Report on
" Mapping spaces of algebras for iterated +-construction for polynomial monads"

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(submitted as a Master Thesis in Mathematics, Charles University)

May 30, 2023

This compact and dense memoir builds on a substantial paper of Batanin and De Leger (2019), which itself places works of Dwyer and Hess (2012) and of Turchin (2014) (establishing that some homotopy limits arising from topological operads are weakly equivalent to double loop spaces) in the conceptual framework of Joyal et al's polynomial functors. Also, Batanin and De Leger make use of a sophisticated homotopical algebra apparatus for categorical / simplicial algebras over (set-based) polynomial monads developped by Batanin and Berger (2017). All this to say that the candidate needed to absorb a lot of material (both deep and technical) before engaging in his own research. Polynomial functors and monads are a very versatile framework, which allows to capture combinatorially the categories of (pointed) operads, bimodules, and infinitesimal bimodules which feature in the two successive delooping results of Turchin and Dwyer-Hess: those respective categories are the categories of algebras for three polynomial monads, which are monads of trees with various decorations on their nodes or edges.

The work presented here is a further generalisation of those of Batanin and De Leger. The latter involve the first two levels in a hierarchy based on the iteration of Baez-Dolan slice construction (called +-construction in its formulation in terms of polynomial monads by Kock, Joyal, Batanin, and Mascari). This hierarchy founds the so-called opetopic approach to higher algebraic structures.

In the presented work, the notion of bimodule is now parameterised by pairs (k, n) $(k \leq n)$, where n refers to the position in the Baez-Dolan hierarchy. The results of Batanin and De Leger correspond exactly to the cases $(1, 2)$ (first delooping) and $(0, 2)$ (second delooping). The generalisation made by Maroš is two-fold:

- He literally extends Batanin De Leger to $(n-1, n)$ and $(n-2, n)$, thus evidencing

(sufficient reducedness conditions for) delooping and double delooping at all levels of the hierarchy.

- He studies thoroughly the case $(n-3, n)$ and gives sufficient conditions for triple delooping.

He also briefly discusses obstructions to giving reducedness conditions for making further quadruple etc. delooping possible at higher dimensions.

This is original and highly skilled work: in fact, in on-going joint work with De Leger, Maroš has more results, and the plan is to integrate the (already impressive) outcome of the master thesis in a forthcoming submission.

The writing of the thesis is generally clear, although a bit dry at places (but this allows the candidate to keep the memoir to a reasonable size). I have been lost in the combinatorics at a certain point, where the terms used are too vague for me to grasp exactly what is going on (especially in the crucial description of the notion of k-dimensional set of vertices, where it is not clear to me what "contracting a vertex", or "lying on a path" mean exactly). More pictures with more explanations would definitely help, e.g. illustrating the many colours introduced: black, white, gray, "of type 1 or 2" (it is not clear to which vertices this dichotomy of types applies), etc. Also colour "white" is mentioned before having been introduced...

Overall, I definitely recommend to give a high grade for this thesis work, and I am looking forward to the oral defence. I refrain from recommending the highest grade right-away, for the exposition reasons explained above, but I definitely think that the work (once cleaned and stream-lined) is worth publication in a very good journal.