depends on the underlying set theory we are working in. From Gödel's first incompleteness theorem, we know that there are statements, that (assuming that the set theory itself is consistent) cannot be proven or disproven. In other words, in some models of set theory, the statement holds and in some other models, it doesn't. Thus in some areas of mathematics, we often add axioms to our basic set theory ZFC to be able to further specify the model we are working with. From Gödel's second incompleteness theorem, we know we will never be able to prove the consistency of ZFC and trying to prove the consistency of some theory with additional axioms is therefore pointless. However, we can sometimes prove equiconsistency of some theories, which means that one theory is consistent if and only if the other one is. Probably the most famous example is the continuum hypothesis - if ZFC is consistent, then $ZFC + CH$ is also (the other implication is trivial) and the same holds for $ZFC + \neg CH$. In our case, we will add axioms about the existence/non-existence of measurable cardinals:

Definition 4.1. *Let S be a set. We say that $U \subseteq \mathcal{P}(S)$ is an ultrafilter, if*

- $u \in U, u \subseteq v \implies v \in U$
- $u_1, u_2 \in U \implies u_1 \cap u_2 \in U$
- $u \in U \iff S \setminus u \notin U$

For a cardinal κ, we say that the ultrafilter U is κ-complete if it is closed not only under finite intersections but under all intersections of cardinality $< \kappa$ (for any $V \subseteq U$, $|V| < \kappa$ it holds that $\bigcap V \in U$).

Example 4.2. *Let S be a set. A principal ultrafilter on S is an ultrafilter of the form $\{u \subseteq S; s \in u\}$ for some fixed element $s \in S$. A principal ultrafilter is κ-complete for any cardinal κ.*

Definition 4.3. *Let κ be a cardinal. We say that κ is measurable if it is uncountable and there exists a non-principal κ-complete ultrafilter on κ. We say that κ is ω-measurable if there exists a non-principal ω_1-complete ultrafilter (i.e. closed under finite and countable intersections).*

Strongly self-slender modules in the case when measurable cardinals don't exist

Measurable cardinals are a kind of large cardinals, i.e. if a cardinal with this property exists, it must be larger than any cardinal we usually work with. And it doesn't have to exist - $ZFC +$ "$\nexists$ measurable cardinal" is equiconsistent with ZFC (the nonexistence of measurable cardinals follows for example from the axiom $V = L$, see [6], Theorem 17.1). In that case, there are many strongly self-slender modules:

Theorem 4.4. *[3, Theorem II.2.11, Corollary II.2.12] Let κ be a cardinal. Then κ is ω-measurable iff there exists a measurable cardinal $\lambda \leq \kappa$.*

Theorem 4.5. *[3, Corollary III.3.3] Let R be a ring, M a sledner R-module and λ be a cardinal, which is not ω-measurable. Then for any collection of R-modules M_α, $\alpha \in \lambda$ the homomorphism $\bigoplus_{\alpha\in\lambda} Hom(M_\alpha, M) \to Hom(\prod_{\alpha\in\lambda} M_\alpha, M)$ is an isomorphism.*

Note. In the book [3], the result is formulated a bit differently, namely, that for any homomorphism $f : \prod_{\alpha\in\lambda} M_\alpha \to M$ there exists $J \subseteq \lambda$, such that $(\forall j \in J : m_j = 0) \implies f\Big((m_\alpha)_{\alpha\in\lambda}\Big) = 0$. But that is exactly the same as saying that f is in the image of the homomorphism $\bigoplus_{\alpha\in\lambda} Hom(M_\alpha, M) \to Hom(\prod_{\alpha\in\lambda} M_\alpha, M)$.

Corollary 4.6. *Let R be a ring, M slender R-module. If there are no measurable cardinals, M is strongly slender (and therefore strongly self-slender).*

Slender modules are not that rare, we present the following two theorems as a source of examples:

Theorem 4.7. *[3, Corollary III.2.4] Let R be a countable integral domain, which is not a field. Then R as a module over itself is slender.*

Theorem 4.8. *[3, Corollary IX.2.4] An abelian group is slender if and only if it doesn't contain any of the following groups as a subgroup:*

- $\mathbb{Q}$
- $\mathbb{Z}^\omega$
- $\mathbb{Z}_p$ *(the cyclic group of order p)*
- J_p *(the group of p-adic integers)*

Strongly self-slender modules in the case when measurable cardinals exist

On the other hand, the consistency of $ZFC +$ "$\exists$ measurable cardinal" is strictly stronger than the consistency of ZFC alone, i.e. even if we assume that ZFC is consistent, we cannot possibly prove that $ZFC+$"$\exists$ measurable cardinal" is. The reason is that a set of such large cardinality can be a model of ZFC, therefore proving the consistency of ZFC (see [6], Theorem 12.12).

Theorem 4.9. *[3, Theorem II.2.13] Let κ be a measurable cardinal. Then κ is strongly inaccessible: it is regular and for any cardinal $\lambda < \kappa$ it holds that $2^\lambda < \kappa$.*

Lemma 4.10. *Let U be a κ-complete ultrafilter on S, $\lambda < \kappa$ and $\{S_\alpha; \alpha \in \lambda\}$ be any partition of S into λ many subsets. Then $S_\alpha \in U$ holds for exactly one $\alpha \leq \lambda$.*

Proof. Since $U_\alpha \cap U_\beta = \emptyset \notin U$, there is at most one such α. On the other hand, since $\bigcap_{\alpha\in\lambda}(S \setminus S_\alpha) = \emptyset \notin U$, there is a α such that $S \setminus S_\alpha \notin U$ and therefore $S_\alpha \in U$.

□

Lemma 4.11. *[3, Chapter II, Exercise 12] Let κ be a measurable cardinal. Then the number of κ-complete ultrafilters on κ is $\geq 2^\kappa$.*

Proof. Let T be the set of partial functions $\{f : \alpha \to \{0,1\}; \alpha < \kappa\}$, from κ being strongly inaccessible follows $|T| = \kappa$ (T can be decomposed into κ-many subsets $T_\alpha = \{f : \alpha \to \{0,1\}\}$ of cardinality $2^{|\alpha|} < \kappa$). We assume that there is a non-principal κ-complete ultrafilter U on T. Fix an ordinal $\alpha < \kappa$. Decompose T into $2^{|\alpha|} + 1$ subsets S_0 and S_g for each $g : \alpha \to \{0,1\}\}$ defined by $S_0 = \{f : \beta \to \{0,1\}; \beta < \alpha\}$, $S_g = \{f : \beta \to \{0,1\}; \beta \geq \alpha, f|_\alpha = g\}$. Since $|S_0| < \kappa$, $S_0 \notin U$ (S_0 is a union of $< \kappa$-many singletons, which are not in U, otherwise U would be principal), therefore using the previous lemma $S_f \in U$ for exactly one $f : \alpha \to \{0,1\}$. It is clear that for $\alpha < \alpha' < \kappa$ it will hold $f'|_\alpha = f$, therefore the union F_U of all such f's is a map $\kappa \to \{0,1\}$. Now for a function $h : \kappa \to \{0,1\}$ define a permutation π_h on T by $f \mapsto f + h|_\alpha \pmod 2$, where α is the domain of f. Then $\pi_h(U) := \{\pi_h(u); u \in U\}$ is clearly a κ-complete non-principal ultrafilter with $F_{\pi_h(U)} = F_U + h \pmod 2$. There are 2^κ possibilities for h and therefore at least 2^κ non-principal κ-complete ultrafilters.

□

Theorem 4.12. *[4, Corollary 2.4] Let κ be a measurable cardinal, R be a ring and M be a non-zero R-module of cardinality less than κ. Then M is not strongly self-slender.*

Proof. We will show that $|Hom(M^\kappa, M)| \geq 2^\kappa > \kappa \geq |End(M)^{(\kappa)}|$. When we view endomorphisms of M as subsets of $M \times M$, we get a bound $End(M) \leq 2^{|M \times M|}$, which together with the fact that κ is a strongly inaccessible implies $|End(M)| < \kappa$ and therefore $|End(M)|^{(\kappa)} \leq \kappa$. $2^\kappa > \kappa$ is a well-known result of Cantor holding for every cardinal. To show the first inequality, define for a κ-complete ultrafilter U a homomorphism $f_U : M^\kappa \to M$ by $f_U((m_\alpha)_{\alpha\in\kappa}) = m \iff \{\alpha; m_\alpha = m\} \in U$. For a nonzero M, those homomorphisms are different for different ultrafilters.

□

Corollary 4.13. *[4, Corollary 2.5] If there exists a proper class of measurable cardinals, there is no non-zero strongly self-slender module.*

5. Open problems

Problem 1. *In the previous chapter, we saw that if our set theory contains a measurable cardinal κ, a module with cardinality $< \kappa$ cannot be strongly self-slender. Can it be proven that the equality $\varprojlim prod(M) = \varprojlim Prod(M)$ doesn't hold, at least for example for $M = \mathbb{Z}$? A possible candidate for a module in $\varprojlim Prod(M) \setminus \varprojlim prod(M)$ could be the kernel of the morphisms f_U defined in Theorem 4.12, as these morphisms are the reason why the module cannot be strongly self-slender. Moreover, these morphisms are split (their one-sided inverse is the diagonal morphism $m \mapsto (m)_{\alpha\in\kappa}$, therefore $Ker(f_U) \in \varprojlim Prod(M)$.*

Problem 2. *As was argued in the proof of Theorem 3.8, for any module M it holds that $\varprojlim Prod(M) = \bigcup_{\kappa \text{ cardinal}} \varprojlim\{M^\kappa\}$. What can be said about the non-decreasing chain $\varprojlim\{M^\kappa\}$? For example, if M is slender, does it increase only at measurable cardinals? Will this chain eventually stabilize for any module M (at least if we assume that measurable cardinals don't exist)?*

Problem 3. *Theorem 3.3 suggests that the question $prod(M) = \varprojlim Prod(M)$ could be easy when M is a module, such that $End(M)$ is a right perfect ring, in that case, flat modules over it are projective and therefore $\varprojlim prod(M)$ contains only direct summands of products of M. Can we, therefore, generalize the result of Corollary 3.4 to all modules with right perfect $End(M)$?*

Problem 4. *As can be seen from Corollary 3.4, for the class $\mathcal{C} := prod(F)$ of finitely-dimensional vector spaces it holds that $\varprojlim \mathcal{C} \not\supseteq \varprojlim \varprojlim \mathcal{C}$. For a ring R and a class $\mathcal{C}$ of R-modules and an ordinal α, we can define inductively classes $\varprojlim^{\alpha+1} \mathcal{C} := \varprojlim \varprojlim^\alpha \mathcal{C}$ and $\varprojlim^\alpha := \bigcup_{\beta<\alpha} \varprojlim^\beta \mathcal{C}$ for α limit. Does the chain $\varprojlim^\alpha \mathcal{C}$ need to stabilize for any class $\mathcal{C}$? If so, can this point of stabilization be bounded by some ordinal dependent on the ring R? Or even independent on the ring?*

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