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Mapping spaces of algebras over iterated +-construction for polynomial monads

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I would like to thank my supervisor Michael Batanin and consultant Florian De Leger for introducing me to this subject and all the encouragement and advice I have received. It is a great experience to explore these topics together.

Title: Mapping spaces of algebras over iterated +-construction for polynomial monads

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Abstract: In a 2019 paper “Polynomial monads and delooping of mapping spaces”, Batanin and De Leger have introduced an extension of Grothendieck homotopy theory from the category of small categories to the category of polynomial monads. As an application (among other), they provided a new proof of a famous Tourchin-Dwyer-Hess theorem on explicit double loop space of mapping spaces between the associativity operad and an arbitrary reduced multiplicative operad.

In this thesis we generalize Batanin-De Leger results to a sequence of polynomial monads produced by iteration of the Baez and Dolan +-construction (the so called opetopic sequence). For the n-th element of the opetopic sequence, we introduce the monads called k-dimensional bimodules, $0 \leq k \leq n$, which generalize the notions of bimodules and infinitesimal bimodules over the associative operad for non-symmetric operads. The 0-dimensional bimodules are a sequence of categories of opetopes, with each the full subcategory of the next, which generalizes the simplicial category Δ and the dentroidal category of planar trees Ω_p.

We show that an explicit double looping of the corresponding mapping space exists for any $n \geq 2$, where $n = 2$ corresponds to the classical case. We provide a further reduceness condition under which the third looping of the mapping space has an explicit expression. We hope this result will be useful for constructing novel models of embedding spaces of manifolds.

Keywords: operads homotopy theory

Contents

Introduction

Let $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ be the space of embeddings of $\mathbb{R}^1 \to \mathbb{R}^n$ with compact support (equal to the canonical embedding outside a compact set), the so called space of *long knots*. It was shown by Sinha [2009] using Goodwille calculus that for $n \geq 4$, there is a weak equivalence (which we denote $\sim$ in this text)

$$\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n) \sim \mathrm{holim}_\Delta \mathcal{C}^\bullet$$

where $\mathcal{C}^n$ is a Fulton-MacPherson completion of the configuration space of n points. In Sinha [2006], it is furthermore shown that

$$\overline{\mathrm{Emb}}(\mathbb{R}^1, \mathbb{R}^n) \sim \mathrm{holim}_\Delta \mathcal{K}^*$$

where $\overline{\mathrm{Emb}}(\mathbb{R}^1, \mathbb{R}^n)$ is the homotopy fiber of the inclusion of $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ to the space of immersions (embeddings with intersections allowed), $\mathcal{K}$ is the desymmetrisation of the Kontsevich operad [Kontsevich, 1999] and $\mathcal{K}^*$ its associated cosimplicial object.

Polynomial monads were introduced in Kock et al. [2010]. It was proved in Gambino and Kock [2013] that the category of finitary polynomial monads is equivalent to the category of Σ-free colored operads in Set. Finitary polynomial monads may have algebras in any symmetric monoidal category $(\mathcal{E}, \otimes, e)$ (Batanin and Berger [2017], see also the Section 1.1.3). For any finitary polynomial monad P, there always exists a special P-algebra, which has the monoidal unit e of $\mathcal{E}$ for each color $i \in I$ of P. We denote this algebra ζ_P (or simply ζ if there is no confusion). If $(\mathcal{E}, \otimes, e)$ is a cartesian symmetric monoidal category, then ζ_P is the terminal algebra of P.

For a model category $\mathcal{C}$, let $\mathrm{Map}_\mathcal{C}(-,-)$ be the homotopy mapping space in $\mathcal{C}$. Turchin [2014] and Dwyer and Hess [2012] have proved that if a multiplicative non-symmetric X operad in topological spaces or simplicial sets is reduced (meaning $X_0 \sim X_1 \sim 1$), then

$$\mathrm{holim}_\Delta X^* \sim \Omega^2 \mathrm{Map}_{\mathrm{NOp}}(\zeta, X)$$

where Ω denotes the loop space (over the base point given by the map $\zeta \to X$ from the structure of multiplicative operad). These results together provide an explicit model of $\overline{\mathrm{Emb}}(\mathbb{R}^1, \mathbb{R}^n)$ as a double loop space.

Now Turchin and Dwyer-Hess theorem proceeds in two steps

$$\begin{aligned} \Omega \mathrm{Map}_{\mathrm{NOp}}(\zeta, X) &\sim \mathrm{Map}_{\mathrm{Bimod}}(\zeta, X) \qquad \text{if } X_1 \sim 1 \\ \Omega \mathrm{Map}_{\mathrm{Bimod}}(\zeta, X) &\sim \mathrm{Map}_{\mathrm{IBimod}}(\zeta, X) \qquad \text{if } X_0 \sim 1 \end{aligned}$$

where Bimod, resp. IBimod are bimodules, resp. infinitesimal bimodules over the associativity operad and we skip writing down the forgetful functors. The result follows from the category of infinitesimal bimodules being equivalent to the category of cosimplicial objects and $\mathrm{Map}_{\mathrm{IBimod}}(\zeta, X) \sim \mathrm{holim}_\Delta X$.

Batanin and De Leger [2019] used the fact that non-symmetric operads, their bimodules and infinitesimal bimodules are algebras of particular polynomial monads to give a more categorical proof of the preceding results, using their extension of Grothendieck homotopy theory to the category of polynomial monads, via the machinery of internal algebra classifiers introduced in Batanin and Berger [2017].

In the terminology of Kock et al. [2010], the polynomial monad for non-symmetric operads is the second iteration of the Baez-Dolan +-construction [Baez and Dolan, 1998]. If we continue to iterate the +-construction, we obtain the so called *opetopic sequence* of polynomial monads. The goal of this thesis is to generalize the results of Turchin [2014], Dwyer and Hess [2012] and Batanin and De Leger [2019] to higher elemens of the opetopic sequence.

In the first chapter, we recall the preliminaries and known results about polynomial monads, opetopic sequence and their homotopy theory.

In the second chapter, for the n-th iteration of the +-construction, we introduce the category of (k, n)-bimodules $\mathrm{Bimod}^{k,n}$, $0 \leq k \leq n$. They generalize bimodules, resp. infinitesimal bimodules over ζ for non-symmetric operads, which are $\mathrm{Bimod}^{1,2}$, resp. $\mathrm{Bimod}^{0,2}$. Moreover, we introduce small categories $\Omega_{\mathbb{R}^n}$, such that for a monoidal category $\mathcal{E}$, $\mathrm{Bimod}^{0,n+1}(\mathcal{E})$ are the presheaves $[\Omega_{\mathbb{R}^n}, \mathcal{E}]$. They generalize the categories Δ for $n = 1$ and Ω_p (the dendroidal category of planar trees) for $n = 2$. We have injections of full subcategories $\Omega_{\mathbb{R}^n} \to \Omega_{\mathbb{R}^{n+1}}$. We construct their colimit $\Omega_{\mathbb{R}^\infty}$ and identify its objects with the stable opetopes of Kock et al. [2010].

In the third chapter, we investigate the possible delooping. The aim is to find a reduceness condition C on a simplicial (k, n)-bimodule X equipped with a map $\zeta \to X$ such that

$$\Omega\mathrm{Map}_{\mathrm{Bimod}^{k,n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}^{k-1,n}}(\zeta, X) \qquad \text{if } X \text{ satisfies } C$$

where we again skip writing down the forgetful functors. It is shown that for the first two deloopings (from n to $n-1$ to $n-2$), the reduceness conditions are straightforward generalizations to the ones for non-symmetric operads. We then show that for any $n \geq 2$, there are $(n-3, n)$-bimodules $\delta^*\tau_n$ such that

$$\Omega\mathrm{Map}_{\mathrm{Bimod}^{n-2,n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}^{n-3,n}}(\zeta, X) \qquad \text{if } \mathrm{Map}_{\mathrm{Bimod}^{n-3,n}}(\delta^*\tau_n, X) \sim 1$$

In particular $\delta^*\tau_3$ can be understood as a functor $\alpha : \Omega_p \to \mathrm{SSet}$ assigning to a planar tree the set of 2-dimensional strata, to which the tree separates the plane. By adjunction, the reduceness condition above is equivalent to

$$\mathrm{Map}_{[\int\alpha, \mathrm{SSet}]}(1, \pi^*X) \sim \mathrm{holim}_{\int\alpha}\pi^*X \sim 1$$

where $\int\alpha$ is the Grothendieck construction of α and $\pi : \int\alpha \to \Omega_p$ the canonical projection.

The proof that this condition is satisfied for the higher "desymmetrizations" of symmetric operads, which have multiplicative structure (in particular the Kontsevich operad), will be published in a joint paper with Florian De Leger containing the results of this thesis.

1. Preliminaries

1.1 Polynomial monads

Definition 1.1. Let C^I be the product of a category C indexed by the set I. For $X \in C^I$ and $i \in I$, X_i will be the component of X with the index i.

Consider a general oriented multidigraph (equivalently a span in Set) given by the diagram

$$I \xleftarrow{s} B \xrightarrow{t} I$$

where I is the set of vertices, B is the set of edges, s is the source map and t is the target map. It induces a functor $G : \mathrm{Set}^I \to \mathrm{Set}^I$ given in the i-th component for $i \in I$ by

$$G(X)_i = \sum_{\substack{b \in B \\ t(b)=i}} X_{s(b)}$$

It turns out that the structure of a category on this graph (the composition of morphisms and units) corresponds to a monad structure on G, with the associative and unit laws corresponding precisely to the monadic laws [Gambino and Kock, 2013]. This motivates the following, more general definition.

1.1.1 Polynomial functors

Definition 1.2 (Gambino and Kock [2013])**.** A polynomial is a diagram of sets of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

A polynomial functor given by the this diagram is the composition $t_! p_* s^*$, where

- s^* is the restriction of s (given by $s^*(X)_e = X_{s(e)}$),
- p_* is the right adjoint of p^* (given by $p_*(X)_b = \prod_{e \in p^{-1}(b)} X_e$)
- $t_!$ is the left adjoint of t^* (given by $t_!(X)_i = \sum_{b \in t^{-1}(i)} X_b$).

In total,

$$t_! p_* s^*(X)_i = \sum_{b \in t^{-1}(i)} \prod_{e \in p^{-1}(b)} X_{s(e)}$$

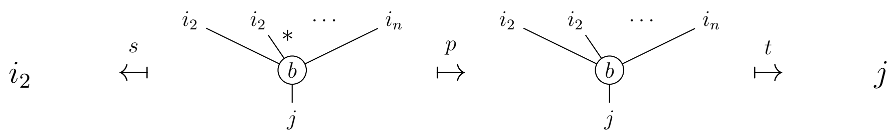
Hence the name "polynomial".

The set I will be called *colors*, B will be called *operations* and E will be called *marked operations*. s will be called *source* map, t *target* map and p *middle* map.

The polynomial is called *finitary* if $p^{-1}(b)$ is finite for all $b \in B$. All polynomials considered here will be finitary.

An element $b \in B$ can be pictured as a corolla, with the edges above in a bijective correspondence with $p^{-1}(b)$, each decorated by $i_k = s(e_k)$ for the corresponding $e_k \in p^{-1}(b)$; they will be called *input* edges. The edge bellow is decorated by $t(b)$; it will be called an *output* edge. $b \in B$ can be thought of as a "multimorphism" having $i_1, \ldots, i_n$ as sources and $t(b)$ as a target.

An element of E can be pictured similarly, but with one input edge marked. The map p forgets the marked edge.

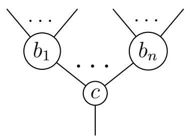


Remark. It is possible to consider more general polynomials where the target set J may be different from I. We will not do it here.

Construction 1.3. Let F be a polynomial functor given by $I \leftarrow E \rightarrow B \rightarrow I$ and G a polynomial functor given by $I \leftarrow D \rightarrow C \rightarrow I$. Their composition $G \circ F$ is isomorphic to a polynomial functor given by

$$I \xleftarrow{s} (C \circ B)^* \xrightarrow{p} C \circ B \xrightarrow{t} I$$

where $C \circ B$ is the set of two level trees of the form



with $c \in C$, $b_1, \ldots, b_n \in B$ being represented by decorated corollas as above, such that the target of b_k is i_k for $k = 1, \ldots, n$. The target of this tree is the target of c. $(C \circ B)^*$ is the set of such trees with one input edge marked. The map s returns the color of the marked edge and p forgets the mark.

Proof. $G \circ F(X)_i = \sum_{c \in t_G^{-1}(i)} \prod_{d \in p_G^{-1}(c)} \sum_{b \in t_F^{-1}(s_G(d))} \prod_{e \in p_F^{-1}(b)} X_{s_F(e)}$, where s_P, p_P, t_P are the source, middle and target map for the polynomial functor P. Distributivity for sums and products in Set gives the sought isomorphism. $\square$

We will denote by 1 the terminal object in Set^I (1 in every component). For a polynomial functor P given by $I \leftarrow E \xrightarrow{p} B \xrightarrow{t} I$,

$$P(1)_i = \sum_{b \in t^{-1}(i)} \prod_{e \in p^{-1}(b)} 1 \simeq t^{-1}(i)$$

So if B is considered an element of Set^I by $B_i = t^{-1}(i)$, then $P(1) \simeq B \in \mathrm{Set}^I$.

Definition 1.4. A natural transformation is called *cartesian* if all the naturality squares are pullbacks.

Proposition 1.5. *Cartesian natural transformations between polynomial functors $P' \Rightarrow P$ correspond to the following diagrams between their respective polynomials*

$$
\begin{tikzcd}
& E' \arrow[ld] \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\lrcorner", very near start] & B' \arrow[rd] \arrow[dd, "v"] & \\
I & & & I \\
& E \arrow[lu] \arrow[r] & B \arrow[ru] &
\end{tikzcd}
$$

where the middle square is a pullback.

Proof. Because the middle square is a pullback, for $b' \in B'$, the source image of its fiber of the middle map is the same as of $v(b')$.

Since $B \simeq P(1)$, $B' \simeq P'(1)$, we have the pullback $P'(X) \times_{P'(1)} P(1) \simeq P'(X) \times_{B'} B$ whose i-th component is $\sum_{b \in t^{-1}(i)} \prod_{e \in p^{-1}(v(b))} X_{s(e)} = P(X)_i$ because the sources of fibers are the same. This gives a natural transformation $P'(X) \to P(X)$. Using the pullback lemma, one can show that it is cartesian.

On the other hand, such natural transformation gives a map $P'(1) \to P(1)$ which by the above yields the pullback square in the given diagram. $\square$

Remark. By the preceding proposition, to give a cartesian natural transformation between polynomial functors, it suffices to give a map $v : B' \to B$ such that for every $b' \in B'$ the sources of fibers of b' are the same as for $v(b')$.

1.1.2 Monads

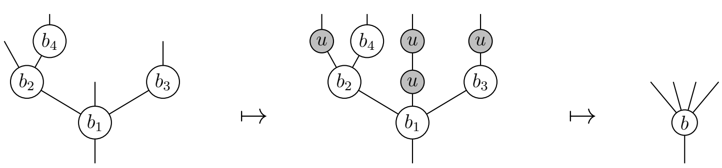
Definition 1.6 (Gambino and Kock [2013])**.** A polynomial monad is a monad whose underlying functor P is polynomial and the natural transformations $1 \Rightarrow P$, $P \circ P \Rightarrow P$ are cartesian.

If the functor P is given by the polynomial $I \leftarrow E \to B \to I$, the polynomial monad structure corresponds to the following diagrams:

$$
\begin{tikzcd}
& (B \circ B)^* \arrow[ld] \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\lrcorner", very near start] & B \circ B \arrow[rd] \arrow[dd] & \\
I & & & I \\
& E \arrow[lu] \arrow[r] & B \arrow[ru] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[ld] \arrow[r] \arrow[dd] \arrow[rdd, phantom, "\lrcorner", very near start] & I \arrow[rd] \arrow[dd] & \\
I & & & I \\
& E \arrow[lu] \arrow[r] & B \arrow[ru] &
\end{tikzcd}
$$

The map $B \circ B \to B$ can be understood as a composition of operations. The middle square being a pullback ensures that the colors of the input edges of the two level tree in $B \circ B$ must correspond to the colors of inputs of its composition. The unit map gives a distinguished operation $u_i \in B$ with unary fiber for each $i \in I$, called *unit*. The unit law says they indeed act as units during composition. The associative law says it doesn't matter in which order we do the composition.

In general, this means than any tree composed of corollas of operations of B can be contracted to a unique operation of B, by pluging units and using the monadic composition.



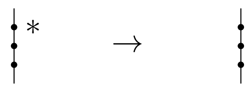
Example (Batanin and Berger [2017], Section 9.1). A small category C is a polynomial monad, where I are objects of C, B are morphisms of C and the middle map is identity. The monadic composition is the composition of morphisms in C and the monadic units are the identities.

Since operations for categories all have unary fibers, categories can be thought of as *linear* monads. In this sense, polynomial monads are a generalization of categories.

Example (Batanin and Berger [2017], Section 9.2). The free monoid monad is a monad for the "geometric series" functor $\sum_{n \in \mathbb{N}} X^n$. It is given by the polynomial

$$1 \leftarrow \mathrm{Ltr}^* \rightarrow \mathrm{Ltr} \rightarrow 1$$

where Ltr is the set of linear trees - the trees where every vertex has one input edge (which are in bijection with $\mathbb{N}$ (including 0)) given by the number of vertices) and Ltr^* are linear trees with one vertex marked. The middle map forgets the marked vertex.



Example (Batanin and Berger [2017], Section 9.2). The non-symmetric operad monad is given by the following polynomial

$$\mathbb{N} \leftarrow \mathrm{Ptr}^* \rightarrow \mathrm{Ptr} \rightarrow \mathbb{N}$$

where Ptr is the set of planar trees, Ptr^* are planar trees with one vertex marked, the source map returns the number of input edges into the marked vertex, the middle map forgets the mark and the target map returns the number of input edges of the whole tree.

It is shown in the subsequent section that these monads are the first interations of the Baez-Dolan +-construction starting from the identity monad

$$1 \leftarrow 1 \rightarrow 1 \rightarrow 1$$

.

Definition 1.7 (Gambino and Kock [2013]). For a polynomial monad P given by the polynomial $I \leftarrow E \rightarrow B \rightarrow I$ and a polynomial monad Q given by the polynomial $J \leftarrow D \rightarrow C \rightarrow J$, a cartesian morphism from P to Q is be the following commutative diagrams

$$\begin{tikzcd}
I \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d] & I \arrow[d] \\
J & D \arrow[l] \arrow[r] & C \arrow[r] & J
\end{tikzcd}$$

where the middle square is a pullback and the vertical maps are compatible with the monadic composition and units.

Finitary polynomial monads and cartesian morphisms form a category **Poly**.

1.1.3 Algebras

Let $\mathcal{E}$ be a cocomplete symmetric monoidal category and let P be a finitary polynomial functor. We can construct a functor $P^{\mathcal{E}} : \mathcal{E}^I \to \mathcal{E}^I$ by a similar formula as in the case of Set:

$$P^{\mathcal{E}}(X)_i = \sum_{b \in t^{-1}(i)} \bigotimes_{e \in p^{-1}(b)} X_{s(e)}$$

If P has a structure of a polynomial monad, this induces a monad on $\mathcal{E}^I$. We will call P-algebras in $\mathcal{E}$ the algebras of this induced monad.

Explicitly, it is given by a collection $A_i \in \mathcal{E}$ for each $i \in I$, along with, for each $b \in B$, the map

$$m_b : \bigotimes_{e \in p_n^{-1}(T)} A_e \to A_{t_n(T)} \tag{1.1}$$

satisfying associativity and unitarity conditions

We will denote the category of $P^{\mathcal{E}}$-algebras and $P^{\mathcal{E}}$-algebra morphisms $\mathrm{Alg}_P(\mathcal{E})$.

Definition 1.8. Let ζ_P be the $P^{\mathcal{E}}$-algebra, which is the unit of $\mathcal{E}$ in every component. Where the index is clear, we will just write ζ.

Remark. In all categories considered here, ζ will also be the terminal P-algebra.

Example. When a small category C is considered as a linear monad, the category of C-algebras is the category of covariant presheaves $[C, \mathcal{E}]$.

Example. Algebras of the free monoid monad are monoids in the monoidal category $\mathcal{E}$.

Example. Algebras of the non-symmetric operad monad are the non-symmetric $\mathcal{E}$-operads. That is an object A_n for each $n \in \mathbb{N}$, along with morphisms $A_k \otimes A_{n_1} \otimes \cdots \otimes A_{n_k} \to A_{n_1 + \cdots + n_k}$ and $e \to A_1$ (where e is the unit of $\mathcal{E}$) satisfying the corresponding associativity and unit conditions.

1.2 Baez-Dolan +-construction for polynomial monads

The +-constuction was originally introduced by Baez and Dolan [1998] as a way to formalize the composition of higher-dimensional operations arising in the theory of higher categories. Later in Kock, Joyal, Batanin, and Mascari [2010], it was defined for polynomial monads and shown that they are a natural setting for this construction. The definition given here follows this paper.

Informally, for a polynomial monad P, the monad P^+ has as colors the operations B of P and as operations the compositions of operations of P.

Definition 1.9 (Kock et al. [2010])**.** Let P be a polynomial monad given by:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

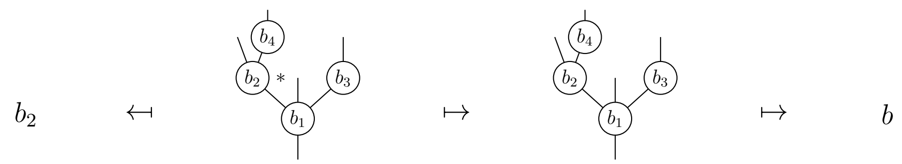
The monad P^+ is given by the following polynomial:

$$B \leftarrow \mathrm{tr}(B)^* \to \mathrm{tr}(B) \to B$$

where $\mathrm{tr}(B)$ (called P-trees) is the set of trees composed of corollas of the operations B and edges decorated by I. In other words, they are trees whose vertices are decorated by the elements of $b \in B$ and edges by $i \in I$, with each having precisely one output edge decorated with $t(b)$ and input edges with a correspondence to the elements $e \in p^{-1}(b)$, each being decorated with $s(e)$. The decorations of inputs and outputs must match.

The set $\mathrm{tr}(B)^*$ is the set of P-trees with one vertex marked. The source returns the decoration of the marked vertex, the middle map forgets the marking and the target returns the monadic composition of this tree or the unit of $i \in I$ if the tree is just a free living edge decorated by i - it is where the monadic structure of P is used.

The monadic composition is given by insertion of decorated trees to marked points.



1.2.1 Opetopic sequence

Definition 1.10. Let $\mathcal{I}$ be the identity polynomial monad given by the polynomial

$$1 \leftarrow 1 \rightarrow 1 \rightarrow 1$$

where all map identities.

Claim 1.11 (Kock et al. [2010]). *$\mathcal{I}^+$ is the free monoid monad.*

Proof. The set $\mathrm{tr}(1)$ is the set of trees composed of unary vertices, with each vertex and edge labeled by 1. Those are exactly the linear trees. □

Definition 1.12. For a polynomial monad P, let P^{+n} be the monad resulting from applying the $+$-construction on P n times.

Claim 1.13 (Kock et al. [2010]). *$\mathcal{I}^{+2}$ is the non-symmetric operad monad.*

Proof. Linear trees are in bijection with $\mathbb{N}$ given by the number of vertices. So for each $n \in \mathbb{N}$, there is one possible marking of the vertex with n input edges. Moreover, the vertices of a linear tree are canonically linearly ordered. As the input edges of a vertex of $\mathrm{tr}(\mathrm{Ltr})$ correspond to the vertices of the linear tree which marks this vertex, this gives the linear order of input edges for each vertex. That means the tree is planar. □

Definition 1.14. The colors of the polynomial monad $\mathcal{I}^{+n}$ will be called I_n. Its operations (which are the colors of $\mathcal{I}^{+(n+1)}$) will be called B_n.

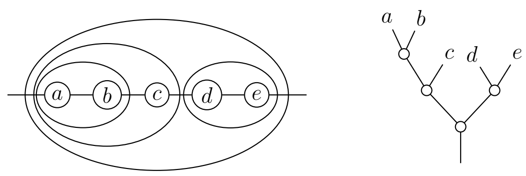
The sequence I_n for $n \in \mathbb{N}$ is called *opetopic sequence* and it elements are called *opetopes* [Baez and Dolan, 1998].

To work represent them, Kock et al. [2010] introduce a notion of a *constellation* and *zoom complex.*

We will call *a nesting* a finite collection of non-intersecting circles or dots, which consists either of a single dot, or has a large circle containing everything else. To every nesting, we can associate a tree, whose vertices correspond to the circles, leaves to the dots and a vertex or a leaf is above another vertex v if the corresponding circle or a dot is inside the circle corresponding to v.

For a polynomial monad P, we will call a P-constellation a tree decorated with operations of P in an appropriate manner, along with a nesting over it, whose dots are the vertices of the tree and the vertices and edges contained in every circle form a subtree. Although this is a rather geometric way to think about constellations, they can be defined in purely combinatorial terms, as in Kock et al. [2010]. We will call the set of P-constellations $\mathrm{const}(P)$.

Example. The constellations of the free monoid monad correspond to planar trees.



For a constellation, we define its target as the underlying tree. For a circle in a constellation, we define its source as the subtree inside this circle, where all circles inside it are contracted into a vertex (using the monadic composition).

Theorem 1.15 (Kock et al. [2010], Theorem 3.6)**.** *There is a bijection* $\mathrm{const}(P) \simeq \mathrm{tr}(P^+)$ *compatible with the source and target maps.*

Proof. Given a constellation, assign to it the associated tree of the underlying nesting; decorate each vertex with the source of the corresponding circle and each input edge with the source of the corresponding dot (as a vertex of a tree in $\mathrm{tr}(P)$) or the target of the circle corresponding to the vertex above it.

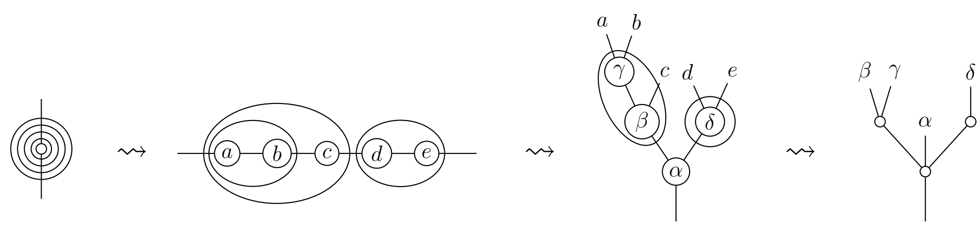
Given a tree $T \in \mathrm{tr}(P^+)$, consider its target, which is a tree $t(T) \in \mathrm{tr}(P)$. Each vertex of T is decorated with tree, which is inserted during the monadic composition to obtain $t(T)$; draw a circle around it. This yields a constellation and these maps are mutually inverse.

For details, see Kock et al. [2010]. □

Constellations are a graphical way to represent the operations of P^{+2}. For the further iterations of the $+$-construction, we can use the fact that a constellation corresponds to a tree in $\mathrm{tr}(P^+)$ and have yet another constellation on this tree.

A *zoom* is a pair of constellations $X \rightsquigarrow Y$ such that the nesting of the former corresponds to the underlying tree of the later. A *zoom complex* is a sequence of zooms $X_1 \rightsquigarrow \cdots \rightsquigarrow X_n$.

An opetope can be represented by a zoom complex like this one (note there is always the top circle containing the whole tree that we skip drawing, corresponding of the root of the next tree):



In total, between the elements $X_k \rightsquigarrow X_{k+1}$ of the zoom complex, we have the correspondences:

X_k	X_{k+1}
vertices	leaves
circles	vertices
underlying tree	target
sources of circles	sources

1.2.2 Stable opetopes

Construction 1.16 (Suspension of opetopes)**.** For all $n \in \mathbb{N}$, we will inductively construct injective maps of sets $\Sigma_n : I_n \to I_{n+1}$ which induce maps of polynomial monads for opetopes

$$\begin{tikzcd}
I_n \arrow[d, "\Sigma_n"] & B_n^* \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_n \arrow[r] \arrow[d, "\Sigma_{n+1}"] & I_n \arrow[d, "\Sigma_n"] \\
I_{n+1} & B_{n+1}^* \arrow[l] \arrow[r] & B_{n+1} \arrow[r] & I_{n+1}
\end{tikzcd}$$

So all the vertical maps are injections compatible with targets and sources.

1. $\Sigma_0 : I_0 \to I_1$ is the unique map between singleton sets
2. $\Sigma_1 : I_1 \to I_2$ maps the single element to the linear tree with one vertex
3. $T \in I_n$ is a tree with vertices decorated by I_{n-1} and edges decorated by I_{n-2}; replace each decoration b of a vertex by $\Sigma_{n-1}b$ and i of an edge by $\Sigma_{n-2}i$. Since Σ_k for $k < n$ are compatible with targets and sources, this decoration is again compatible. Directly by this definition, Σ_n preserves targets and sources.

Σ_0 and Σ_1 are injections, so inductively, each Σ_n maps a distinct tree into a distinct tree and it is an injection as well.

Remark. Kock et al. [2010, Section 4] define suspension of opetopes equivalently in terms of zoom complexes as prepending a new Φ in front of the zoom complex, raising indices of all of its components.

Definition 1.17 (Kock et al. [2010])**.** Let I_∞ be the colimit of the sequence of maps Σ_n for $n \in \mathbb{N}$, called *stable opetopes.*

So I_∞ is a set of opetopes of all dimensions, where we identify two opetopes if one is a suspension of the other.

1.3 Homotopy theory of algebras of polynomial monads

1.3.1 Internal algebras and classifiers

Definition 1.18. Let P be a polynomial monad and A a P-algebra in the category of small categories (a so called categorical P-algebra). An internal P-algebra is a lax-mophism from the terminal P-algebra 1 to A, i.e. a functor f and a natural transformation ϕ

$$\begin{tikzcd} P(1) \arrow[r, "P(f)"] \arrow[d] & P(A) \arrow[d] \arrow[dl, Rightarrow, "\phi"'] \\ 1 \arrow[r, "f"] & A \end{tikzcd}$$

satisfying appropriate coherence conditions [Batanin and Berger, 2017].

Theorem 1.19 (Batanin and Berger [2017], Theorem 5.4)**.** *For a polynomial functor P with the polynomial monad structure given by the unit η and multiplication μ, there is a categorical P-algebra P^P, called an absolute classifier of P, such that for every categorical P-algebra A, there is a natural bijection*

$$\frac{1 \to_{\text{lax}} A}{P^P \to A}$$

i.e. the internal P-algebras in A correspond to (strict) P-algebra morphisms $P^P \to A$.

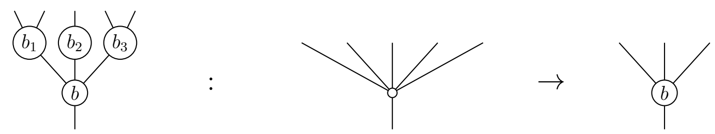
P^P is given by the truncated simplicial bar resolution of the terminal P-algebra:

$$\begin{tikzcd} P(1) \arrow[r, "P\eta" description] & P^2(1) \arrow[l, shift right=2, "\mu"'] \arrow[l, shift left=2, "P!"] & P^3(1) \arrow[l, shift right=2, "\mu P"'] \arrow[l, "P\mu" description] \arrow[l, shift left=2, "P^2!"] \end{tikzcd}$$

where $! : P(1) \to 1$ is the unique map.

For the proof, see Batanin and Berger [2017].

Let P be given by the polynomial $I \leftarrow E \to B \xrightarrow{t} I$. Then P^P is a collection of categories indexed by $i \in I$ and the objects of P^P are $P(1) \simeq B$ as an element of Set^I where $B_i = t^{-1}(i)$ for $i \in I$. Thus the objects correspond to the operations B. In the component $i \in I$, the objects are the operations with the target i. The morphisms are given by the set of operations of $P^2(1)$, i.e. two level trees composed by the elements of B. The source of the morphism is the monadic composition of this tree and the target is the decoration of the root vertex.



The morphisms can be thought of as the opposite of plugging in operations of P into an opperation of P. The composition of morphism given by $P^3(1) \xrightarrow{P\mu} P^2(1)$ ensures that doing this twice is the same as plugging in the composition of the plugged in operations.

By definition, the categorical algebra P^P has a universal internal algebra $e : 1 \to P^P$ by which it is freely generated. The components of e pick up and object $e(i)$ in each category $P^P(i)$ for $i \in I$. This object is the terminal object of $P^P(i)$.

Example. The absolute classifier of the free monoid monad is the augmented simplex category Δ_a of finite ordinals and order-preserving maps. The universal internal algebra is the ordinal $[0]$.

Example. The absolute classifier for the non-symmetric operad monad is the non-symmetric categorical operad of planar trees, where in the component $n \in \mathbb{N}$, the objects are trees with n inputs and the morphisms are generated by contractions of internal edges and insertion of a vertex to an edge. The universal internal algebra is given by the corolla in each component.

Definition 1.20. Let $f : S \to P$ be a cartesian map of polynomial monads and let $f^* : \mathrm{Alg}_P(\mathrm{Cat}) \to \mathrm{Alg}_S(\mathrm{Cat})$ be the corresponding restriction functor. Let A be a categorical P-algebra. An internal S-algebra in A is a lax mophism from the terminal S-algebra to $f^*(A)$.

Theorem 1.21 (Batanin and Berger [2017], Theorem 5.10)**.** *Let $f : S \to P$ be a cartesian map between polynomial monads, where P has the polynomial monad structure given by the unit η and multiplication μ. There is a categorical P-algebra P^S, called a relative classifier of S in P, such that for every categorical P-algebra A, there is a natural bijection*

$$\frac{1 \quad \to_{\mathrm{lax}} f^*(A)}{P^S \quad \to A}$$

i.e. internal S-algebras in A correspond to (strict) P-algebra morphisms $P^S \to A$.

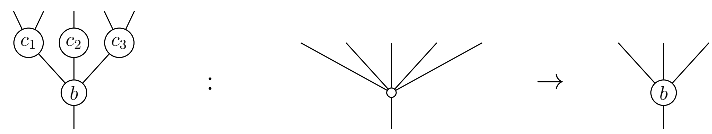
P^S is given by the truncated simplicial bar resolution of the terminal S-algebra:

$$\begin{tikzcd} P\phi_!(1) \arrow[r, "P\eta" description] & P\phi_!S(1) \arrow[l, shift right=2, "\mu\circ Pf"'] \arrow[l, shift left=2, "P!"] & P\phi_!S^2(1) \arrow[l, shift right=2, "\mu P\circ PfP"'] \arrow[l, "P\mu_S" description] \arrow[l, shift left=2, "PS!"] \end{tikzcd}$$

where ϕ is the underlying map on collections induced by f, $\phi_!$ is the left adjoint its restriction functor ϕ^*, $! : \phi_!S(1) \to 1$ is the unique map and μ_S is the monadic composition for S.

For the proof, see Batanin and Berger [2017].

The objects of P^S are again the operations of P, but for each input edge, with its color i, in addition decorated by j such that $\phi(j) = i$, representing the source color. The morphisms are given by two level trees, where the root is decorated by an operation of P with edges colored tihs way and other vertices by operations of S (where the colors of edges are compatible). The target of the morphism is the decoration of the root and the source is given by first mapping every vertex into an operation of P by f and then applying the monadic composition.



Proposition 1.22 (Batanin and De Leger [2019], Proposition 4.6)**.** *Let $f : S \to P$ be a map of polynomial monads and let $f_!$ be the left adjoint to f^*. Then $f_!(S^S) \simeq P^S$.*

Proof. For an S-algebra A, by universal properties of classifiers,

$$\mathrm{Alg}_P(P^S, A) \simeq \mathrm{Alg}_S(S^S, f^*(A))$$

where $\mathrm{Alg}_P(-,-)$ denotes the categorical hom-set of P-algebras. □

Proposition 1.23 (Batanin and De Leger [2019], Proposition 4.7). *A commutative square of maps of polynomial monads*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F} \\ C & \xrightarrow{G} & D \end{array}$$

induces a map between the classifiers $G^f : C^A \to G^*(D^B)$ functorial with respect to horizontal pasting of squares.

Proof. By adjunction, such a map corresponds to a map $G_!(C^A) \to D^B$. But by the Proposition 1.22, $G_!(C^A) \simeq G_!(g_!(A^A)) \simeq D^A$ and we have a map $D^A \to D^B$ induced by f. $\square$

1.3.2 Homotopically cofinal maps of polynomial monads

Let SSet be the category of simplicial sets and let P be a polynomial monad. We will denote $\mathrm{Alg}_P(\mathrm{SSet})$ (or simply Alg_P if no confusion can arise) the simplicial P-algebras. It was proven in Batanin and Berger [2017] that simplicial P-algebras form a simplicial model category, with the model structure transfered from the projective model structure on the collections of simplicial sets along the forgetful functor $\mathcal{U}_P : \mathrm{Alg}_P(\mathrm{SSet}) \to \mathrm{SSet}^I$ (where I is the set of colors of P).

If A is a categorical P-algebra, we obtain a simplicial P-algebra $N(A)$ by taking the nerve in each component.

The following theorem establishes an important connection between the homotopy theory of algebras and the theory of internal algebra classifiers:

Theorem 1.24 (Batanin and Berger [2017], Corollary 8.4). *Let $f : S \to P$ be a map of polynomial monads and P^S the corresponding relative clasifier. Then $N(P^S)$ is a cofibrant simplicial P-algebra and moreover, there is a weak equivalence*

$$\mathbb{L}f_!(\zeta) \sim N(P^S)$$

where $\mathbb{L}f_!$ is the left derived functor of $f_!$.

For a model category $\mathcal{M}$ and its objects X, Y, let $\mathrm{Map}_{\mathcal{M}}(X, Y)$ be the homotopy mapping space between X and Y. If $\mathcal{M}$ is a simplicial model category, it can be calculated as $\underline{\mathcal{M}}(\mathrm{cof}(X), \mathrm{fib}(Y))$, where $\underline{\mathcal{M}}$ denotes the simplicial hom and cof and fib denote the cofibrant, resp. fibrant replacement.

Using classifiers, Batanin and De Leger [2019] proved the following generalization of Quillen Theorem A for small categories:

Theorem 1.25 (Batanin and De Leger [2019], Theorem 5.3). *In the following diagram of maps of polynomial monads*

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "h"'] & & T \arrow[dl, "g"] \\ & R \arrow[d, "\phi"] & \\ & P & \end{tikzcd}$$

if $N(R^S) \to N(R^T)$ *is a weak equivalence then* $N(P^S) \to N(P^T)$ *is a weak equivalence.*

Proof. $N(P^S) \simeq (\phi \circ h)_!(N(S^S)) \simeq \phi_!(N(R^S))$, analogously for $N(P^T)$. So for a fibrant P-algebra X, $N(P^S) \to N(P^T)$ induces a morphism of simplicial homs

$$\underline{\mathrm{Alg}_P}(\phi_!(N(R^S)), X) \leftarrow \underline{\mathrm{Alg}_P}(\phi_!(N(R^T)), X)$$

which by adjunction corresponds to a morphism

$$\underline{\mathrm{Alg}_R}(N(R^S), \phi^*(X)) \leftarrow \underline{\mathrm{Alg}_R}(N(R^T), \phi^*(X))$$

Because $N(R^S)$, $N(R^T)$ are cofibrant and $\phi^*(X)$ is fibrant, it is a weak equivalence, so $N(P^S) \to N(P^T)$ is a weak equivalence as well. $\square$

Corollary 1.26 (Batanin and De Leger [2019], Corollary 5.5)**.** *For a map of polynomial monads* $f : S \to T$*, the following statements are equivalent:*

1. $N(T^S)$ *is contractible*

2. *For any commutative triangle of polynomial monads* $\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "h"'] & & T \arrow[dl, "g"] \\ & R & \end{tikzcd}$

   $N(R^S) \to N(R^T)$ *is a weak equivalence.*

3. *For any triangle as above and a simplicial* R*-algebra* X*,* f *induces a weak equivalence*

$$\mathrm{Map}_{\mathrm{Alg}_S}(\zeta, h^*(X)) \to \mathrm{Map}_{\mathrm{Alg}_T}(\zeta, g^*(X))$$

Proof. $1 \Rightarrow 2$: In the diagram $\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f"'] & & T \arrow[dl, "\mathrm{id}"] \\ & T \arrow[d, "g"] & \\ & R & \end{tikzcd}$

$N(T^S) \to N(T^T)$ is a weak equivalence because T^T has terminal object. The result follows from the Theorem 1.25.

$2 \Rightarrow 1$ follows by taking $R = T$ and $g = \mathrm{id}$ since $N(T^T)$ is contractible.

$2 \Leftrightarrow 3$: $\mathrm{Map}_{\mathrm{Alg}_S}(\zeta, h^*(X))$ can be computed as the simplicial hom

$$\underline{\mathrm{Alg}_S}(\mathrm{cof}(1), h^*(X)) \sim \underline{\mathrm{Alg}_R}(N(R^S), X)$$

by adjunction and $\mathbb{L}h_!(1) \sim N(R^S)$. Analogously

$$\underline{\mathrm{Alg}_T}(\mathrm{cof}(1), g^*(X)) \sim \underline{\mathrm{Alg}_R}(N(R^T), X)$$

. So the induced map between mapping spaces for any X is a weak equivalence iff $N(R^S) \to N(R^T)$ is a weak equivalence. $\square$

Using this result, to prove a weak equivalence of mapping spaces, one can show contractibility of the nerve of the relative classifier, which is often more tractable.

Definition 1.27. A map $f : S \to T$ between polynomial monads is called *homotopy cofinal* if the equivalent conditions of the preceding Corollary are satisfied.

1.3.3 Formal delooping

For a polynomial monad P, let P_* be a monad for pointed P-algebras, so $\mathrm{Alg}_{P_*} \simeq \zeta/\mathrm{Alg}_P$. Similarly, let P_{**} be a monad fo double pointed P-algebras - $\mathrm{Alg}_{P_{**}} \simeq \zeta \sqcup \zeta/\mathrm{Alg}_P$.

There is a map of monads $u : P \to P_*$ whose restriction u^* "forgets the point". There is also a map $U : P_{**} \to P_*$ whose restriction U^* "doubles the point".

Overall this data give a pushout of monads over P_*:

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner"] & P_* \arrow[d] \\ P_* \arrow[r] & P_{**} \end{tikzcd}$$

To prove the following theorem about delooping by Batanin and De Leger [2019], one needs this square to induce a homotopy pushout of classifiers.

Definition 1.28 (Batanin and De Leger [2019]). A commutative square of polynomial monads below is called *homotopically cofinal* if for any map of polynomial monads $D \to R$ the square of nerves of classifiers induced by Proposition 1.23 is a homotopy pushout square.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "F"] \\ C \arrow[r, "G"] & D \end{tikzcd} \qquad \begin{tikzcd} N(R^A) \arrow[r, "N(R^f)"] \arrow[d, "N(R^g)"] & N(R^B) \arrow[d, "N(R^F)"] \\ N(R^C) \arrow[r, "N(R^G)"] & N(R^D) \end{tikzcd}$$

Theorem 1.29 (Batanin and De Leger [2019], Theorem 8.1). *Suppose P_* is a polynomial monad and the corresponding pushout square of monads over P_* is homotopically cofinal. For a simplicial P_*-algebra X, there is a weak equivalence of simplicial sets*

$$\Omega \mathrm{Map}_{\mathrm{Alg}_P}(\zeta, u^*(X)) \to \mathrm{Map}_{\mathrm{Alg}_{P_{**}}}(\zeta, U^*(X))$$

where $\Omega \mathrm{Map}_{\mathrm{Alg}_P}(\zeta, u^*(X))$ is the loop space with the base point given by the map $\zeta \to X$ from the P_*-algebra X.

Proof. By the assumption, we have a homotopy pushout of nerves of classifiers over P^*. Assume X is fibrant (otherwise take its fibrant replacement). We get a homotopy pullback of simplicial sets

$$\begin{tikzcd} \underline{\mathrm{Alg}_{P_*}}(N(P_*^{P_{**}}), X) \arrow[r] \arrow[d] & \underline{\mathrm{Alg}_{P_*}}(N(P_*^{P_*}), X) \arrow[d] \\ \underline{\mathrm{Alg}_{P_*}}(N(P_*^{P_*}), X) \arrow[r] & \underline{\mathrm{Alg}_{P_*}}(N(P_*^{P}), X) \end{tikzcd}$$

By adjunctions

$$\underline{\mathrm{Alg}_{P_*}}(N(P_*^{P_{**}}), X) \sim \mathrm{Map}_{P_{**}}(1, U^*(X))$$
$$\underline{\mathrm{Alg}_{P_*}}(N(P_*^{P}), X) \sim \mathrm{Map}_{P}(1, u^*(X))$$
$$\underline{\mathrm{Alg}_{P_*}}(N(P_*^{P_*}), X) \sim \mathrm{Map}_{P_*}(1, X) \sim 1$$

where the contractibility of $\mathrm{Map}_{P_*}(1, X)$ follows from the fact that the terminal P_*-algebra is also the initial object.

So this square is exactly the homotopy pullback square giving the loop space. □

1.3.4 Algebras with chosen points

Let P be a polynomial monad with the set of colors I and let $J \subseteq I$. Let Id_K be the identity polynomial on Set^K. The inclusion $J \to I$ yields a map of polynomial monads $i : \mathrm{Id}_J \to \mathrm{Id}_I$.

Let now Id_{+J} be a monad with a "chosen point" in each component, given by

$$J \leftarrow J \xrightarrow{p} J \times \{0, 1\} \to J$$

where $p(j) = (j, 1)$ and $(j, 0)$ is a nullary map for $j \in J$. Let now P_{+J} be the pushout of polynomial monads

$$\begin{tikzcd}
\mathrm{Id}_J \arrow[r, "i"] \arrow[d] & \mathrm{Id}_I \arrow[r, "\eta"] & P \arrow[d] \\
\mathrm{Id}_{+J} \arrow[rr] & & P_{+J} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where η is the unit of P. Its algebras are the algebras of P along with a chosen point for each $j \in J$. We will call them *J-pointed P-algebras.*

The following formal fibration sequence theorem is a slight generalization of a Theorem 8.3 from Batanin and De Leger [2019].

Theorem 1.30. *Let X be a J-pointed simplicial P-algebra. If the square above is homotopy cartesian, there is a fibration sequence*

$$\mathrm{Map}_{\mathrm{Alg}_{P_{+J}}}(\zeta, X) \to \mathrm{Map}_{\mathrm{Alg}_P}(\zeta, X) \to \prod_{j \in J} \mathrm{fib}(X_j)$$

where $\mathrm{fib}(X_j)$ *is a fibrant replacement of X_j and we skip the notation of a forgetful functor from J-pointed P-algebras to P-algebras.*

Proof. By the assumption, we have a homotopy pushout of nerves of classifiers over P_{+J}. Assume X is fibrant (otherwise take its fibrant replacement); since we work with projective model structure on P-algebras, X_i is a fibrant simplicial set for $i \in I$. We get a homotopy pullback of simplicial sets

$$\begin{tikzcd}
\underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{P_{+J}}), X) \arrow[r] \arrow[d] & \underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{P}), X) \arrow[d] \\
\underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{\mathrm{Id}_{+J}}), X) \arrow[r] & \underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{\mathrm{Id}_J}), X)
\end{tikzcd}$$

By adjunction

$$\underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{\mathrm{Id}_J}), X) \sim \underline{\mathrm{Alg}_{\mathrm{Id}_J}}(N(\mathrm{Id}_J^{\mathrm{Id}_J}), i^*\eta^* X) \sim \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Id}_J}}(\zeta, i^*\eta^* X) \sim \prod_{j \in J} X_j$$

since $\mathrm{Alg}_{\mathrm{Id}_J}$ are just collections indexed by J. Similarly

$$\underline{\mathrm{Alg}_{P_{+J}}}(N(P_{+J}^{\mathrm{Id}_{+J}}), X) \sim \underline{\mathrm{Alg}_{\mathrm{Id}_{+J}}}(N(\mathrm{Id}_{+J}^{\mathrm{Id}_{+J}}), \gamma^* X)$$

where $\gamma : \mathrm{Id}_{+J} \to P_{+J}$. By the description given by the Theorem 1.21, the classifier $\mathrm{Id}_{+J}^{\mathrm{Id}_{+J}}$ is in each component a category with two objects 0, 1 and an arrow $0 \to 1$. Its nerve is thus a pointed simplicial interval and the simplicial hom above is in each component the space of path from the chosen point, which is contractible.

Now by adjunction, the space in the upper right corner of the homotopy pullback square is weakly equivalent to $\mathrm{Map}_{\mathrm{Alg}_P}(\zeta, X)$. □

A direct consequence is the following.

Corollary 1.31. *Let X be a J-pointed P-algebra satisfying the assumption of the preceding theorem, such that X_j is contractible for each $j \in J$. Then*

$$\mathrm{Map}_{\mathrm{Alg}_{P_{+J}}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Alg}_P}(\zeta, X)$$

2. Higher dimensional bimodules

To generalise the results from Batanin and De Leger [2019] to n-th iteration of the $+$-construction, we need a notion of k-dimensional bimodules for $0 \leq k \leq n$. Bimodules, resp. infinitesimal bimodules for non-symmetric operads are going to be special cases for $n = 2$, $k = 1$, resp. $k = 0$.

For what follows, let $\mathcal{I}^{+n}$ be given by the polynomial

$$I_n \xleftarrow{s_n} B_n^* \xrightarrow{p_n} B_n \xrightarrow{t_n} I_n$$

Definition 2.1. Let ζ_n be the $\mathcal{I}^{+n}$-algebra $\zeta_{\mathcal{I}^{+n}}$.

2.1 k-dimensional sets of vertices

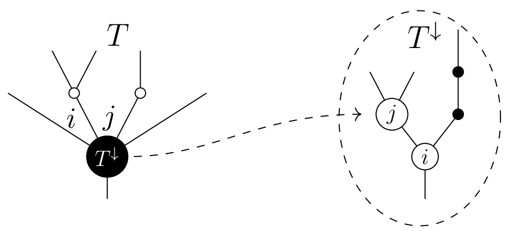
Recall that $T \in B_n$ is a tree, whose vertices are decorated by B_{n-1} and edges by I_{n-1} in a compatible way. Recall also that its vertices are in a 1-to-1 correspondence with the elements of $p_n^{-1}(T)$. For a subset of its vertices, we will associate a tree in B_{n-1} and a corresponding subset of vertices of this tree.

Construction 2.2 ($\downarrow$-construction)**.** For a $T \in B_n$ and $V \subseteq p_n^{-1}(T)$, construct $T^\downarrow \in B_{n-1}$ and $V^\downarrow \subseteq p_{n-1}^{-1}(T^\downarrow)$ in the following way:

Assume all the vertices of T not in V are contracted (otherwise use the monadic composition to contract them), as well as that the root is not in V (otherwise add a unit as the new root). Let $T^\downarrow$ be the decoration of the root vertex. By the $+$-construction, there is a bijection between the vertices of $T^\downarrow$ and the incoming edges to the root in T.

Let $V^\downarrow$ be the set of vertices of $T^\downarrow$ with the corresponding edges connected to the vertices of V. We will say V *covers* $V^\downarrow$.

Note that if V doesn't contain any pair of vertices lying on a path, the $\downarrow$-construction gives a bijection between V and $V^\downarrow$.



Definition 2.3. For $T \in B_n$, $V \subseteq p_n^{-1}(T)$ and $0 \leq k \leq n$, we will define the property of V being k-dimensional by induction on $n - k$:

1. V is n-dimensional if $V = p_n^{-1}(T)$, i.e. if it contains every vertex of T.

2. V is k dimensional for $k < n$ if it does not contain any pair of vertices lying on a path and $V^\downarrow \subseteq p_{n-1}^{-1}(T^\downarrow)$ given by the Construction 2.2 is k dimensional in $T^\downarrow \in B_{n-1}$.

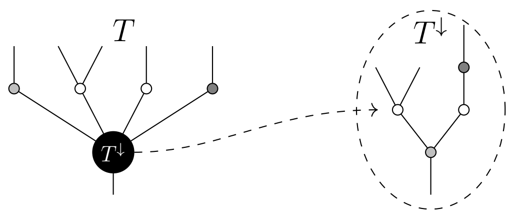
So V is k-dimensional if by applying the Construction 2.2 $(n - k)$-times, we get a bijective correspondence between V and all vertices in a tree in B_k.

Remark. If we consider $T \in B_n$ as a constellation (via the Theorem 1.15), the $\downarrow$-construction amounts to contracting white circles to white vertices and erasing other circles.

Remark. Let V be an $(n-1)$-dimensional subset of $T \in B_n$. Since there are no vertices of V lying above each other and V covers every vertex of the tree which decorates the root vertex, there must be precisely one white vertex lying on each path from root to leaf in T.

Definition 2.4. Let $T \in B_n$, $V \subseteq p_n^{-1}(T)$ be k-dimensional and $U \subseteq V$ be $(k-1)$-dimensional for $0 < k \leq n$. That means vertices of V are in a bijective correspondence with all vertices of a tree $\tilde{T} \in B_k$. Moreover, the vertices of U correspond to a $(k-1)$-dimensional set of vertices $\tilde{U}$ of $\tilde{T}$. By the preceding remark, every path from root to leaf in $\tilde{T}$ crosses a vertex in $\tilde{U}$.

We will call vertices *below* U (or U_-), resp. vertices *above* U (or U_+) the subset of vertices of V such that the corresponding vertices of $\tilde{T}$ are all the ones lying below, resp. above $\tilde{U}$.



Remark. Since 0-dimensional sets of vertices have to be in a bijective correspondence with vertices in a tree in B_0 and there is precisely one tree with one vertex, they are precisely the singleton sets of vertices.

Remark. The singleton set containing the root vertex is k-dimensional for every $0 \leq k \leq n$, since the iterations of the $\downarrow$-construction always yield trees with only one vertex.

2.2 Polynomial monads for (k, n)-bimodules

Recall that an $\mathcal{I}^{+n}$-algebra A is given by a collection of A_i for $i \in I_n$, along with, for each $T \in B_n$, a multiplication morphism

$$\bigotimes_{i \in p_n^{-1}(T)} A_i \to A_{t_n(T)} \tag{2.1}$$

satisfying the associativity and unit conditions.

We are now going to define k-dimensional ζ_n-bimodules as algebras of certain polynomial monad. We can think of the multiplication for their algebras as the preceding one, but where everything outside a k-dimensional set of vertices is replaced by a unit (represeting the bimodule action of ζ_n, which is unit in every component). That means, for each $T \in B_n$ and a k-dimensional set of vertices $V \subseteq p_n^{-1}(T)$, there will be a multiplication morphism

$$\bigotimes_{i \in V} A_i \to A_{t_n(T)} \tag{2.2}$$

Definition 2.5. Let $\mathbf{Bimod}^{k,n}$ be the polynomial monad given by the polynomial

$$I_n \leftarrow B^*_{k,n} \to B_{k,n} \to I_n$$

where $B_{k,n}$ is the set of trees of B_n with vertices colored by two colors: white (representing the bimodule) and black (representing the action of ζ_n), with the set of white vertices k-dimensional, factored by the equivalence contracting the black vertices and adding or removing black units (the operations of ζ_n can always be uniquely contracted). We can always take a canonical representative of an equivalence class, where the black vertices are maximally contracted and the units removed.

$B^*_{k,n}$ is the same, but with one white vertex marked and the middle map forgets the marking. The source and the target maps are the same as for $\mathcal{I}^{+n}$. The monadic composition is given by inserting the given tree into a white vertex (and contracting all black vertices for the canonical representative).

Definition 2.6. The algebras of $\mathbf{Bimod}^{k,n}$ will be called (k, n)-bimodules, or just k-dimensional bimodules where n is clear, and denoted $\mathrm{Bimod}^{k,n}$.

It is straightforwrad to see that the algebras of k-dimensional bimodules have the multiplication morphisms described above.

Remark. $\mathbf{Bimod}^{n,n} \simeq \mathcal{I}^{+n}$, since in $B_{n,n}$, the set of white vertices is n-dimensional, meaning every vertex is white and they are just the trees in B_n.

Example. For $n = 2$, the 1-dimensional bimodules are usually called simply ζ-bimodules for a non-symmetric operads and 0-dimensional weak or infinitesimal ζ-bimodules [Batanin and De Leger, 2019]. The notion given here is a direct generalisation of these.

Definition 2.7. We define $\zeta_{k,n} = \zeta_{\mathbf{Bimod}^{k,n}}$.

2.3 0-dimensional bimodules as categories

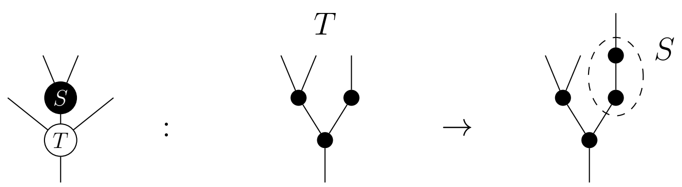
Since the operations of 0-dimensional bimodules $B_{0,n}$ are trees where the set of white vertices is 0-dimensional, there is precisely one white vertex. So all the operations are unary, meaning these polynomial monads are actually categories.

Definition 2.8. Let $\Omega_{\mathbb{R}^n}$ be the category given by the linear monad $\mathbf{Bimod}^{0,n+1}$.

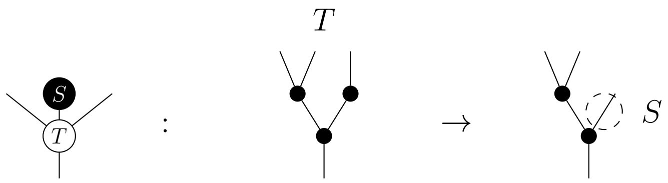
That means for the category of $(0, n+1)$-bimodules in the category $\mathcal{E}$,

$$\mathrm{Bimod}^{0,n+1}(\mathcal{E}) \simeq [\Omega_{\mathbb{R}^n}, \mathcal{E}]$$

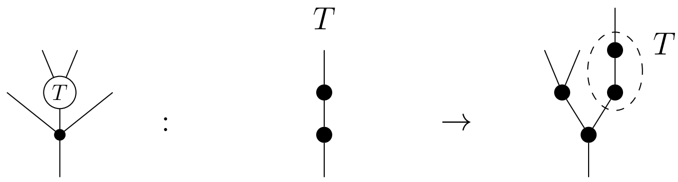
The objects of the category $\Omega_{\mathbb{R}^n}$ are the trees $I_{n+1} = B_n$. From the description of the polynomial monad, we can see the morphisms are generated by the *active* part, corresponding to black vertices above the white vertex, i.e. the blowing up of vertices (opposite of contraction)



or deletion of a unary vertex



and the *inert* part, corresponding to the white vertex above a black vertex, i.e. the inclusion of a subtree.



Proposition 2.9. *The subcategory of active morphisms of $\Omega_{\mathbb{R}^n}$ has a different connected component for each target $S \in I_{n-1}$, which is isomorphic to the opposite of the component of the absolute classifier $(\mathcal{I}^{+n})^{\mathcal{I}^{+n}}$ indexed by S.*

Proof. The active part doesn't chang the target. The rest follows from the description of the classifiers in the Theorem 1.19. □

Example. $\Omega_{\mathbb{R}^0}$ is the terminal category.

Example. $\Omega_{\mathbb{R}^1}$ is isomorphic to the simplicial category Δ. Its objects can be considered as linear trees.

Example. $\Omega_{\mathbb{R}^2}$ is isomorphic to the category Ω_p, the so called dendroidal category of planar trees [Moerdijk and Toën, 2010].

Proposition 2.10. *The opetope suspension maps $\Sigma_n : I_n \to I_{n+1}$ extend to fully faithful functors $\Omega_{\mathbb{R}^{n-1}} \to \Omega_{\mathbb{R}^n}$.*

Proof. We know Σ_n is injective on objects. Since the morphisms are trees in $B_n = I_{n+1}$ with vertices of two colors, apply Σ_{n+1} to them, preserving the colors; as it is an injective map, we have an injection on morphisms.

Now consider one of the generating morphisms given above in $\Omega_{\mathbb{R}^n}$ between objects which are suspensions of some objects of $\Omega_{\mathbb{R}^{n-1}}$. As every vertex of these trees is decorated by a suspension of an element of I_{n-2}, this morphism is a suspension of a morphism of $\Omega_{\mathbb{R}^n}$. □

Definition 2.11. Let $\Omega_{\mathbb{R}^\infty}$ be the colimit of the sequence of functors Σ_n for $n \in \mathbb{N}$.

The objects of $\Omega_{\mathbb{R}^\infty}$ are the stable opetopes I_∞. For two trees from $S \in I_m$ and $T \in I_n$ representing the elements of I_∞, $m < n$, we take suspensions of S enought times to get $S' \in I_n$. Now the set of morphisms between S and T in $\Omega_{\mathbb{R}^\infty}$ is in bijection with the set of morphisms between S' and T in $\Omega_{\mathbb{R}^n}$. In particular, the morphisms of $\Omega_{\mathbb{R}^\infty}$ have an active-inert factorization system.

Remark. Moerdijk and Weiss [2007] introduced the dendroidal category Ω, whose objects are abstract trees and the morphisms are generated by blowing vertices up, deletion of unary vertices and inclusion of subtrees. For every n, there is a functor $\Omega_{\mathbb{R}^n} \to \Omega$, which forgets the decorations of the trees. These maps form a cone over Ω and hence, we obtain a funcor $\Omega_{\mathbb{R}^\infty} \to \Omega$. We are going to study this functor in a future work.

3. Delooping

Recall that an algebra of $\mathbf{Bimod}_{*}^{k,n}$ is (k,n)-bimodule A along with a map $\zeta \to A$. Similarly, for an algebra of $\mathbf{Bimod}_{**}^{k,n}$, there are two such maps.

Explicitly, $\mathbf{Bimod}_{**}^{k,n}$ is given by

$$I_n \leftarrow (B_{k,n}^{*})_{**} \to (B_{k,n})_{**} \to I_n$$

where $(B_{k,n})_{**}$ is a set of trees of $B_{k,n}$, where further assign to some white vertices gray color of type 1 or 2 (representing the distinguished operations given by two maps from $\zeta_{k,n}$), factored by the equivalence retation generated by contractions of a tree with only black vertices without type and gray vertices of the same type (1 or 2), where the set of gray vertices is k-dimensional, to a gray corolla of this type (since the operations of $\zeta_{k,n}$ bimodules can always be uniquely contracted). $(B_{k,n}^{*})_{**}$ is the same with one white vertex marked.

$\mathbf{Bimod}_{*}^{k,n}$ is similar, but with gray vertices only of one type.

The category of algebras of $\mathbf{Bimod}_{**}^{k,n}$ will be denoted $\mathrm{Bimod}_{**}^{k,n}$.

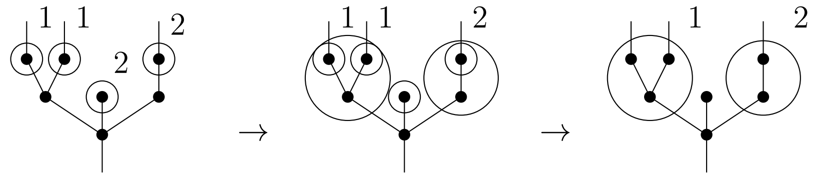
Construction 3.1. For $k < n$, to each equivalence class of a tree $T \in (B_{k,n})_{**}$, we will assign a canonical representative where the gray vertices are contracted as much as possible, except when in some subtree, the empty set of vertices is k-dimensional - these subtrees are equivalent to same ones with black vertices.

T can be pictured as a constellation, i.e. a tree in I_n with circles, where each circle is either black, white or gray of type 1 or 2. No white or gray circles can be inside each other, since they correspond to a k-dimensional set of vertices. Moreover, black circles without type inside black or gray circles can be erased (which corresponds to contraction of black vertices without type or to a tree where the root is gray).

We will show this tree is equivalent to the one where the gray circles are the maximal possible. If $k = n - 1$ and there are gray circles of the same type above each other, insert a black circle containing them both and contract them to one gray circle - the set of gray vertices there is $(n-1)$-dimensional. For $k < n - 1$, let R be the set of vertices of T that have above them vertices from at least two colors from white, gray 1 or gray 2 circles. Let c be a gray circle. By the opposite of contraction, we can add a black circle without type that contains all the vertices above c and all the vertices below, up to some vertex from R; if there is another branch going from those vertices, it shall be contained in the circle as well. Then it can be contracted to a gray circle of the same color as c.

Now, if the empty set of vertices inside a gray circle is k-dimensional, the subtree inside this circle decorates a vertex of T and this vertex is equivalent to a black one with the same decoration, so we can change the circe to a black one and erase it (contracting the black vertex of T).

Here is an illustration for the case $(n,k) = (3,1)$:



Remark. For a tree in $(B_{n,n})_{**}$, one can obtain the canonical representative by simply contracting all gray vertices of the same type and removing gray units, yielding a terminal object of the category above.

We want to use the Theorem 1.29. For that, we neet to show its assumption.

Lemma 3.2. *For all* $0 \le k \le n$*, the square*

$$\begin{array}{ccc} \mathbf{Bimod}^{k,n} & \longrightarrow & \mathbf{Bimod}^{k,n}_{*} \\ \downarrow & & \downarrow \\ \mathbf{Bimod}^{k,n}_{*} & \longrightarrow & \mathbf{Bimod}^{k,n}_{**} \end{array}$$

is homotopically cofinal.

Proof. Batanin and De Leger [2019] give a proof of this statement for non-symmetric operads, which is $\mathbf{Bimod}^{2,2}$. We follow their argument, replacing the planar trees with the trees in $B_{k,n}$. □

So by the general result from the Theorem 1.29, we have for $X \in \mathrm{Bimod}^{k,n}_{*}$ $(0 < k \le n)$ the following weak equivalence

$$\Omega\mathrm{Map}_{\mathrm{Bimod}^{k,n}}(\zeta, u^*(X)) \sim \mathrm{Map}_{\mathrm{Bimod}^{k,n}_{**}}(\zeta, U^*(X))$$

where u and U are defined as in the Theorem 1.29.

3.1 Cofinality

We will now show that there is a homotopically cofinal map to $\mathbf{Bimod}^{k,n}_{**}$ from the monad where in operations, white vertices are confined to some $(k-1)$-dimensional subset. This is the crucial step in going from k-dimensional subsets to $(k-1)$-dimensional subsets.

Definition 3.3. For $0 \le k < n$, let $\mathbf{Bimod}^{k,n}_{\odot}$ be the polynomial monad given by

$$I_n \leftarrow (B^*_{k,n})_{\odot} \to (B_{k,n})_{\odot} \to I_n$$

where $(B_{k,n})_{\odot}$ is a subset of $(B_{k+1,n})_{**}$ satisfying the following condition: if the $(k+1)$-dimensional set of white and gray vertices is non-empty, there is a k-dimensional subset such that all vertices below are gray of type 1 and all vertices above are gray of type 2.

The category of algebras of $\mathbf{Bimod}^{k,n}_{\odot}$ will be denoted $\mathrm{Bimod}^{k,n}_{\odot}$.

Our aim is to prove the following.

Theorem 3.4. *There map of polynomial monads given by inclusion of sets*

$$\mathbf{Bimod}^{k-1,n}_{\odot} \to \mathbf{Bimod}^{k,n}_{**}$$

is homotopically cofinal.

For the proof, we will closely follow what is done in Batanin and De Leger [2019]. We will prove a somehow more general statement.

Definition 3.5. Let $\mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet}$ be a polynomial monad for cospans of k-dimensional bimodules $A \to C \leftarrow B$. Explicitly, it is given by the polynomial

$$\{A,B,C\} \times I_n \leftarrow (B^*_{k,n})_{\bullet\to\bullet\leftarrow\bullet} \to (B_{k,n})_{\bullet\to\bullet\leftarrow\bullet} \to \{A,B,C\} \times I_n$$

where the elements of $(B_{k,n})_{\bullet\to\bullet\leftarrow\bullet}$ are the trees of $B_{k,n}$ equipped with a label in $\{A,B,C\}$ called *target label* and for each white vertex a label in $\{A,B,C\}$ called *source label* such that if the target label is A, resp. B, then each source label must be A, resp. B. $(B^*_{k,n})_{\bullet\to\bullet\leftarrow\bullet}$ is the same, but with one vertex marked. The middle map forgets the marking and source and target maps are given by the source and target maps of $\mathbf{Bimod}^{k,n}$, along with the corresponding source or target label.

We also have a version where the vertices with the label C are confined to some $(k-1)$-dimensional subset, below which are only vertices of B and above which are only vertices of A.

Definition 3.6. Let $\mathbf{Bimod}^{k-1,n}_{\bullet+\bullet}$ be the polynomial monad with the same description as $\mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet}$, but with a restriction on the set of operations: there must be a $(k-1)$-dimensional subset of white vertices such that all white vertices below it have label B and all white vertices above it have label A.

We are going to show the following.

Theorem 3.7. *The map of polynomial monads given by inclusion of sets*

$$\mathbf{Bimod}^{k-1,n}_{\bullet+\bullet} \to \mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet}$$

is homotopically cofinal.

We have a commutative square of polynomial monads

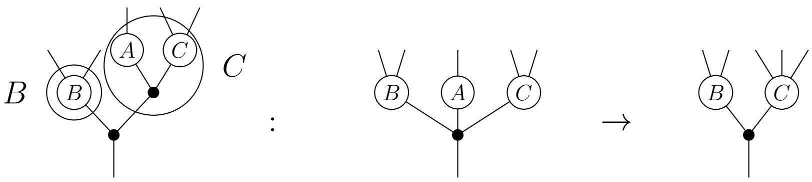
$$\begin{tikzcd} \mathbf{Bimod}^{k-1,n}_{\bullet+\bullet} \arrow[r] \arrow[d] & \mathbf{Bimod}^{k,n} \arrow[d, "id"] \\ \mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet} \arrow[r, "u"] & \mathbf{Bimod}^{k,n} \end{tikzcd}$$

where the horizontal maps are given by projections forgetting the labels. By the Proposition 1.23, this induces a map of classifiers

$$F : (\mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet})^{\mathbf{Bimod}^{k-1,n}_{\bullet+\bullet}} \to (\mathbf{Bimod}^{k,n})^{\mathbf{Bimod}^{k,n}}$$

To simplify notation, we will write $\mathcal{X}$ and $\mathcal{Y}$ for the domain and codomain of F.

By the description in the Theorem 1.21, in each component of $\mathcal{X}$, the objects are the operations of $\mathbf{Bimod}^{k,n}_{\bullet\to\bullet\leftarrow\bullet}$ and morphisms correspond to nested trees with two levels of nesting, with circles labeled by A, B or C in a compatible way. The source of the morphism is given by removing the circles and contracing black vertices and the target by contracting the circles.



In particular, there are morphisms with turn vertices with the label A or B to a vertex with the label C. We will need them in the proof and that is why we need to assume that in the $(k-1)$-dimensional set in the definition of $\mathbf{Bimod}^{k,n}_{\bullet+\bullet}$ there can be also vertices with labels A or B. We will then get cofinality $\mathbf{Bimod}^{k,n}_{\odot}$ and not for $\mathbf{Bimod}^{k,n}$ - there would be no such morphisms.

$\mathcal{Y}$ is similar, but without the labels. F forgets the labels. It is an absolute classifier, so it has a terminal object and its nerve is contractible. To show that the nerve of $\mathcal{X}$ is contractible (meaning the corresponding map is homotopy cofinal), we will prove that F induces a weak equivalence of nerves.

By the Quillen theorem A for small categories [Quillen, 1973], this is the case if for any $y \in \mathcal{Y}$, the comma category y/F is contractible.

Definition 3.8. For a functor $F : C \to D$ and $y \in D$, we will denote by by F_y the fiber of F over y (the full subcategory of objects $x \in C$ such that $F(x) = y$).

A functor $f : C \to D$ is *smooth* if for all $y \in D$, the canonical functor $F_y \to y/F$ is a weak equivalence between nerves.

For every $y \in \mathcal{Y}$, the fiber F_y is contractible, since it has a terminal object - y with every white vertex with the same label as the target label. So it is enought to prove that F is smooth. For that, we will use the Cisinski lemma.

Lemma 3.9 (Cisinski [2006], Proposition 5.3.4)**.** *A functor $F : C \to D$ is smooth if and only if for all maps $f : y_0 \to y_1$ in D and objects x_1 in C such that $F(x_1) = y_1$, the lifting category of f over x_1 $C(x_1, f)$, whose objects are arrows $g : x \to x_1$ with $F(g) = f$ and the morphisms are commutative triangles*

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "g"'] & & x' \arrow[dl, "g'"] \\ & x_1 & \end{tikzcd}$$

with the horizontal arrow from F_{y_0}, has contractible nerve.

Proof of the Theorem 3.7. Let $f : y_0 \to y_1$ be a map in (a component of) $\mathcal{Y}$, x_1 an object of $\mathcal{X}$ such that $F(x_1) = y_1$ and consider the lifting category $\mathcal{X}(x_1, f)$. It is isomorphic to the product of the lifting categories $\mathcal{X}(x_1^v, f^v)$, where v ranges over white vertices of y_1, x_1^v is the corolla around the vertex v with the color of v in x_1 and $f^v : y_0^v \to y_1^v$ is the contraction to the corolla y_1^v around v from its preimage under f.

So to prove that every lifting category is contractible, it suffices to show it for the case when y_1 is a corolla. If its label is A or B, the label in the sources must be the same and the lifting category is the terminal category. Now assume the label is C. The morphisms in the lifting category just change the labels of white vertices of y_0 - that is why we need to have this kind of morphism.

By the $\downarrow$-construction 2.2, the white vertices of y_0 correspond to all vertices of a tree $y_0' \in B_k$. The labels must be such that there is $(k-1)$-dimensional set with all vertices below having label B and all vertices above label A - otherwise, we wouldn't be able to contract it to a corolla by the operations from $\mathbf{Bimod}^{k,n}_{\bullet+\bullet}$.

Proceed by induction on number of white vertices of y'_0. If there is none, $\mathcal{X}(x_1, f)$ is the terminal category, thus contractible. Now for a general y_0, let $\mathcal{B}$ be the full subcategory of trees where the root of y'_0 has label B and $\mathcal{A}$ the full subcategory where everything above the root has label A. The union of $\mathcal{A}$ and $\mathcal{B}$ is the full subcategory, the intersection is the terminal category and $\mathcal{A}$ is a cospan. $\mathcal{B}$ is isomorphic to the product of analogous categories for the subtrees above the root of y'_0, which are contractible by induction. So $\mathcal{X}(x_1, f)$ is contractible as well. □

Proof of the Theorem 3.4. Let $\mathcal{Z}$ be the classifier for the map in the Theorem 3.4 and $\mathcal{X}_C$ be the subcategory of $\mathcal{X}$ for objects with the target label C. There is a map $E : \mathcal{X}_C \to \mathcal{Z}$ that factors the trees by the defining equivalence of $\mathbf{Bimod}^{k,n}_{**}$.

We are again going to use the Cisinski lemma 3.9 to prove that E is smooth. Consider a map $f : y_0 \to y_1$ in (a component of) $\mathcal{Z}$ and $x_1 \in \mathcal{X}_C$ such that $E(x_1) = y_1$. So y_1 is obtained from x_1 by bimodule contratcion of vertices with labels A or B. f can be represented as a nested tree T; blow up everything in T outside the circles in the opposite way as x_1 is contracted to y_1 to obtain a tree T'. Now the objects of $\mathcal{X}_C(x_1, f)$ correspond to nested trees, same as T outside the circles, where there is a zigzag of contractions between subtrees in their circles and the subtrees in the circles of T'. The morphisms are such contractions. So $\mathcal{X}_C(x_1, f)$ is the product of fibers E_S for subtrees S in the circles in T.

Let E_T be a fiber of T, E'_T be E_T factored by those generating equivalences of $(B_{k,n})_{**}$ where the k-dimensional set in nonempty and $\pi_T : E_T \to E'_T$ the projection onto it. By the Construction 3.1, the fibers of π_T have terminal objects, so they are contractible. Let now $g : y_0 \to y_1$ be a morphism in E'_T turning a subtree Q with the empty set of vertices k-dimensional to a gray subtree and x_1 an object of E_T such that $\pi_T(x_1) = y_1$. The lifting category $E_T(x_1, g)$ has again a terminal object, where the source of the lift of g is x_1 with the subtree corresponding to Q turned black. So π_T is smooth and induces a weak equivalence of nerves.

But E'_T has an initial object (the class where there are no gray subtrees with the empty set of vertices k-dimensional). In conclusion, E is smooth and by the same reasoning as before, E induces a weak equivalence of nerves. □

3.2 First two deloopings

Definition 3.10. For $b \in B_n$, we mean by its arity the cardinality of the fiber $p_n^{-1}(b)$. By the +-constuction, this is equivalent to the number of vertices when T is considered as an $\mathcal{I}^{+n}$-tree.

Specifically, $i \in I_n = B_{n-1}$ is unary if $|p_{n-1}^{-1}(i)| = 1$, i.e. as an $\mathcal{I}^{+n}$ tree, i is a corolla with single vertex. Similarly, i is nullary if $|p_{n-1}^{-1}(i)| = 0$, i.e. as an $\mathcal{I}^{+n}$-tree, i is a free living edge without vertices.

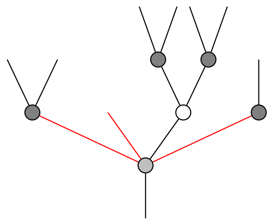
Proposition 3.11. *The category* $\mathrm{Bimod}_{\odot}^{n-1,n}$ *is isomorphic to the category of* $(n-1, n)$*-bimodules with a chosen point in each* $i \in I_n = B_{n-1}$ *unary.*

Proof. By the +-construction, whe have a free living edge in B_n decorated by the target of i whose target is i. The empty set of vertices there is n-dimensional, so it is also an operation of $\mathbf{Bimod}_{\odot}^{n-1,n}$ - in pictures, we will color it red. This represents the chosen point in the component i.

Now, the operations of $\mathbf{Bimod}_{\odot}^{n-1,n}$ consist of trees of white vertices and gray vertices of type 1 and 2, such that there is a $(n-1)$-dimensional set with all vertices below it gray of type 1 and all vertices above it gray of type 2.

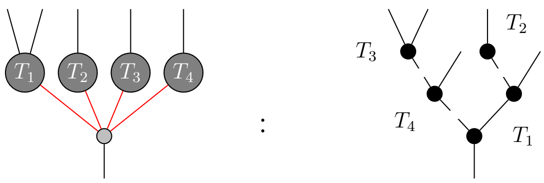
Let T be an operation of $\mathbf{Bimod}_{\odot}^{n-1,n}$. The preceding means every path from root to leaf either crosses a white vertex, or there is an edge where gray vertices change from type 1 to type 2 (or the whole path is gray of the same type - in that case, choose the edge above the leaf for type 1 and below root for type 2). Consider the tree $\tilde{T}$, which is T with white vertices on these edges - its set of white vertices is thus $(n-1)$-dimensional, so it is an operation of $\mathbf{Bimod}^{n-1,n}$. Now substitute the red edges into those vertices.

We see the set of operations is the same as when we add the free living edges representing the chosen points in the unary components (which we color red).



$\square$

By the description above, the constant operations of $\mathbf{Bimod}_{\odot}^{n-1,n}$ (where no vertex is white) correspond to decompositions of the target tree into subtrees (given by the decorations of gray vertices of type 2).



Let $\mathcal{F}_{k,n}$ be the *free algebra functor* - left adjoint to the forgetful functor to collections $\mathcal{U}_{k,n} : \mathrm{Bimod}^{k,n} \to \mathrm{SSet}^{I_n}$. Let e_1 be the collection indexed by I_n, which is 1 in every component indexed by a unary tree and $\emptyset$ elsewhere. Let $\sigma_n = \mathcal{F}_{n-1,n}(e_1)$. The preceding proposition says having an algebra of $\mathrm{Bimod}_{\odot}^{n-1,n}$ is the same an algebra X of $\mathrm{Bimod}^{n-1,n}$, along with a map of collections $e_1 \to \mathcal{U}(X)$. By adjunction, this means

$$\mathrm{Bimod}_{\odot}^{n-1,n} \simeq \sigma_n/\mathrm{Bimod}^{n-1,n}$$

σ_n is, by the previous proposition, in component T the set of decomposition of T into subtrees, with the natural $(n-1)$-dimensional bimodule composition.

Theorem 3.12. *For $n \geq 2$ and $X \in \mathrm{Alg}_{\mathcal{I}_*^{+n}}$, if X_i is contractible for each $i \in I_n = B_{n-1}$ unary, we have a delooping*

$$\Omega \mathrm{Map}_{\mathrm{Bimod}^{n,n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}^{n-1,n}}(\zeta, X)$$

where we skip writing down the forgetful functors.

Proof. By the Theorems 1.29 and 3.4, $\Omega\mathrm{Map}_{\mathrm{Alg}_{\mathcal{I}+n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}_{\odot}^{n-1,n}}(\zeta, X)$ where the forgetful functors are skipped. In order for the Corollary 1.31 to apply, the square in its assumption needs to be homotopically cofinal. Batanin and De Leger [2019] give a proof for the case $\mathbf{Bimod}^{1,2}$. We follow the same argument in our case. The result follows from the Proposition 3.11 and Corollary 1.31. □

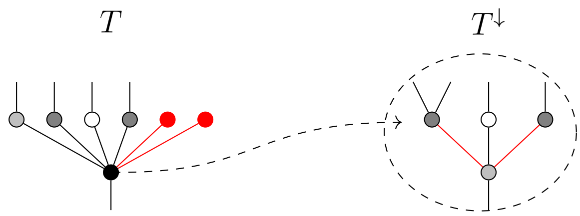
Proposition 3.13. *The category* $\mathrm{Bimod}_{\odot}^{n-2,n}$ *is isomorphic to the category of* $(n-2,n)$*-bimodules with a chosen point in each* $i \in I_n = B_{n-1}$ *nullary (each free living edge).*

Proof. Consider a trunk (a tree with no leaves) in B_n with a black vertex. Its target is a free living edge, i.e. some i nullary. The empty set of its vertices is $(n-1)$-dimensional, so it is an operation of $\mathbf{Bimod}_{\odot}^{n-2,n}$ - in pictures, we will color it red. It represents the chosen point in the component i.

The empty set of vertices being $(n-1)$-dimensional also means this trunk is equivalent to a trunk with a gray vertex of type 1 or 2 (see the Construction 3.1).

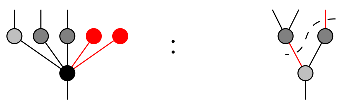
Now for a tree $T \in B_n$, we are in a similar situation as in the previous proposition, but for a tree $T^{\downarrow} \in B_{n-1}$ obtained by the $\downarrow$-construction 2.2, which also assigns its vertices white or gray colors. It means every path in $T^{\downarrow}$ from root to leaf either crosses a white vertex, or there is an edge where gray vertices change from type 1 to type 2 (or the whole path is gray of the same type - we deal with it the same way as before). Now consider a tree $\tilde{T}^{\downarrow} \in B_{n-1}$, which is $T^{\downarrow}$ with white vertices inserted inside those edges. T is equivalent a tree in B_n with the root decorated by $\tilde{T}^{\downarrow}$, the edges corresponding to the added white vertices connected to red trunks and the edges corresponding to the other vertices connected to the same vertices as in T.

This shows $\mathbf{Bimod}_{\odot}^{n-2,n}$ has the same operations as when we add trunks representing chosen points in each nullary component, which replace units in $T^{\downarrow}$ by red free living edges.



□

By the description above, the constant operations of $\mathbf{Bimod}_{\odot}^{n-2,n}$ (where no vertex is white) correspond to cuts of the target tree on every path from root edge to a leaf edge.



Let e_0 be the collection indexed by I_n, which is 1 in every component indexed by a nullary tree and $\emptyset$ elsewhere. Let $\tau_n = \mathcal{F}_{n-2,n}(e_0)$ - the algebra freely generated by 1 in nullary components. In component T, it is the set of cuts of T as above with the natural $(n-1)$-dimensional bimodule composition. By adjunction as before, the preceding proposition says

$$\mathrm{Bimod}_{\odot}^{n-2,n} \simeq \tau_n/\mathrm{Bimod}^{n-2,n}$$

Theorem 3.14. *For $n \geq 2$ and $X \in \mathrm{Bimod}_{*}^{n-1,n}$, if X_i is contractible for each $i \in I_n = B_{n-1}$ nullary, we have a delooping*

$$\Omega\mathrm{Map}_{\mathrm{Bimod}^{n-1,n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}^{n-2,n}}(\zeta, X)$$

where we skip writing down the forgetful functors.

Proof. By the same virtue as in the proof of the Theorem 3.12, we can use the Corollary 1.31, along with the Proposition 3.13. □

Using the Theorems 3.12 and 3.14 together, for X multiplicative (n, n)-bimodule, if X_i is contractible for every i unary and nullary, we have a double delooping

$$\Omega^2\mathrm{Map}_{\mathrm{Bimod}^{n,n}}(\zeta, X) \sim \mathrm{Map}_{\mathrm{Bimod}^{n-2,n}}(\zeta, X)$$

Corollary 3.15 (Turchin [2014], Dwyer and Hess [2012])**.** *For a multiplicative non-symmetric operad X such that X_1 and X_0 are contractible,*

$$\Omega^2\mathrm{Map}_{\mathrm{NOp}}(\zeta, X) \sim \mathrm{holim}_{\Delta}X$$

where we skip writing down the forgetful functors.

Proof. Apply the Theorems 3.12 and 3.14 for $n = 2$, since the $(2, 2)$-bimodules are non-symmetric operads, $\mathbf{Bimod}^{0,2}$ is the small category $\Omega_{\mathbb{R}^1} = \Delta$ and

$$\mathrm{Map}_{\mathrm{Alg}_{\Delta}}(\zeta, X) \sim \mathrm{holim}_{\Delta}X$$

□

3.3 Third delooping

For $n \geq 3$ and $k < n-2$, we no longer get a characterisation of $\mathrm{Bimod}_{\odot}^{k,n}$ in terms of bimodules with chosen points in some components.

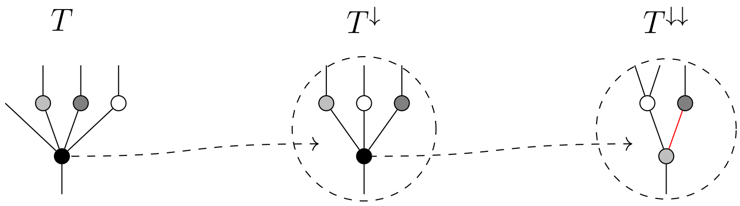
Proposition 3.16. *An operation of $\mathbf{Bimod}_{\odot}^{n-3,n}$ can be characterised as a tree T from B_n with black and white vertices, where black vertices can be assumed to be contracted, allong with the following data:*

Applying the $\downarrow$-construction 2 times to get a tree $T^{\downarrow\downarrow} \in B_{n-2}$, we never get two white vertices lying on the path and there is a distinguished set of edges E of $T^{\downarrow\downarrow}$ such that if white units were added on these edges, the set of white vertices would become $(n-3)$-dimensional.

Proof. If T is an operation of $\mathbf{Bimod}_{\odot}^{n-3,n}$, we are in a similar situation in $T^{\downarrow\downarrow}$ as before: E is the set of edges that connect gray vertices of different types, or root edge if the root is gray of type 2, or leaf edge if the leaf is of type 1.

In the opposite direction, having a choice of edges E in the tree $T^{\downarrow\downarrow}$, color the vertices below by gray of type 1 and the above by type 2. Let $\tilde{T}^{\downarrow}$ be $T^{\downarrow}$ with black vertices contracted. $\downarrow$-construction gives a correspondence of edges above the root of $\tilde{T}^{\downarrow}$ and vertices in $T^{\downarrow\downarrow}$; insert units on the edges corresponding to gray vertices and expand the tree back the same way it was contracted from $T^{\downarrow}$. Add vertices to T which replace those units with free living edges and color them accordingly, yielding a tree $T' \in (B_{n-3,n})_{\odot}$.

Construction 3.1 ensures that every T with the choice of edges E is equivalent to the maximal representative of its equivalence class.



□

By the description above, the constant operations of $\mathbf{Bimod}_{\odot}^{n-3,n}$ (where no vertex is white) correspond to cuts of the target of the target tree on every path from root edge to a leaf edge. So it is a similar situation as before, but one level below.

Construction 3.17. There is a map of polynomial monads $\delta_{k,n} : \mathbf{Bimod}^{k,n} \to \mathbf{Bimod}^{k,n-1}$ for $k < n$ given by the diagram

$$\begin{tikzcd}
I_n \arrow[d, "t_{n-1}"] & B^*_{k,n} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{k,n} \arrow[r] \arrow[d, "(-)^{\downarrow}"] & I_n \arrow[d, "t_{n-1}"] \\
I_{n-1} & B^*_{k,n-1} \arrow[l] \arrow[r] & B_{k,n-1} \arrow[r] & I_{n-1}
\end{tikzcd}$$

where t_{n-1} is the target map and the map $(-)^{\downarrow}$ is given by Construction 2.2 from the definition of the k-dimensional set of vertices; by definition, the set of white vertices in the image is k-dimensional, so the map is well defined.

To avoid clutter, we will omit the indices and just write δ.

So the algebra $\delta^*\tau_{n-1}$ is in the component T the set of cuts of $t_{n-1}(T)$ and the Proposition 3.16 says

$$\mathrm{Bimod}_{\odot}^{n-2,n} \simeq \delta^*\tau_{n-1}/\mathrm{Bimod}^{n-2,n}$$

According to a general theorem of De Leger [2022][Theorem 4.13], for $X \in \delta^*\tau/\mathrm{Bimod}^{n-2,n}$, there is a fibration sequence

$$\mathrm{Map}_{\delta^*\tau/\mathrm{Bimod}^{n-2,n}}(1, X) \to \mathrm{Map}_{\mathrm{Bimod}^{n-2,n}}(1, X) \to \mathrm{Map}_{\mathrm{Bimod}^{n-2,n}}(\delta^*\tau, X)$$

where we skip writing down the forgetful functors. Note we don't have to assume $\delta^*\tau$ to be cofibrant, since according to Batanin and Berger [2017], algebras for the polynomial monads considered form a left proper model category, so thanks to Rezk [2002, Proposition 2.7], a weak equivalence $\alpha \to \beta$ induces a Quillen equivalence of the slice categories over them.

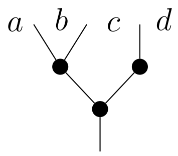
So contractibility of the last space of the fibration sequence would yield the third delooping.

Unfortunately, for further delooping, one cannot apply the construction from the preceding proposition, since we cannot replace units by free living edges for the tree $T^{\downarrow\downarrow}$. We can only modify the tree one level below, in limited ways.

3.4 The case $n = 3$

Recall that the category $\Omega_{\mathbb{R}^2} = \mathbf{Bimod}^{0,3}$ is the category Ω_p, i.e. the category, whose objects are planar trees and morphisms are generated by the active part, which is blowing up of vertices (the opposite of contraction) or deleting a unary vertex, and the inert part, which are inclusions of subtrees.

Denote by α the presheaf $\delta^*\tau_2 \in \mathrm{Bimod}^{0,3} = [\Omega_p, \mathrm{SSet}]$. By the Proposition 3.16, $\mathrm{Bimod}^{0,3}_{\odot}$ are the α-pointed presheaves $\alpha/[\Omega_p, \mathrm{SSet}]$. α assigns to a tree T a cut in its target, which for a linear tree is just a single edge. Because the leaves of T correspond to the vertices of its target, α can be pictured as assigning spaces between leaves, which is in bijection with 2-dimensional strata generated by emedding the planar tree in the plane.



$\overset{\alpha}{\mapsto} \{a, b, c, d\}$

Let $\int \alpha$ be the Grothendieck construction of α, i.e. the category, whose objects are pairs (T, t) with T an object of Ω_p and $t \in \alpha(T)$ and morphisms $(T, t) \to (S, s)$ given by each $f : T \to S$ in Ω_p such that $\alpha(f)(T) = S$.

According to De Leger [2022][Theorem 4.13], for $X \in \alpha/[\Omega_p, \mathrm{SSet}]$, as in the previous section, there is a fibration sequence

$$\mathrm{Map}_{\alpha/[\Omega_p,\mathrm{SSet}]}(1, X) \to \mathrm{Map}_{[\Omega_p,\mathrm{SSet}]}(1, X) \to \mathrm{Map}_{[\Omega_p,\mathrm{SSet}]}(\alpha, X)$$

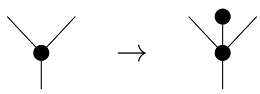
where we skip writing down the forgetful functors. By adjunction, the rightmost space is equivalent to

$$\mathrm{Map}_{[\int \alpha,\mathrm{SSet}]}(1, \pi^* X) \sim \mathrm{holim}_{\int \alpha} \pi^* X$$

where $\pi : \int \alpha \to \Omega_p$ is the projection given by the Grothendieck construction. So in order to have the third delooping, we need to have this space contractible.

3.4.1 Contractibility of homotopy limit over $\int \alpha$

Definition 3.18. Let $\mathfrak{b}$ be set of morphisms in Ω_p which blow up the trunk, i.e. maps in the active part of Ω_p which only add a nullary vertex.



Let X be a presheaf in $[\Omega_p, \mathrm{SSet}]$. Florian De Leger has proven that if image of every map in $\mathfrak{b}$ has a retraction, i.e. there is a map r such that $r \circ X(s) = \mathrm{id}$ for every $s \in \mathfrak{b}$, then $\mathrm{holim}_{\int \alpha} X \sim X_{⫯}$, where ⫯ denotes the component of the trunk - the tree with one vertex and no leaves. The proof constructs an explicit homotopy in this homotopy limit. It will be published in our joint paper containing the results of this thesis.

Here, we give just an argument why such result is plausible.

Proposition 3.19. *For $X \in [\Omega_p, \mathrm{SSet}]$, if every map which is the image of blowing up of a trunk has a retraction, every natural transformation $\alpha \Rightarrow X$ is uniquely determined by the map $* = \alpha_{\bullet} \to X_{\bullet}$.*

In other words, the natural (in X) morphism

$$\mathrm{Hom}_{[\Omega_p,\mathrm{SSet}]}(\alpha, X) \to \mathrm{Hom}_{\mathrm{SSet}}(1, X_{\bullet})$$

is an injection.

Proof. Label the 2-strata left to right by numbers from 0. So $\alpha_T = \mathbf{n}$, where n is the number of strata of T and $\mathbf{n}$ is the set $\{0, \dots, n-1\}$. Note the active part of Ω_p does not change the value of α. For trees whose number of strata is n, there is a unique active map from the corolla of $n-1$ inputs.

Now suppose we have a map $\mathbf{1} \to X_{\bullet}$. Consider a corolla C_n with $n-1$ inputs and C'_n having in addition a trunk in every strata. There are n maps from the trunk to it, which determine the value of $\alpha_{C'_n} = \mathbf{n} \to X_{C'_n}$. Since blowing up trunks has a retraction, this determines $\alpha_{C_n} = \mathbf{n} \to X_{C_n}$. The active map from C_n to any tree T with $n-1$ leaves determines $\alpha_T = \mathbf{n} \to X_T$.

□

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & \mathbf{n} \arrow[r, equal] \arrow[d] & \mathbf{n} \arrow[d] \\
X_{\bullet} \arrow[r] & X_{C'_n} & X_{C_n} \arrow[l]
\end{tikzcd}$$

3.4.2 Desymmetrizations of symmetric operads

Batanin and Berger [2017, Definition 9.4] introduce the monad $\mathcal{S}$, whose algebras are symmetric operads. It is given by the polynomial

$$\mathbb{N} \leftarrow \mathrm{Otr}^* \to \mathrm{Otr} \to \mathbb{N}$$

where Otr are isomorphism classes of ordered rooted trees [Batanin and Berger, 2017, Section 13.3]. Each such isomorphism class has a unique representative by a planar rooted tree with a linear order on its leaves.

Otr^* are ordered rooted trees with one vertex marked. The monadic composition is given by inserting the tree into the marked vertex, reordering the subtrees above the marked vertex so that they match the linear order of the leaves of the inserted tree.

We will denote by SOp the category $\mathrm{Alg}_{\mathcal{S}}$.

Construction 3.20 (Desymmetrization)**.** For every $n > 0$, there is map $\mathcal{I}^{+n} \to \mathcal{S}$, given by the diagram

$$\begin{tikzcd}
I_n \arrow[d, "|-|"] & B^*_n \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_n \arrow[r] \arrow[d, "o"] & I_n \arrow[d, "|-|"] \\
\mathbb{N} & \mathrm{Otr}^* \arrow[l] \arrow[r] & \mathrm{Otr} \arrow[r] & \mathbb{N}
\end{tikzcd}$$

where $|i|$ is the cardinality of the fiber $p_{n-1}^{-1}(i)$ for $i \in I_n = B_{n-1}$ (which by the +-constuction is the number of vertices of i if i is considered as an $\mathcal{I}^{+n-1}$-tree). The map o is given by choosing for each $i \in I_n$ a linear order on $p_{n-1}^{-1}(i)$, inducing a linear order on the incoming edges of each vertex of an I_n-tree $b \in B_n$. Moreover, since its leaves are in a bijective correspondence with the fiber of its target, this induces a linear order on them. So we get an ordered rooted tree.

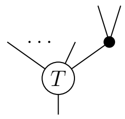
Denote by des_n the restriction functor $\mathrm{SOp} = \mathrm{Alg}_{\mathcal{S}} \to \mathrm{Alg}_{\mathcal{I}^{+n}}$ induced by this map, which we will call n-desymmetrization.

In particular, for a $X \in \mathrm{SOp}$ and $T \in I_n$, $\mathrm{des}_n(X)_T = X_{|T|}$.

Remark. des_2 is the classical desymmetrization of symmetric operads to non-symmetric operads.

Proposition 3.21. *Let O be a symmetric operad such that its 3-desymmetrization $\mathrm{des}_3(O)$ is multiplicative and O^* the presheaf in $[\Omega_p, \mathrm{SSet}]$ induced by $\mathrm{des}_3(O)$. Then the image of every morphism of $\mathfrak{b}$ under O^* has a retraction.*

Proof. Let $s : T \to T^{\bullet}$ be a map in $\mathfrak{b}$. As an operation of $\Omega_p = \mathbf{Bimod}^{0,3}$, it is given by a tree $b \in B_n$ with two colors of the form

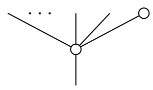


$O^*(s)$ is induced by the morphism of symmetric operads $o(b)$, where o is the map from the Construction 3.20.

$$O_{|T|} \otimes O_2 \to O_{|T|+1} = O_{|T^{\bullet}|}$$

There is also the following map for the symmetric operad O:

$$O_{|T^{\bullet}|} \otimes O_0 \to O_{|T|}$$



Precomposition with the map $1 \to O_0$ (given by the multiplicative structure on $\mathrm{des}_3(O)$) in the second component yields the sought retraction. $\square$

Since for the 2-desymmetrization of Kontsevich operad K [Kontsevich, 1999], we get the result [Sinha, 2006]

$$\overline{\mathrm{Emb}}(\mathbb{R}^1, \mathbb{R}^n) \sim \mathrm{holim}_{\Delta}(\mathrm{des}_2 K)^*$$

we expect its analogues in opetopic context to be also significant regarding the applications in geometry. Specifically, we conjecture that $\mathrm{des}_3 K$ has a multiplicative structure, so the preceding results apply to it. Its geometric meaning remains an open question.

Conclusion

We have defined $\mathrm{Bimod}^{k,n}$ for $n \in \mathbb{N}$, $0 \leq k \leq n$, which generalize bimodules, resp. infinitesimal bimodules for non-symmetric operads. We showed that in the cases $k = n-1$ or $k = n-2$, analogous delooping results hold as in the Turchin-Dwyer-Hess theorem. Already for $k = n-3$, the reduceness condition turned out to be more complex and we still don't have full understanding of what it means geometrically. For $k \leq n-4$, the conditions require further investigation.

The hope is that there is always an algebra $\alpha_{k,n} \in \mathrm{Bimod}^{k,n}$ such that

$$\mathrm{Bimod}_{\odot}^{k,n} \simeq \alpha_{k,n}/\mathrm{Bimod}^{k,n}$$

meaning the $\alpha_{k,n}$ is representing a sort of obstruction for the delooping to be possible. It can also be suspected that these algebras stabilize, meaning that $\alpha_{k,n} = (\delta^*)^{n-m}\alpha_{k,m}$, where δ is the map given by the Construction 3.17, however this is completely hypothetical at this stage. We have seen this behavior with $\alpha_{n-2,n} = \tau_n$ and $\alpha_{n-3,n} = \delta^*\alpha_{n-2,n-1}$. Unfortunately, the argument seems to break at $k = n-4$.

As there are non-symmetric operads coming from the study of configuration spaces, we expect to be able to construct algebras of $\mathcal{I}^{+n}$ that have significance to their geometry. We conjecture to have the result of triple delooping for the higher desymmetrization of the Kontsevich operad, but the question of its geometric meaning remains open.

The properties of the introduced categories $\Omega_{\mathbb{R}^n}$ for $n \in \mathbb{N} \cup \{\infty\}$ also open some paths for future research. On one hand, there is a relationship to the the dendroidal category Ω of Moerdijk and Weiss [2007] made explicit by the maps $\Omega_{\mathbb{R}^n} \to \Omega$ forgetting the labels. On the other hand, there are various notions of categories of opetopes and opetopic sets, introduced in Baez and Dolan [1998] and investigated for example in Cheng [2004]. In future, we would like to examine their relation to presheaves on $\Omega_{\mathbb{R}^n}$ (satisfying certain properties) and the potential of utilizing them as models of weak n-categories. In particular, as opetopes are certain trees, they are amenable for a straightforward computer representation.

The results of this thesis are going to be submitted to a journal in a joint article with Florian De Leger.

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