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Convergence in Banach spaces

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Abstract: The thesis consists of three articles. The common theme of the first two articles is the possibility of iterating weak* derived sets in dual Banach spaces. In the first article we prove that in the dual of any non-reflexive Banach space we can always find a convex set of order n for any $n \in \mathbb{N}$, and a convex set of order $\omega + 1$. This result extends Ostrovskii's characterization of reflexive spaces as those spaces for which weak* derived sets coincide with weak* closures for convex sets. In the second article we prove an iterated version of another result of Ostrovskii, that a dual to a Banach space X contains a subspace whose weak* derived set is proper and norm dense, if and only if X is non-quasi-reflexive and contains an infinite-dimensional subspace with separable dual. In the third article we study quantitative results concerning ξ-Banach-Saks sets and weak ξ-Banach-Saks sets. We provide quantitative analogues to characterizations of weak ξ-Banach-Saks sets using $\ell_1^{\xi+1}$ spreading models and a quantitative version of the relation of ξ-Banach-Saks sets, weak ξ-Banach-Saks sets, norm compactness and weak compactness. We use these results to define a new measure of weak non-compactness and finally give some relevant examples.

Keywords: Weak* derived sets, Banach-Saks property, ℓ_1-spreading model

Contents

Introduction

This thesis consists of a compilation of three papers by the author as well as this introductory section. Each paper constitutes one chapter:

1. Weak* derived sets of convex sets in duals of non-reflexive spaces, *J. Funct. Anal.* 281 (2021), no. 12, Paper No. 109259, 19 pp.;

2. On subspaces whose weak* derived sets are proper and norm dense, accepted in *Studia Mathematica*, *arXiv:2203.00288*;

3. Quantification of Banach-Saks properties of higher orders, submitted, *arXiv:2111.12773.*

The papers are presented in the original form, with the change that the list of references is unified and moved to the end of the thesis. Any new remarks and comments not present in the original papers are added to footnotes.

Let us briefly introduce the topics of the thesis. As its name suggests, the main focus of this thesis is the study of properties of Banach spaces which are defined by some notion of convergence.

First of these notions is a *weak* derived set.* Recall that the weak* derived set of a subset A of a dual Banach space X^* is the set $A^{(1)}$ consisting of all limits of bounded weak* convergent nets in A. This notion is closely tied to another – the *weak* sequential closure.* Indeed, if the predual space X is separable, the weak* derived set $A^{(1)}$ is the weak* sequential closure of A as bounded sets in X^* are weak* metrizable and thus limits of bounded nets can be attained by sequences. Weak* derived sets and sequential closures have many applications in Banach space theory (see the introductions of the first two papers of this thesis). Taking a weak* derived set is not a closure operation as it is not idempotent – it can happen that $A^{(1)}$ is a proper subset of $\left(A^{(1)}\right)^{(1)}$. Hence, it makes sense to define iterated weak* derived sets in the natural recursive way. Those will be denoted by $A^{(\alpha)}$ for an ordinal α. One of key aspects of the study of weak* derived sets is that they can be used to characterize reflexivity and quasi-reflexivity. A Banach space X is reflexive (resp. quasi-reflexive) if and only if $A^{(1)} = \overline{A}^{w^*}$ for every convex subset (resp. every vector subspace) A of X^*. Let us define the order of A to be the least ordinal α such that $A^{(\alpha)} = A^{(\alpha+1)}$. It follows that in duals of quasi-reflexive spaces the only possible orders of a subspace A are 0, if A is weak* closed, or 1, if it is not. However, if the space is not quasi-reflexive, iterating weak* derived sets of subspaces can stabilize much later – in the dual of any non-quasi-reflexive space there are subspaces of any countable non-limit order [25] (and if the predual is moreover separable, these orders are the only possible ones). The aim of the first paper was to provide a partial analogue to this statement for convex subsets in duals of non-reflexive spaces – we managed to show that in the dual of any non-reflexive space we can find a convex set of any finite order and a convex set of order $\omega + 1$. Let us note that this result has been already generalized for any countable non-limit ordinals in [24]. It still remains open if the order of a convex set can be a countable limit ordinal. The second paper deals with a more special result which is motivated by the study of extensions of holomorphic functions on dual Banach spaces. We showed that in the dual of any non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual we can, for any countable non-limit ordinal α, find a subspace A, such that $A^{(\alpha)} \subsetneq \overline{A^{(\alpha)}} = \overline{A}^{w^*}$.

The second notion is *Cesàro summability.* Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is Cesàro summable (or *Cesàro limitable* in some literature) if the sequence or arithmetic means $\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}}$ is convergent. A Banach space X has the *Banach-Saks property* if every bounded sequence in X admits a Cesàro summable subsequence, and the *weak Banach-Saks property* if every weakly convergent sequence in X admits a Cesàro summable subsequence. The Banach-Saks property is a notion weaker than super-reflexivity but stronger than reflexivity. There are also localized versions of these properties – a subset A of a Banach space X is a *Banach-Saks set* (resp. *weak Banach-Saks set*) if every (resp. every weakly convergent) sequence in A has a Cesàro summable subsequence. It follows from the Mazur theorem that if we have a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X, then there is a sequence of convex combinations which converges to zero in norm. The weak Banach-Saks property of X then means that these convex combinations can be chosen to be the Cesàro means of some subsequence. In [3] the authors investigated how regular these convex combinations can be in spaces failing the weak Banach-Saks property and defined the *ξ-Banach-Saks property* and the *weak ξ-Banach-Saks property* for a countable ordinal ξ. Roughly speaking, a Banach space X has the ξ-Banach-Saks property if any bounded sequence in X has a subsequence, whose ξ-times iterated Cesàro means are convergent (for precise definition see the introduction to the third paper). X is said to have the weak ξ-Banach-Saks property if the same holds for any weakly convergents sequence in X. Some quantitative results concerning the weak Banach-Saks sets and Banach-Saks sets were given in [7]. We provided analogous quantitative results for the Banach-Saks properties of higher orders in the third paper of this thesis. This investigation led to a new measure of weak compactness.

1. Weak* derived sets of convex sets in duals of non-reflexive spaces

Abstract: We investigate weak* derived sets, that is the sets of weak* limits of bounded nets, of convex subsets of duals of non-reflexive Banach spaces and their possible iterations. We prove that a dual space of any non-reflexive Banach space contains convex subsets of any finite order and a convex subset of order $\omega + 1$.

1.1 Introduction and formulation of main results

Let A be a subset of a dual Banach space X^*. The weak* derived set $A^{(1)}$ of the set A is the set of all weak* limits of bounded convergent nets in A, i.e.

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*},$$

where B_{X^*} denotes the closed unit ball of X^* and $\overline{M}^{w^*}$ denotes the weak* closure of M for any subset M of X^*. If X is separable, the weak* derived set $A^{(1)}$ coincides with the weak* sequential closure of A. Indeed, the weak* topology of the dual of any separable Banach space restricted to any bounded set is metrizable, and thus $A^{(1)}$ is the set of all weak* limits of bounded sequences in A, which is, by the uniform boundedness principle, the set of all weak* limits of sequences in A.

The study of weak* derived sets of subspaces in duals of separable spaces (or rather weak* sequential closures, but as we have seen, for duals of separable spaces these notions coincide) was initiated by Banach [5] and his school in 1930's. It can be, however, natural to suppose that their interest in weak* derived sets was due the lack of acquaintance with the concepts of general topology. Later weak* derived sets found significant applications. To name a few, they were applied by Piatetski-Shapiro [30] for characterization of sets of uniqueness in harmonic analysis, used by Saint-Raymond [32] for Borel and Baire classification of inverses of continuous injective linear operators, by Dierolf and Moscatelli [12] in the structure theory of Fréchet spaces, or by Plichko [29] to solve a problem on universal Markushevich bases posed by Kalton. For additional information and a historical account, see the survey on weak* sequential closures by Ostrovskii [26].

The theory of weak* derived sets of subspaces was essentially completed by Ostrovskii [25]. On the other hand, the study of weak* derived sets of convex subsets was initiated much later by Garcia, Kalenda and Maestre [15] in 2010 in their study of extension problems for holomorphic functions on dual Banach spaces, where they asked whether the theory will remain the same if we consider convex sets instead of subspaces. This question was answered negatively by Ostrovskii [27]. Let us explain the situation in more detail.

Recall that a Banach space X is called quasi-reflexive if its canonical embedding into its bidual X^{**} is of finite codimension. All reflexive spaces are also quasi-reflexive and there are non-reflexive quasi-reflexive spaces, e.g. the James' space [17]. Reflexivity or quasi-reflexivity of X is closely related to the behaviour of weak* derived sets of convex subsets of X^*. We summarize known results in the following theorem. We use the notation $A \subset\subset X$ to say that A is a subspace of X.

Theorem A. *Let X be a Banach space.*

1. *X is reflexive if and only if $A^{(1)} = \overline{A}^{w^*}$ for every convex set $A \subseteq X^*$.*
2. *X is quasi-reflexive if and only if $A^{(1)} = \overline{A}^{w^*}$ for every subspace $A \subset\subset X^*$.*

The proof of *(2)*, using the notion of norming subspaces, can be done using the results of [11]. The implication from left to right of *(1)* can be easily shown using the Mazur's theorem. The other implication, i.e. the existence of a convex subset A of the dual space of every non-reflexive space for which $A^{(1)} \subsetneq \overline{A}^{w^*}$, is the mentioned result of Ostrovskii [27]. In [27] Ostrovskii also proved a stronger version of *(2)*: A Banach space X is quasi-reflexive if and only if $A^{(1)} = \overline{A}^{w^*}$ for every absolutely convex set $A \subseteq X^*$.

A convex subset A of a dual Banach space X^* is weak* closed if and only if it equals its weak* derived set, i.e. $A = \overline{A}^{w^*}$ if and only if $A = A^{(1)}$. This is a formulation of the Krein-Šmulyan theorem. The existence of subsets $A \subseteq X^*$ such that $A^{(1)} \neq \overline{A}^{w^*}$ inspires the definition of weak* derived sets of higher orders: For a successor ordinal α, the weak* derived set of A of order α is $A^{(\alpha)} = \left(A^{(\alpha-1)}\right)^{(1)}$. For a limit ordinal α we define $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)}$. The order of A is the least ordinal α, such that $A^{(\alpha)} = A^{(\alpha+1)}$. We use the convention that $A^{(0)} = A$.

In [25] it is shown that for every non-quasi-reflexive separable Banach space X and every countable ordinal α we can find a subspace $A \subset\subset X^*$ of order $\alpha+1$. It also holds, that in separable Banach spaces countable non-limit ordinals are the only possible orders of subspaces [16]. This gives a complete description of possible orders of subspaces of duals of non-quasi-reflexive separable Banach spaces.

In this paper we prove some partial results regarding orders of convex subsets of duals of non-reflexive Banach spaces. The main results are:

Theorem B. *Let X be a non-reflexive Banach space and $n \in \mathbb{N}$. Then there is a convex subset of X^* of order n.*

Theorem C. *Let X be a non-reflexive Banach space. Then there is a convex subset of X^* of order $\omega + 1$.*

These results are proved below in Theorems 1.8 and 1.13. Note that we can restrict ourselves to the case of non-reflexive quasi-reflexive Banach spaces. In the case of reflexive spaces the only possible orders of convex sets are 0, if the set is already weak* closed, or 1, if the set is not weak* closed. The case of non-quasi-reflexive separable spaces is already solved in [25]. The proofs of Theorems B and C use a modified construction of Ostrovskii used in [27].

1.2 Proofs of main results

Lemma 1.1. *Let X be a Banach space and $Z \subset\subset X$ its closed subspace. Denote by $E : Z \to X$ the identity embedding. Then for every ordinal α and $A \subseteq Z^*$ we have*

$$(E^*)^{-1}(A^{(\alpha)}) = ((E^*)^{-1}(A))^{(\alpha)}.$$

This lemma is proved in [27, Lemma 1] for $\alpha = 1$. For general α the lemma follows by transfinite induction. Note that the weak* derived set $A^{(\alpha)}$ is taken in Z^* and $((E^*)^{-1}(A))^{(\alpha)}$ is taken in X^*.

We say that a sequence $(z_n)_{n=1}^{\infty}$ in a Banach space X is seminormalized if there are constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ we have $C_1 \leq ||z_n|| \leq C_2$, and we say that $(z_n)_{n=1}^{\infty}$ has bounded partial sums if there is $C > 0$ such that for all $N \in \mathbb{N}$ we have $||\sum_{n=1}^{N} z_n|| \leq C$.

Lemma 1.2. *Let X be a non-reflexive Banach space. Then X contains a seminormalized basic sequence $(z_n)_{n=1}^{\infty}$ which has bounded partial sums.*

Proof. This lemma is proved for a non-reflexive space with a basis in [34, Theorem 3, ($1° \Leftrightarrow 3°$)]. As any non-reflexive space contains a non-reflexive subspace with a basis [28, Theorem 1], the lemma follows. $\square$

For the rest of this paper we pick and fix such seminormalized basic sequence $(z_n)_{n=1}^{\infty}$ in X with bounded partial sums and denote its closed linear span by Z. For further reference, we also fix the constants $C, C_1, C_2 > 0$ such that $||\sum_{n=1}^{N} z_n|| \leq C$ for all $N \in \mathbb{N}$ and $C_1 \leq ||z_n|| \leq C_2$ for all $n \in \mathbb{N}$. Let us denote by $(z_n^*)_{n=1}^{\infty}$ the biorthogonal functionals of $(z_n)_{n=1}^{\infty}$ and by K the positive cone of Z^*. That is the weak* closed convex set

$$K = \{z^* \in Z^*;\ z^*(z_j) \geq 0 \text{ for each } j \in \mathbb{N}\}.$$

Note that as Z is separable, weak* derived sets in Z^* coincide with weak* sequential closures. Also note, that as the basis $(z_n)_{n=1}^{\infty}$ is seminormalized, we get that $z_n^* \xrightarrow{w^*} 0$.

Lemma 1.3. *For every $z^* \in K$ we have $z^* = \sum_{n=1}^{\infty} z^*(z_n) z_n^*$, where the series converges absolutely. Further, we have $||z^*|| \geq C^{-1} \sum_{n=1}^{\infty} z^*(z_n)$.*

Proof. For each $N \in \mathbb{N}$ we have

$$||z^*|| \geq C^{-1} z^* \left(\sum_{n=1}^{N} z_n \right) = C^{-1} \sum_{n=1}^{N} z^*(z_n).$$

Hence, as $z^*(z_n) \geq 0$ for each $n \in \mathbb{N}$, we get $||z^*|| \geq C^{-1} \sum_{n=1}^{\infty} z^*(z_n)$ and the series $\sum_{n=1}^{\infty} z^*(z_n) z_n^*$ converges absolutely in Z^*. As it also converges to z^* in the weak* topology, we get that $\sum_{n=1}^{\infty} z^*(z_n) z_n^* = z^*$. $\square$

Now, let us partition $\mathbb{N}$ into countably many subsequences: There will be the set $\mathbf{N}_0 = \{i_1 < i_2 < \dots\}$. Then for each $n \in \mathbb{N}$ there will be the set $\mathbf{N}(i_n)$, for each $j_1 \in \mathbf{N}(i_n)$ there will be the set $\mathbf{N}(i_n, j_1)$ and so on up to for each $j_n \in \mathbf{N}(i_n, j_1, \dots, j_{n-1})$ there will be the set $\mathbf{N}(i_n, j_1, \dots, j_n)$.

Fix a sequence of positive numbers $(\beta_k)_{k=1}^{\infty}$, such that $0 \neq \beta_k \nearrow \infty$, and a sequence $(\alpha_k)_{k=1}^{\infty}$ of numbers in the interval $[0, 1)$, such that for each $n \in \mathbb{N}$ we have that $(\alpha_{j_1})_{j_1 \in \mathbf{N}(i_n)}$ is a sequence increasing monotonically to 1 with the first element equal to 0. We will say that a finite sequence of positive integers $(n_1, \ldots, n_k)$ is admissible if $n_1 \in \mathbf{N}_0$ and for each $2 \leq i \leq k$ we have that $n_i \in \mathbf{N}(n_1, \ldots, n_{i-1})$.

For each $n \in \mathbb{N}$ define

$$A_n = \operatorname{conv}\left\{\alpha_{j_1} z_{i_n}^* + \sum_{k=1}^{n} \beta_{j_k} z_{j_{k+1}}^*;\ (i_n, j_1, \ldots, j_{n+1}) \text{ is admissible}\right\}$$

and, moreover, define

$$A = \operatorname{conv} \bigcup_{n=1}^{\infty} A_n.$$

Let us further denote by $\mathbf{N}_n$ the support of A_n, i.e.

$$\mathbf{N}_n = \{i_n\} \cup \bigcup \{\mathbf{N}(i_n, j_1) \cup \mathbf{N}(i_n, j_1, j_2) \cup \cdots \cup \mathbf{N}(i_n, j_1, \ldots, j_n); \\ (i_n, j_1, \ldots, j_n) \text{ is admissible}\}.$$

Later we will prove that those A_n's are the desired convex sets of order $n+1$ and A is the desired convex set of order $\omega + 1$.

Proposition 1.4. *A_n is the set of those $x^* \in Z^*$ which have finite support in $\mathbf{N}_n$ and which satisfy the following equations:*

$$\begin{aligned}
1 &= \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} \\
x^*(z_{i_n}) &= \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2}) \alpha_{j_1}}{\beta_{j_1}} \\
x^*(z_{j_2}) &= \sum_{j_3 \in \mathbf{N}(i_n, j_1, j_2)} \frac{x^*(z_{j_3}) \beta_{j_1}}{\beta_{j_2}} && j_1 \in \mathbf{N}(i_n),\ j_2 \in \mathbf{N}(i_n, j_1) \\
x^*(z_{j_3}) &= \sum_{j_4 \in \mathbf{N}(i_n, j_1, j_2, j_3)} \frac{x^*(z_{j_4}) \beta_{j_2}}{\beta_{j_3}} && j_1 \in \mathbf{N}(i_n), \ldots,\ j_3 \in \mathbf{N}(i_n, j_1, j_2) \\
&\vdots \\
x^*(z_{j_n}) &= \sum_{j_{n+1} \in \mathbf{N}(i_n, j_1, \ldots, j_n)} \frac{x^*(z_{j_{n+1}}) \beta_{j_{n-1}}}{\beta_{j_n}} && j_1 \in \mathbf{N}(i_n), \ldots, j_n \in \mathbf{N}(i_n, j_1, \ldots, j_{n-1}).
\end{aligned}$$

Proof. Each element of A_n has finite support in $\mathbf{N}_n$ and satisfies the required equations, as the vectors $\alpha_{j_1} z_{i_n}^* + \sum_{k=1}^{n} \beta_{j_k} z_{j_{k+1}}^*$ satisfy them and the validity of these equations is preserved by taking convex combinations. To prove the converse inclusion, let us have $x^* \in Z^*$ with finite support in $\mathbf{N}_n$ and satisfying these equations. Set $c_{j_k} = x^*(z_{j_k})^{-1}$, if $x^*(z_{j_k}) \neq 0$, and $c_{j_k} = 0$ otherwise. Then it follows from convexity of A_n and the choice of c_{j_k} that for each admissible $(i_n, j_1, \ldots, j_n)$ we have

$$x^*_{j_n} := \sum_{j_{n+1} \in \mathbf{N}(i_n, j_1, \ldots, j_n)} \frac{c_{j_n} x^*(z_{j_{n+1}}) \beta_{j_{n-1}}}{\beta_{j_n}} \left(\alpha_{j_1} z^*_{i_n} + \sum_{k=1}^{n} \beta_{j_k} z^*_{j_{k+1}} \right) \in A_n.$$

In a similar way it follows that for each $1 < m < n$ and each admissible $(i_n, j_1, \ldots, j_m)$ we have

$$x^*_{j_m} := \sum_{j_{m+1} \in \mathbf{N}(i_n, j_1, \ldots, j_m)} \frac{c_{j_m} x^*(z_{j_{m+1}}) \beta_{j_{m-1}}}{\beta_{j_m}} x^*_{j_{m+1}} \in A_n$$

and finally that

$$y^* := \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} x^*_{j_2} \in A_n.$$

Hence, we just need to show that $y^* = x^*$. If $k \notin \mathbf{N}_n$ we have that $x^*(z_k) = y^*(z_k) = 0$. Let $m \leq n$ and fix an admissible $(i_n, j_1, \ldots, j_m)$. Then for each admissible $(i_n, \widetilde{j_1}, \ldots, \widetilde{j_n})$ we have

$$x^*_{\widetilde{j_n}}(z_{j_m}) = \sum_{\widetilde{j_{n+1}} \in \mathbf{N}(i_n, \widetilde{j_1}, \ldots, \widetilde{j_n})} \frac{c_{\widetilde{j_n}} x^*(z_{\widetilde{j_{n+1}}}) \beta_{\widetilde{j_{n-1}}}}{\beta_{\widetilde{j_n}}} \beta_{j_{m-1}}$$

if $\widetilde{j_m} = j_m$ (and therefore $\widetilde{j_i} = j_i$ for each $i \leq m$) and $x^*_{\widetilde{j_n}}(z_{j_m}) = 0$ otherwise. Hence, for each admissible $(i_n, \widetilde{j_1}, \ldots, \widetilde{j_{n-1}})$, we have

$$x^*_{\widetilde{j_{n-1}}}(z_{j_m}) = \sum_{\substack{\widetilde{j_n} \in \mathbf{N}(i_n, \widetilde{j_1}, \ldots, \widetilde{j_{n-1}}) \\ \widetilde{j_{n+1}} \in \mathbf{N}(i_n, \widetilde{j_1}, \ldots, \widetilde{j_n})}} \frac{c_{\widetilde{j_{n-1}}} x^*(z_{\widetilde{j_n}}) \beta_{\widetilde{j_{n-2}}}}{\beta_{\widetilde{j_{n-1}}}} \frac{c_{\widetilde{j_n}} x^*(z_{\widetilde{j_{n+1}}}) \beta_{\widetilde{j_{n-1}}}}{\beta_{\widetilde{j_n}}} \beta_{j_{m-1}}$$

if $\widetilde{j_m} = j_m$ and $x^*_{\widetilde{j_{n-1}}}(z_{j_m}) = 0$ otherwise. Iterating this, we get for $m+1 \leq k \leq n$ and admissible $(i_n, \widetilde{j_1}, \ldots, \widetilde{j_k})$

$$x^*_{\widetilde{j_k}}(z_{j_m}) = \sum_{\substack{\widetilde{j_{k+1}} \in \mathbf{N}(i_n, \widetilde{j_1}, \ldots, \widetilde{j_k}), \ldots, \\ \widetilde{j_{n+1}} \in \mathbf{N}(i_n, \widetilde{j_1}, \ldots, \widetilde{j_n})}} \frac{c_{\widetilde{j_k}} x^*(z_{\widetilde{j_{k+1}}}) \beta_{\widetilde{j_{k-1}}}}{\beta_{\widetilde{j_k}}} \cdots \frac{c_{\widetilde{j_n}} x^*(z_{\widetilde{j_{n+1}}}) \beta_{\widetilde{j_{n-1}}}}{\beta_{\widetilde{j_n}}} \beta_{j_{m-1}}$$

if $\widetilde{j_k} = j_k$ and $x^*_{\widetilde{j_k}}(z_{j_m}) = 0$ otherwise. Hence, we get

$$x^*_{j_m}(z_{j_m}) = \sum_{\substack{j_{m+1} \in \mathbf{N}(i_n, j_1, \ldots, j_m), \ldots, \\ j_{n+1} \in \mathbf{N}(i_n, j_1, \ldots, j_n)}} \frac{c_{j_m} x^*(z_{j_{m+1}}) \beta_{j_{m-1}}}{\beta_{j_m}} \cdots \frac{c_{j_n} x^*(z_{j_{n+1}}) \beta_{j_{n-1}}}{\beta_{j_n}} \beta_{j_{m-1}}$$

and for admissible $(i_n, \widetilde{j_1}, \ldots, \widetilde{j_m})$ such that $j_m \neq \widetilde{j_m}$ we get $x^*_{\widetilde{j_m}}(z_{j_m}) = 0$. We can then inductively prove that if $2 \leq k \leq m-1$, the only admissible $(i_n, \widetilde{j_1}, \ldots, \widetilde{j_k})$

with nonzero $x^*_{\tilde{j}_k}(z_{j_m})$ are the initial segments of $(i_n, j_1, \ldots, j_m)$ and for them we have

$$x^*_{j_k}(z_{j_m}) = \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m),\ldots,\\ j_{n+1}\in\mathbf{N}(i_n,j_1,\ldots,j_n)}} \frac{c_{j_k}x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} \cdots \frac{c_{j_n}x^*(z_{j_{n+1}})\beta_{j_{n-1}}}{\beta_{j_n}}\beta_{j_{m-1}}.$$

Then, as $c_{j_k} = x^*(z_{j_k})^{-1}$, we can finally show that

$$\begin{aligned}
y^*(z_{j_m}) &= \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m),\ldots,\\ j_{n+1}\in\mathbf{N}(i_n,j_1,\ldots,j_n)}} \frac{x^*(z_{j_2})}{\beta_{j_1}}\frac{c_{j_2}x^*(z_{j_3})\beta_{j_1}}{\beta_{j_2}} \cdots \frac{c_{j_n}x^*(z_{j_{n+1}})\beta_{j_{n-1}}}{\beta_{j_n}}\beta_{j_{m-1}} \\
&= \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m),\ldots,\\ j_{n+1}\in\mathbf{N}(i_n,j_1,\ldots,j_n)}} \frac{\beta_{j_{m-1}}}{\beta_{j_n}}x^*(z_{j_{n+1}}).
\end{aligned}$$

Now, by consecutive application of the equations of the proposition, we get

$$\begin{aligned}
x^*(z_{j_m}) &= \sum_{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m)} \frac{x^*(z_{j_{m+1}})\beta_{j_{m-1}}}{\beta_{j_m}} \\
&= \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m)\\ j_{m+2}\in\mathbf{N}(i_n,j_1,\ldots,j_{m+1})}} \frac{x^*(z_{j_{m+2}})\beta_{j_{m-1}}}{\beta_{j_m}}\frac{\beta_{j_m}}{\beta_{j_{m+1}}} \\
&= \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m)\\ j_{m+2}\in\mathbf{N}(i_n,j_1,\ldots,j_{m+1})}} \frac{x^*(z_{j_{m+2}})\beta_{j_{m-1}}}{\beta_{j_{m+1}}} = \cdots \\
\cdots &= \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m),\ldots,\\ j_{n+1}\in\mathbf{N}(i_n,j_1,\ldots,j_n)}} \frac{\beta_{j_{m-1}}}{\beta_{j_n}}x^*(z_{j_{n+1}}).
\end{aligned}$$

Hence, $x^*(z_{j_m}) = y^*(z_{j_m})$. Analogically

$$x^*(z_{i_n}) = y^*(z_{i_n}) = \sum_{\substack{j_{m+1}\in\mathbf{N}(i_n,j_1,\ldots,j_m),\ldots,\\ j_{n+1}\in\mathbf{N}(i_n,j_1,\ldots,j_n)}} \frac{\alpha_{j_1}}{\beta_{j_n}}x^*(z_{j_{n+1}})$$

and for admissible $(i_n, j_1, \ldots, j_{n+1})$ we have that $x^*(z_{j_{n+1}}) = y^*(z_{j_{n+1}})$. Hence, $x^* = y^*$ and we are done. $\square$

Proposition 1.5. *Let $0 \le m \le n-1$ and x^* be an element of $A_n^{(m)}$. Then x^* satisfies the equations of Proposition 1.4 possibly except for the equations on the bottom m lines. Precisely:*

$$1 = \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}}$$

$$x^*(z_{i_n}) = \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}}$$

and for $2 \le k \le n-m$ and admissible $(i_n, j_1, \ldots, j_k)$

$$x^*(z_{j_k}) = \sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\ldots,j_k)} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}.$$

Proof. We shall proceed by induction. We have already shown in Proposition 1.4 that the proposition holds for $m = 0$. Now, let us suppose that the proposition holds for $m-1$ and take $x^* \in A_n^{(m)}$. There is a sequence $(x_i^*)_{i=1}^\infty$ in $A_n^{(m-1)}$, such that $x_i^* \overset{w^*}{\to} x^*$. Take admissible $(i_n, j_1, \dots, j_k)$ where $k \le n-m$. Suppose, for a contradiction, that

$$x^*(z_{j_k}) \neq \sum_{j_{k+1} \in \mathbf{N}(i_n, j_1, \dots, j_k)} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}.$$

Take

$$\delta = x^*(z_{j_k}) - \sum_{j_{k+1} \in \mathbf{N}(i_n, j_1, \dots, j_k)} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}.$$

By the induction hypothesis, as $x_i^* \in A_n^{(m-1)}$, we have

$$\begin{aligned} x_i^*(z_{j_{k+1}}) &= \sum_{j_{k+2} \in \mathbf{N}(i_n, j_1, \dots, j_{k+1})} \frac{x_i^*(z_{j_{k+2}})\beta_{j_k}}{\beta_{j_{k+1}}} \\ x_i^*(z_{j_k}) &= \sum_{j_{k+1} \in \mathbf{N}(i_n, j_1, \dots, j_k)} \frac{x_i^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}. \end{aligned}$$

Hence, by Fatou's lemma, we get that $\delta \ge 0$ and as δ is nonzero we get $\delta > 0$.

For $c > 0$ take

$$\begin{aligned} F_c &= \{j_{k+1} \in \mathbf{N}(i_n, j_1, \dots, j_k);\ \beta_{j_{k+1}} \le c\} \\ G_c &= \mathbf{N}(i_n, j_1, \dots, j_k) \setminus F_c. \end{aligned}$$

Then F_c is a finite set and $x_i^* \overset{w^*}{\to} x^*$, therefore there is $i_0 \in \mathbb{N}$, such that for $i \ge i_0$ we have

$$\begin{aligned} \sum_{j_{k+1} \in F_c} \frac{x_i^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} &< \sum_{j_{k+1} \in F_c} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} + \delta/2 \\ &\le \sum_{j_{k+1} \in \mathbf{N}(i_n, j_1, \dots, j_k)} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} + \delta/2 \\ &= x^*(z_{j_k}) - \delta/2, \end{aligned}$$

and therefore

$$\sum_{\substack{j_{k+1} \in F_c \\ j_{k+2} \in \mathbf{N}(i_n, j_1, \dots, j_{k+1})}} \frac{x_i^*(z_{j_{k+2}})\beta_{j_{k-1}}}{\beta_{j_{k+1}}} = \sum_{j_{k+1} \in F_c} \frac{x_i^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} < x^*(z_{j_k}) - \delta/2.$$

Then there is $i_1 \ge i_0$ such that

$$\sum_{\substack{j_{k+1} \in G_c \\ j_{k+2} \in \mathbf{N}(i_n, j_1, \dots, j_{k+1})}} \frac{x_{i_1}^*(z_{j_{k+2}})\beta_{j_{k-1}}}{\beta_{j_{k+1}}} = \sum_{j_{k+1} \in G_c} \frac{x_{i_1}^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}} > \delta/4,$$

as otherwise $x_i^*(z_{j_k}) < x^*(z_{j_k}) - \delta/4$ for all $i \geq i_0$, which would contradict $x_i^* \xrightarrow{*} x^*$. But then it follows from Lemma 1.3 that

$$\left\|x_{i_1}^*\right\| \geq C^{-1} \sum_{\substack{j_{k+1}\in G_c \\ j_{k+2}\in \mathbf{N}(i_n,j_1,\dots,j_{k+1})}} x_{i_1}^*(z_{j_{k+2}}) > C^{-1}\beta_{j_{k-1}}^{-1} c\, \delta/4.$$

As $c > 0$ was chosen arbitrarily, we get that $(x_i^*)_{i=1}^\infty$ is unbounded. But this contradicts the Banach-Steinhaus theorem. Hence,

$$x^*(z_{j_k}) = \sum_{j_{k+1}\in \mathbf{N}(i_n,j_1,\dots,j_k)} \frac{x^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}.$$

Now suppose for a contradiction that

$$x^*(z_{i_n}) - \sum_{\substack{j_1\in \mathbf{N}(i_n) \\ j_2\in \mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} = \delta \neq 0.$$

By the same argument as above we get that $\delta > 0$. As $x_i^* \in A_n^{(m-1)}$, we get by the induction hypothesis that

$$x_i^*(z_{i_n}) = \sum_{\substack{j_1\in \mathbf{N}(i_n) \\ j_2\in \mathbf{N}(i_n,j_1)}} \frac{x_i^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}}$$

and for admissible (i_n, j_1, j_2)

$$x_i^*(z_{j_2}) = \sum_{j_3\in \mathbf{N}(i_n,j_1,j_2)} \frac{x^*(z_{j_3})\beta_{j_1}}{\beta_{j_2}}.$$

Now, for $c > 0$ set

$$\begin{aligned} F_c &= \{j_2;\ \beta_{j_2} \leq c \text{ and } (i_n, j_1, j_2) \text{ is admissible}\} \\ G_c &= \bigcup\{\mathbf{N}(i_n, j_1);\ j_1 \in \mathbf{N}(i_n)\} \setminus F_c. \end{aligned}$$

Then, as F_c is finite and $x_i^* \xrightarrow{w^*} x^*$, we get in the same way as above that there is $i_0 \in \mathbb{N}$ such that for $i \geq i_0$

$$\sum_{j_2\in F_c} \frac{x_i^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} < x^*(z_{i_n}) - \delta/2,$$

and therefore there is $i_1 \geq i_0$ such that

$$\sum_{\substack{j_2\in G_c \\ j_3\in \mathbf{N}(i_n,j_1,j_2)}} \frac{x_{i_1}^*(z_{j_3})\alpha_{j_1}}{\beta_{j_2}} = \sum_{j_2\in G_c} \frac{x_{i_1}^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} > \delta/4.$$

But then again by Lemma 1.3 we have, for j being the second element of $\mathbf{N}(i_n)$,

$$||x_{i_1}^*|| \geq C^{-1}\alpha_j^{-1} c\, \delta/4,$$

which contradicts boundedness of the sequence $(x_i^*)_{i=1}^\infty$. Hence,

$$x^*(z_{i_n}) = \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}}.$$

In exactly the same way we can show that

$$1 - \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} = \delta > 0$$

leads to the fact that for all $c > 0$ there is $i_1 \in \mathbb{N}$ such that

$$||x_{i_1}^*|| \geq C^{-1} c\, \delta/4,$$

and contradicts boundedness of the sequence $(x_i^*)_{i=1}^\infty$. Hence,

$$1 = \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}}.$$

$\square$

Lemma 1.6. *The order of A_n is at least $n+1$. Specifically $z_{i_n}^* \in A_n^{(n+1)} \setminus A_n^{(n)}$.*

Proof. First, observe that $z_{i_n}^* \in A_n^{(n+1)}$ as

$$z_{i_n}^* = w^* \lim_{j_1} \cdots w^* \lim_{j_{n+1}} \left(\alpha_{j_1} z_{i_n}^* + \sum_{k=1}^{n} \beta_{j_k} z_{j_{k+1}}^* \right).$$

Now, suppose for a contradiction that $z_{i_n}^* \in A_n^{(n)}$. There is a sequence $(x_i^*)_{i=1}^\infty$ in $A_n^{(n-1)}$ which weak* converges to $z_{i_n}^*$. By Proposition 1.5 we have

$$x_i^*(z_{i_n}) = \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}}$$

$$1 = \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}},$$

Now fix an arbitrary $M \in \mathbb{N}$, then

$$\begin{aligned}
1 = z_{i_n}^*(z_{i_n}) &= \lim_i x_i^*(z_{i_n}) \\
&= \lim_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 \le M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} + \sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 > M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} \right) \\
&\le \liminf_i \left(\alpha_M \sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 \le M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} + \sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 > M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right) \\
&= \liminf_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} + (\alpha_M - 1) \sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 \le M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right) \\
&\le 1 + (\alpha_M - 1)\ \liminf_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 \le M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right).
\end{aligned}$$

As $\alpha_M - 1 < 0$, we get, up to passing to a subsequence if necessary,

$$\lim_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 \le M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right) = 0.$$

Then

$$\lim_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 > M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right) = \lim_i \left(\sum_{\substack{j_1 \in \mathbf{N}(i_n), \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} \right) = 1.$$

Hence, there is $i \in \mathbb{N}$, such that

$$\sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 > M \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{x_i^*(z_{j_2})}{\beta_{j_1}} > 1/2.$$

But then it follows from Lemma 1.3 that

$$\|x_i^*\| \ge C^{-1} \sum_{\substack{j_1 \in \mathbf{N}(i_n),\, j_1 > M \\ j_2 \in \mathbf{N}(i_n, j_1)}} x_i^*(z_{j_2}) > C^{-1}\beta_M/2.$$

Hence, as M was chosen arbitrarily, we get that $(x_i^*)_{i=1}^\infty$ is unbounded, which is a contradiction. $\square$

Lemma 1.7. *The order of A_n is at most $n+1$. Specifically $\overline{A_n^{(n)}} = A_n^{(n+1)} = \overline{A_n}^{w^*}$.*

Proof. As $\overline{A_n^{(n)}} \subseteq A_n^{(n+1)} \subseteq \overline{A_n}^{w^*}$, we just need to show that each element of $\overline{A_n}^{w^*}$ is a norm limit of elements of $A_n^{(n)}$. For $y^* \in K$ (recall that K is the positive cone of Z^*) we define

$$\delta(y^*) = 1 - \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{y^*(z_{j_2})}{\beta_{j_1}}$$

$$\gamma(y^*) = y^*(z_{i_n}) - \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \frac{y^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}}.$$

Take any $x^* \in K$ with finite support in $\mathbf{N}_n$ and which satisfies $\delta(x^*) > \gamma(x^*) \geq 0$. Let us consider

$$D = \operatorname{conv}\left\{ a\left(\frac{\alpha_{j_1}}{\beta_{j_1}}, \frac{1}{\beta_{j_1}}\right);\ a \geq 0, j_1 \in \mathbf{N}(i_n) \right\}.$$

Then D is the cone formed by rays with gradients in $[\min \alpha_{j_1}, \sup \alpha_{j_1}) = [0, 1)$. Hence, $(\gamma(x^*), \delta(x^*))$ is in D and we can write it as a convex combination

$$(\gamma(x^*), \delta(x^*)) = \sum_{j_1 \in \mathbf{N}(i_n)} a_{j_1} \left(\frac{\alpha_{j_1}}{\beta_{j_1}}, \frac{1}{\beta_{j_1}}\right).$$

We now introduce some new notation. For $j_k \in \mathbf{N}(i_n, j_1, \ldots, j_{k-1})$ we define $j_k(l)$ to be the l^{th} element of $\mathbf{N}(i_n, j_1, \ldots, j_k)$. We write $j_k(l_1, l_2)$ instead of $(j_k(l_1))(l_2)$ for shortness. Now let us inductively define for $l, l_1, \ldots, l_n \in \mathbb{N}$

$$y^*(l) = x^* + \sum_{j_1 \in \mathbf{N}(i_n)} a_{j_1} z^*_{j_1(l)}$$

$$y^*(l_1, l_2) = y^*(l_1) + \sum_{\substack{j_1 \in \mathbf{N}(i_n) \\ j_2 \in \mathbf{N}(i_n, j_1)}} \left(y^*(l_1)(z_{j_2}) - \sum_{j_3 \in \mathbf{N}(i_n, j_1, j_2)} \frac{y^*(l_1)(z_{j_3})\beta_{j_1}}{\beta_{j_2}} \right) \frac{z^*_{j_2(l_2)}\beta_{j_2}}{\beta_{j_1}}$$

$$\vdots$$

$$y^*(l_1, \ldots, l_n) = y^*(l_1, \ldots, l_{n-1}) + \sum_{\substack{j_1 \in \mathbf{N}(i_n), \ldots \\ j_n \in \mathbf{N}(i_n, j_1, \ldots, j_{n-1})}} \Bigg(y^*(l_1, \ldots, l_{n-1})(z_{j_n}) -$$

$$\sum_{j_{n+1} \in \mathbf{N}(i_n, j_1, \ldots, j_n)} \frac{y^*(l_1, \ldots, l_{n-1})(z_{j_{n+1}})\beta_{j_{n-1}}}{\beta_{j_n}} \Bigg) \frac{z^*_{j_n(l_n)}\beta_{j_n}}{\beta_{j_{n-1}}}.$$

It is easily proved by induction over $k = 1, \ldots, n$ that $y^*(l_1, \ldots, l_k)$ have finite support in $\mathbf{N}_n$. Further,

$$y^*(l_1, \ldots, l_k) \xrightarrow[l_k]{w^*} y^*(l_1, \ldots, l_{k-1}) \qquad 2 \leq k \leq n$$

$$y^*(l) \xrightarrow[l]{w^*} x^*.$$

To see this, consider

$$y^*(l_1,\dots,l_k)-y^*(l_1,\dots,l_{k-1})=\sum_{\substack{j_1\in\mathbf{N}(i_n),\dots\\ j_k\in\mathbf{N}(i_n,j_1,\dots,j_{k-1})}} c_{j_k} z^*_{j_k(l_k)},$$

where

$$c_{j_k}=\left(y^*(l_1,\dots,l_{k-1})(z_{j_k})-\sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)}\frac{y^*(l_1,\dots,l_{k-1})(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}\right)\frac{\beta_{j_k}}{\beta_{j_{k-1}}}.$$

Then only finitely many c_{j_k} are nonzero, as $y^*(l_1,\dots,l_{k-1})$ has finite support, and c_{j_k} are independent of l_k. Hence, $y^*(l_1,\dots,l_k)-y^*(l_1,\dots,l_{k-1})$ is a finite linear combination of $z^*_{j_k}(l_k)$. Now, we just need to notice that the sequences $(z^*_{j_k(l_k)})_{l_k=1}^\infty$ are subsequences of $(z^*_r)_{r=1}^\infty$, which is weak* null as the basis $(z_r)_{r=1}^\infty$ is seminormalized. Similarly we get that $y^*(l)$ weak* converges to x^* as $y^*(l)-x^*$ is a finite linear combination of weak* null sequences.

Hence, if we prove that the elements $y^*(l_1,\dots,l_n)\in A_n$, we get that $x^*\in A_n^{(n)}$. We will prove this using Proposition 1.4. For the sake of brevity let us denote $y^*=y^*(l_1,\dots,l_n)$ and show that $y^*\in A_n$. As we have already shown that y^* has finite support in $\mathbf{N}_n$, we just need to prove that the equations of Proposition 1.4 hold for y^*. Take admissible $(i_n,j_1,\dots,j_k)$ for $2\le k\le n$. Then

$$y^*(z_{j_k})=y^*(l_1,\dots,l_{n-1})(z_{j_k})=\cdots=y^*(l_1,\dots,l_{k-1})(z_{j_k})$$

as for $k\le m\le n$ we have that $y^*(l_1,\dots,l_m)-y^*(l_1,\dots,l_{m-1})$ has support in the sets of type $\mathbf{N}(i_n,\tilde{j}_1,\dots,\tilde{j}_m)$ (that is indexed by sequences of length $m+1$) and $j_k\in\mathbf{N}(i_n,j_1,\dots,j_{k-1})$, which is not a set of this type as $k\le m$. Likewise for $j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)$ we have

$$y^*(z_{j_{k+1}})=y^*(l_1,\dots,l_k)(z_{j_{k+1}}).$$

Then

$$\begin{aligned}\sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)}\frac{y^*(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}&=\sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)}\frac{y^*(l_1,\dots,l_k)(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}\\&=\sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)}\frac{y^*(l_1,\dots,l_{k-1})(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}+\\ +\left(y^*(l_1,\dots,l_{k-1})(z_{j_k})-\right.&\left.\sum_{j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)}\frac{y^*(l_1,\dots,l_{k-1})(z_{j_{k+1}})\beta_{j_{k-1}}}{\beta_{j_k}}\right)\\&=y^*(l_1,\dots,l_{k-1})(z_{j_k})=y^*(z_{j_k}).\end{aligned}$$

The second equality holds by the definition of $y^*(l_1,\dots,l_k)$ and the fact that $j_k(l_k)=j_{k+1}$ for exactly one $j_{k+1}\in\mathbf{N}(i_n,j_1,\dots,j_k)$.

Recall that the coefficients $(a_{j_1})_{j_1\in\mathbf{N}(i_n)}$ were chosen in such a way that

$$\delta(x^*)=\sum_{j_1\in\mathbf{N}(i_n)}\frac{a_{j_1}}{\beta_{j_1}}\quad\text{and}\quad\gamma(x^*)=\sum_{j_1\in\mathbf{N}(i_n)}\frac{a_{j_1}\alpha_{j_1}}{\beta_{j_1}}.$$

Hence, as $y^*(z_{j_2}) = y^*(l_1)(z_{j_2})$, we get by the definition of $y^*(l_1)$ and the fact that $z^*_{j_1(l)}(z_{j_2}) = 1$ for exactly one $z_{j_2} \in \mathbf{N}(i_n, j_1)$ and is zero otherwise that

$$\begin{aligned}
\sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{y^*(z_{j_2})}{\beta_{j_1}} &= \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} + \sum_{j_1\in\mathbf{N}(i_n)} \frac{a_{j_1}}{\beta_{j_1}} \\
&= \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} + \delta(x^*) = 1. \\
\sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{y^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} &= \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} + \sum_{j_1\in\mathbf{N}(i_n)} \frac{a_{j_1}\alpha_{j_1}}{\beta_{j_1}} \\
&= \sum_{\substack{j_1\in\mathbf{N}(i_n)\\ j_2\in\mathbf{N}(i_n,j_1)}} \frac{x^*(z_{j_2})}{\beta_{j_1}} + \gamma(x^*) = x^*(z_{i_n}) = y^*(z_{i_n}).
\end{aligned}$$

The last equality holds as z_{i_n} is not in the support of $y^* - x^*$. Therefore $y^* \in A_n$ and $x^* \in A_n^{(n)}$.

Now take $z^* \in \overline{A_n}^{w^*}$. As $\overline{A_n}^{w^*} \subseteq K$, z^* is norm limit of its partial sums by the virtue of Lemma 1.3. Therefore we just need to show that the partial sums of z^* are elements of $\overline{A_n^{(n)}}$. For any such partial sum v^* we have that $\delta(v^*) \geq \gamma(v^*) \geq 0$, as this holds on $\overline{A_n}^{w^*}$ and taking partial sums increases δ more than it increases γ. Then $v_k^* = (1 - k^{-1})v^*$ has finite support in $\mathbf{N}_n$ and $\delta(v_k^*) > \gamma(v_k^*) \geq 0$. Hence, by the previous part of the proposition, $v_k^* \in A_n^{(n)}$, $v^* = \lim v_k^* \in \overline{A_n^{(n)}}$ and $z^* \in \overline{A_n^{(n)}}$. □

Now we can formulate and prove the first main result of this paper.

Theorem 1.8. *Let X be a non-reflexive Banach space and $n \in \mathbb{N}$. Then there is a convex subset $B \subseteq X^*$ of order n.*

Proof. Lemma 1.2 gives us a subspace Z of the space X with semi-normalized basis $(z_n)_{n=1}^{\infty}$ with bounded partial sums. By Lemmata 1.6 and 1.7 there is a convex subset A_{n-1} of Z^* for which $A_n^{(n-1)} \subsetneq A_n^{(n)} = \overline{A_n}^{w^*}$. Let us consider the identity embedding $E : Z \to X$ and define $B = (E^*)^{-1}(A_{n-1})$. Then Lemma 1.1 gives us

$$B^{(n-1)} \subsetneq B^{(n)} = \overline{B}^{w^*}.$$

□

Now we prove that the convex set $A \subseteq Z^*$ has order $\omega + 1$. First we show that the positive cone K behaves nicely with respect to restrictions on subsets of $\mathbb{N}$.

For this sake we define, for $x^* = \sum_{k=1}^{\infty} x^*(z_k) z_k^* \in Z^*$ and $n \in \mathbb{N}$, the restriction of x^* on $\mathbf{N}_n$ as the formal sum $\sum_{k\in\mathbf{N}_n} x^*(z_k) z_k^*$. Note that in general this sum is not necessarily convergent in Z^*. If it is, we denote by $x^* \restriction \mathbf{N}_n$ its limit.

Lemma 1.9. *Let y^* be an element of K and $n \in \mathbb{N}$. Then $y^* \restriction \mathbf{N}_n$ is a well-defined element of K.*

Further, if we have a sequence $(x_k^*)_{k=1}^\infty \subseteq K$ which weak* converges to $x^* \in K$, then for all $n \in \mathbb{N}$ we have that $x_k^* \restriction \mathbf{N}_n \xrightarrow[k]{w^*} x^* \restriction \mathbf{N}_n$.

Proof. For the first part we notice that $\sum_{k \in \mathbf{N}_n} y^*(z_k) z_k^*$ is a subseries of the series $\sum_{k=1}^\infty y^*(z_k) z_k^*$, which is absolutely convergent by Lemma 1.3. Hence, it is also absolutely convergent. The fact that its limit is an element of K is clear by the definitions.

For the second part we first prove that the sequence $(x_k^* \restriction \mathbf{N}_n)_{k=1}^\infty$, which is well defined by the first part of the lemma, is bounded. Recall that, as the basis $(z_n)_{n=1}^\infty$ is seminormalized, the biorthogonal basic sequence $(z_n^*)_{n=1}^\infty$ is bounded by some constant $C_3 > 0$. Then for $k \in \mathbb{N}$

$$||x_k^* \restriction \mathbf{N}_n|| \leq \sum_{l \in \mathbf{N}_n} ||x_k^*(z_l) z_l^*|| \leq C_3 \sum_{l \in \mathbf{N}_n} x_k^*(z_l) \leq C_3 \sum_{l=1}^\infty x_k^*(z_l) \leq C\, C_3 ||x_k^*||.$$

We used that $x_k^* \in K$ and Lemma 1.3. Boundedness of $(x_k^* \restriction \mathbf{N}_n)_{k=1}^\infty$ now follows from the boundedness of the weak* convergent sequence $(x_k^*)_{k=1}^\infty$. Notice that the sequence $(x_k^* \restriction \mathbf{N}_n)_{k=1}^\infty$ converges to $x^* \restriction \mathbf{N}_n$ in the topology of pointwise convergence (that is the topology on Z^* generated by $\{z_k;\ k \in \mathbb{N}\} \subseteq Z$). Hence, as the topology of pointwise convergence is a weaker Hausdorff topology than the weak* topology, they coincide on bounded subsets of Z^*. Therefore, as the sequence of restrictions $(x_k^* \restriction \mathbf{N}_n)_{k=1}^\infty$ is bounded, it converges to $x^* \restriction \mathbf{N}_n$ also in the weak* topology. $\square$

Now let us recall that the set $A \subseteq K$ was defined as

$$A = \operatorname{conv} \bigcup_{n=1}^\infty A_n.$$

and that the sets A_n have support in the sets $\mathbf{N}_n$, which form a partition of $\mathbb{N}$.

Lemma 1.10. *Let x^* be an element of $A^{(k)}$ for some $k \in \omega$. Then for all $n \in \mathbb{N}$ there is $t_n \in [0,1]$ and $x_n^* \in A_n^{(k)}$ such that $x^* \restriction \mathbf{N}_n = t_n x_n^*$ and $\sum_{n=1}^\infty t_n \leq 1$.*

Proof. We will proceed by induction. For $k = 0$ the result follows by the definition of A, as any $x^* \in A$ is a convex combination $\sum_{n=1}^\infty t_n x_n^*$, where $x_n^* \in A_n$. Then $x^* \restriction \mathbf{N}_n = t_n x_n^*$ as the sets $\mathbf{N}_n$ are pairwise disjoint.

Now let us suppose that the lemma holds for $k \in \omega$ and take any $x^* \in A^{(k+1)}$. Then we can find a sequence $(x_l^*)_{l=1}^\infty \subseteq A^{(k)}$ which weak* converges to x^*. By the induction hypothesis we have

$$x_l^* \restriction \mathbf{N}_n = t_{l,n}\, x_{l,n}^*, \qquad x_{l,n}^* \in A_n^{(k)},\ t_{l,n} \in [0,1],\ \sum_{n=1}^\infty t_{l,n} \leq 1.$$

By Lemma 1.9 we have for each $n \in \mathbb{N}$

$$t_{l,n}\, x_{l,n}^* = x_l^* \restriction \mathbf{N}_n \xrightarrow[l]{w^*} x^* \restriction \mathbf{N}_n.$$

Now we can, up to passing to a subsequence, assume that $t_{l,n} \xrightarrow[l]{} t_n \in [0,1]$. If $t_n = 0$, we set x_n^* to be any element of $A_n^{(k+1)}$. Otherwise we set $x_n^* = \frac{x^* \restriction \mathbf{N}_n}{t_n}$, which is the weak* limit of the sequence $(x_{l,n}^*)_{l=1}^{\infty}$. In either case we have $x^* \restriction \mathbf{N}_n = t_n x_n^*$ where $t_n \in [0,1]$ and $x_n^* \in A_n^{(k+1)}$. It remains to show that $\sum_{n=1}^{\infty} t_n \leq 1$. For this we use the Fatou's lemma:

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \lim_{l\to\infty} t_{l,n} \leq \liminf_{l\to\infty} \sum_{n=1}^{\infty} t_{l,m} \leq 1.$$

$\square$

Lemma 1.11. *The order of A is at least $\omega + 1$.*

Proof. Consider the element $z^* = \sum_{n=1}^{\infty} 2^{-n} z_{i_n}^*$. Then $z^* \in \overline{A^{(\omega)}}$ as it is an infinite convex combination of the elements $z_{i_n}^*$ and by Lemma 1.6 we have that $z_{i_n}^* \in A_n^{(n+1)} \subseteq A^{(\omega)}$. Hence, we need to prove that z^* is not an element of $A^{(\omega)}$, that is to prove that it is not an element of any $A^{(m)}$, $m \in \mathbb{N}$. Suppose for a contradiction that $z^* \in A^{(m)}$ for some $m \in \mathbb{N}$. Then by Lemma 1.10 we have that

$$2^{-m-1} z_{i_{m+1}}^* = z^* \restriction \mathbf{N}_{m+1} = t\, z_{m+1}^* \qquad \text{for some } t \in (0,1],\ z_{m+1}^* \in A_{m+1}^{(m)}.$$

In other words, $z_{i_{m+1}}^*$ is a positive multiple of an element of $A_{m+1}^{(m)}$. But then by Proposition 1.5 we have

$$1 = z_{i_{m+1}}^*(z_{i_{m+1}}) = \sum_{\substack{j_1 \in \mathbf{N}(i_{m+1}) \\ j_2 \in \mathbf{N}(i_{m+1}, j_1)}} \frac{z_{i_{m+1}}^*(z_{j_2})\alpha_{j_1}}{\beta_{j_1}} = 0,$$

as $m \leq (m+1) - 1$. But this is a contradiction. Hence, $z^* \notin A^{(\omega)}$. $\square$

Lemma 1.12. *The order of A is at most $\omega + 1$. Specifically $\overline{A}^{w^*} = \overline{A^{(\omega)}}$.*

Proof. First we notice that for each $n \in \mathbb{N}$ it holds that 0 is an element of $A_n^{(\omega)}$ as $0 = \alpha_j z_{i_n}^* \in A_n^{(n)}$, where j is the first element of $\mathbf{N}(i_n)$ (see the paragraph preceding the definition of A_n).

Set

$$B = \left\{ \sum_{n=1}^{\infty} t_n x_n^*;\ x_n^* \in \overline{A_n}^{w^*},\ t_n \in [0,1],\ \sum_{n=1}^{\infty} t_n \leq 1 \right\}.$$

Then B is a subset of $\overline{A^{(\omega)}}$. To see this, cosider $x^* = \sum_{n=1}^{\infty} t_n x_n^* \in B$, where $x_n \in \overline{A_n}^{w^*}$, $t_n \in [0,1]$ and $\sum_{n=1}^{\infty} t_n \leq 1$. Now we will show that for each $N \in \mathbb{N}$ the partial sum $y_N^* = \sum_{n=1}^{N} t_n x_n^*$ is an element of $A^{(\omega)}$. By Lemma 1.7 we have for each $n = 1, \ldots, N$ that $\overline{A_n}^{w^*} = A_n^{(N+1)}$. Hence, for these $n = 1, \ldots, N$ we have $x_n^* \in \overline{A_n}^{w^*} = A_n^{(N+1)} \subseteq A^{(N+1)} \subseteq A^{(\omega)}$. But then, as $0 \in A^{(\omega)}$,

$$y_N^* = \sum_{n=1}^{N} t_n x_n^* + \left(1 - \sum_{n=1}^{N} t_n\right) \cdot 0 \in \operatorname{conv} A^{(\omega)} = A^{(\omega)}.$$

Therefore $x^* = \lim_{N\to\infty} y_N^* \in \overline{A^{(\omega)}}$ and $B \subseteq \overline{A^{(\omega)}}$.

Now we show that B is actually already weak* closed. As B is convex, it is enough to show that $B = B^{(1)}$ by the Krein-Šmulyan theorem. Let us have a sequence $(x_k^*)_{k=1}^\infty$ in B which weak* converges to $x^* \in B^{(1)}$. We want to show that $x^* \in B$. As $x_k^* \in B$, we can write it as $x_k^* = \sum_{n=1}^\infty t_{k,n} x_{k,n}^*$ with $x_{k,n}^* \in \overline{A_n}^{w^*}$, $t_{k,n} \in [0,1]$ and $\sum_{n=1}^\infty t_{k,n} \leq 1$. By Lemma 1.9 it holds for each $n \in \mathbb{N}$ that

$$t_{k,n} x_{k,n}^* = x_k^* \restriction \mathbf{N}_n \xrightarrow[k\to\infty]{w^*} x^* \restriction \mathbf{N}_n.$$

Now we can, using the diagonal argument to pass to a subsequence if necessary, assume that for each $n \in \mathbb{N}$ it holds that $t_{k,n} \underset{k\to\infty}{\longrightarrow} t_n$ for some $t_n \in [0,1]$. Set $y_n^* = 0$ if $t_n = 0$ and otherwise set $y_n^* = \frac{x^* \restriction \mathbf{N}_n}{t_n}$, which is the weak* limit of the sequence $(x_{k,n}^*)_{k=1}^\infty$. In either case we get that $x^* \restriction \mathbf{N}_n = t_n y_n^*$, where $y_n^* \in \left(\overline{A_n}^{w^*}\right)^{(1)} = \overline{A_n}^{w^*}$, $t_n \in [0,1]$ and $\sum_{n=1}^\infty t_n \leq 1$ (where the last inequality follows again from the Fatou's lemma). Now we notice that $x^* = \sum_{n=1}^\infty (x^* \restriction \mathbf{N}_n)$, as the series $x = \sum_{n=1}^\infty x^*(z_n) z_n^*$ is absolutely convergent and the sets $\mathbf{N}_n$, $n \in \mathbb{N}$, form a partition of $\mathbb{N}$.

Now, as obviously $A \subseteq B$, we have

$$B \subseteq \overline{A^{(\omega)}} \subseteq \overline{A}^{w^*} \subseteq \overline{B}^{w^*} = B.$$

Therefore we have equalities and specifically $\overline{A}^{w^*} = \overline{A^{(\omega)}}$. $\square$

Now we are all prepared to prove the second main theorem of this paper.

Theorem 1.13. *Let X be a non-reflexive Banach space. Then there is a convex set $B \subseteq X^*$ of order $\omega + 1$.*

Proof. Lemma 1.2 gives us a subspace Z of the space X with semi-normalized basis $(z_n)_{n=1}^\infty$ with bounded partial sums. By Lemmata 1.11 and 1.12 there is a convex subset A of Z^* for which $A^{(\omega)} \subsetneq A^{(\omega+1)} = \overline{A}^{w^*}$. Let us consider the identity embedding $E : Z \to X$ and define $B = (E^*)^{-1}(A_{n-1})$. Then Lemma 1.1 gives us

$$B^{(\omega)} \subsetneq B^{(\omega+1)} = \overline{B}^{w^*}.$$

$\square$

1.3 Remarks and open problems

The order of any subset of the dual of a separable space must be a countable ordinal (see e.g. [16]). It follows from the Baire category theorem, that the order of a subspace of the dual of a separable Banach space cannot be a limit ordinal. This approach, however, cannot be used for convex sets. So the following question still remains open.

Question 1. *Can the order of a convex subset of the dual to a separable Banach space be a limit ordinal?*

Ostrovskii proved in [25] that in the dual of any non-quasi-reflexive separable Banach space we can find for any non-limit ordinal $\alpha < \omega_1$ a subspace of order α. Can we prove an analogous statement for convex subsets of duals of non-reflexive quasi-reflexive Banach spaces?

Question 2. *Let X be a non-reflexive quasi-reflexive Banach space. Are there any convex subsets of X^* with order higher than $\omega + 1$?*[1]

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[1] This question was answered positively by Ostrovskii [24] – there are indeed convex subsets of arbitrary countable non-limit order in dual of any non-reflexive space.

2. On subspaces whose weak* derived sets are proper and norm dense

Abstract: We study long chains of iterated weak* derived sets, that is sets of all weak* limits of bounded nets, of subspaces with the additional property that the penultimate weak* derived set is a proper norm dense subspace of the dual. We extend the result of Ostrovskii and show, that in the dual of any non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual, we can find for any countable successor ordinal α a subspace, whose weak* derived set of order α is proper and norm dense.

2.1 Introduction

The *weak* derived set* of a subset A of a dual space X^* is defined as the set of all weak* limits of bounded nets from A, i.e.

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{w^*}.$$

If X is separable, bounded sets of X^* are weak* metrizable, and thus the weak* derived set $A^{(1)}$ coincides with the *weak* sequential closure* of A, that is the set of all weak* limits of sequences from A. The study of weak* sequential closures of subspaces in duals of separable spaces was initiated by Banach [5] and his school in 1930's. Later, weak* derived sets and weak* sequential closures found significant applications. To name a few, they were applied by Piatetski-Shapiro [30] to characterization of sets of uniqueness in harmonic analysis, used by Saint-Raymond [32] for Borel and Baire classification of inverses of continuous injective linear operators (see also [31] for application for non-separable spaces), by Dierolf and Moscatelli [12] in the structure theory of Fréchet spaces, or by Plichko [29] to solve a problem on universal Markushevich bases posed by Kalton. For additional information and a historical account, see the survey on weak* sequential closures by Ostrovskii [26] and the introduction to his new paper [24].

The weak* derived set needs not to be closed under taking weak* limits of bounded nets, that is $A^{(1)}$ can be a proper subset of $\left(A^{(1)}\right)^{(1)}$, even if A is a subspace. This inspires the definition of weak* derived sets of higher order. We use the convention that $A^{(0)} = A$. For a successor ordinal α, the weak* derived set of A of order α is $A^{(\alpha)} = \left(A^{(\alpha-1)}\right)^{(1)}$. For a limit ordinal α we define $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)}$. The order of A is defined to be the least ordinal α, such that $A^{(\alpha)} = A^{(\alpha+1)}$. Note that it follows from the Krein-Šmulyan theorem that if A is convex, then $A = A^{(1)}$ if and only if A is weak* closed. Hence, if A is convex, the order of A is the least ordinal α such that $A^{(\alpha)} = \overline{A}^{w^*}$.

It is readily proved that a subspace A of X^* is norming, if and only if $A^{(1)} =$

X^*. Recall that A is said to be *norming* if

$$q_A(x) = \sup\{|f(x)| : \; f \in A \cap B_{X^*}\}$$

defines an equivalent norm on X. Davis and Lindenstrauss [11] have shown that a Banach space is quasi-reflexive if and only if every subspace of its dual which separates points is also norming. Recall that a Banach space X is *quasi-reflexive*, if it is of finite codimension in its bidual. It thus follows by a quotient argument that a Banach space X is quasi-reflexive if and only if $A^{(1)} = \overline{A}^{w^*}$ for every subspace A of X^*, or in other words, if and only if the order of any subspace of X^* is at most one. The study of possible orders of subspaces in duals of separable non-quasi-reflexive spaces was completed by the following theorem of Ostrovskii [25]:

Theorem 2.1. *Let X be a separable non-quasi-reflexive Banach space. Then for every countable successor ordinal α there is a subspace A of X^* of order α.*

Further, the orders of subspaces in a dual to a separable Banach space must be countable and cannot be limit, see for example [16]. Later, García, Kalenda and Maestre [15] asked the following questions in their paper about extension problems for holomorphic functions on dual Banach spaces.

- Let X be a quasi-reflexive Banach space. Is it true that $A^{(1)} = \overline{A}^{w^*}$ for every convex set A in X^*?
- For which Banach space X does there exist a subspace A of X^* such that $A^{(1)}$ is a proper norm dense subspace of X^*?

Both of these questions were solved by Ostrovskii in his paper [27]. He showed that $A^{(1)} = \overline{A}^{w^*}$ for every convex subset A of X^*, if and only if X is reflexive. This result was later developed by the author [33][1] and Ostrovskii [24] in the spirit of Theorem 2.1. Let us note that it is still an open problem if the order of a convex set can be a countable limit ordinal. Regarding the second question, Ostrovskii proved the following theorem [27, Theorem 1]:

Theorem 2.2. *The dual Banach space X^* contains a linear subspace A such that $A^{(1)}$ is a proper norm dense subset of X^*, if and only if X is a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual.*

The aim of this paper is to extend the result of Theorem 2.2 for higher ordinals in the spirit of Theorem 2.1. The results will be valid for both real and complex scalars.

We will use the following notation. We write $\mathbb{F}$ for the underlying field $\mathbb{R}$ or $\mathbb{C}$. For a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X we denote its closed linear span by $[x_n]_{n=1}^{\infty}$. Analogically, $[x_n]_{n=1}^{N}$ will denote the linear span of $(x_n)_{n=1}^{N}$. Recall that a couple $(\{x_i\}_{i\in I}, \{x_i^*\}_{i\in I})$ is called a *biorthoganal system* in X if for all $i, j \in I$ we have $x_j^*(i) = \delta_{i,j}$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. An indexed family $\{x_i\}_{i\in I}$ in X is said to be *minimal* if there is an indexed family $\{x_i^*\}_{i\in I}$ in X^*, such that $(\{x_i\}_{i\in I}, \{x_i^*\}_{i\in I})$ forms a biorthoganal system. For a subset A of a Banach space X we denote by $A^{\perp}$ the *annihilator* of A in X^*, that is $A^{\perp} = \{x^* \in X^* : \; x^*(a) = 0 \text{ for all } a \in A\} \subseteq X^*$. For a subser B of X^* we denote by $B_{\perp}$ the *preannihilator* of B in X, that is $B_{\perp} = \bigcap_{b\in B} \operatorname{Ker} b \subseteq X$.

[1] The first paper of this thesis.

2.2 The results

The statement of the main theorem is the following.

Theorem 2.3. *Let X be a Banach space. Then the following are equivalent:*

1. *X is non-quasi-reflexive and contains an infinite-dimensional subspace with separable dual.*
2. *There is a subspace A in X^*, such that $A^{(1)}$ is a proper norm dense subspace of X^*.*
3. *For each countable successor ordinal α there is a subspace A in X^*, such that $A^{(\alpha)}$ is a proper norm dense subspace of X^*.*

The equivalence *(1)* $\iff$ *(2)* follows from Theorem 2.2 and clearly *(3)* $\implies$ *(2)*. The strategy to prove the implication *(1)* $\implies$ *(3)* is to combine the construction from [25], that is the proof of Theorem 2.1, and the construction from [27], which is the proof of Theorem 2.2. We will first find a subspace W of X spanned by a nice biorthogonal system and find a suitable subspace K in W^*. In the following lemma we use the notation $n_k = \frac{k(k+1)}{2}$, $k \in \mathbb{N}$. Then the sequences $\{(n_m + 0)_{m=1}^{\infty}\} \cup \{(n_m + i - 1)_{m=i-1}^{\infty} : i \geq 2\}$ form a partition of $\mathbb{N}$, as illustrated in the following table.

$(n_m+0)_{m=1}^{\infty}$	$(n_m+1)_{m=1}^{\infty}$	$(n_m+2)_{m=2}^{\infty}$	$(n_m+3)_{m=3}^{\infty}$	$(n_m+4)_{m=4}^{\infty}$	$\dots$
1	2				
3	4	5			
6	7	8	9		
10	11	12	13	14	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lemma 2.4. *[27, Lemma 2] Let X be a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual. Then there is a minimal system*

$$\mathcal{W} = \{x_n\}_{n\in\mathbb{N}} \cup \{u_n\}_{n\in\mathbb{N}}$$

satisfying:

(i) *$\mathcal{W}$ and its biorthogonal functionals are uniformly bounded.*

(ii) *The sequence $(u_n)_{n\in\mathbb{N}}$ is a shrinking basic sequence.*

(iii) *The sequences $(x_{n_p})_{p=1}^{\infty}$ and $(x_{n_p+j-1})_{p=j-1}^{\infty}$, $j \geq 2$, have uniformly bounded partial sums.*

(iv) *The sequence of subspaces*

$$[x_1, x_2, u_1], [x_3, x_4, x_5, u_2], \dots, [x_{n_p}, x_{n_p+1} \dots, x_{n_p+p}, u_p], \dots$$

forms a finite-dimensional decomposition of $W := \overline{\operatorname{span}}\mathcal{W}$.

Note that assertions $(i) - (iii)$ follow from the statement of [27, Lemma 2]. Assertion (iv) is shown within its proof. We will now prove that assertion *(3)* of Theorem 2.3 is satisfied for the space W. For information about shrinking bases see Section 3.2. of [1] or Section 1.b. of [20]. For more information about finite-dimensional decompositions see Section 1.g. od [20].

Notation 2.5. We will further use the following notation.

1. For $p \in \mathbb{N}$ we set $x_p^1 = x_{n_p}$. For $j \geq 2$ and $p \in \mathbb{N}$ we set $x_p^j = x_{n_{p+j-2}+j-1}$. In this notation, the sequences $(x_p^j)_{p=1}^\infty$, $j \in \mathbb{N}$, form a partition of the set $\{x_n\}_{n=1}^\infty$ and have uniformly bounded partial sums by Lemma 2.4 *(iii)*.

2. For $w \in \mathcal{W}$ we shall denote by $\widetilde{w}$ the biorthogonal functional of w (with respect to $\mathcal{W}$). We differ from the usual notation w^* since we use upper indices for some elements of $\mathcal{W}$.

3. For each $n \in \mathbb{N}$ we fix a weak* cluster point f_n of the sequence $\left(\sum_{j=1}^k x_j^n\right)_{k=1}^\infty$ in W^{**}. Note that its existence is guaranteed by Lemma 2.4 (iii) and 1. and that those elements are also uniformly bounded.

4. For $n \in \mathbb{N}$ we write P_n for the canonical projection onto

$$[x_1, x_2, u_1, x_3, x_4, x_5, u_2, \dots, x_{n_n}, x_{n_n+1} \dots, x_{n_n+n}, u_n].$$

   The projections P_n, $n \in \mathbb{N}$, are uniformly bounded by Lemma 2.4 (iv) and the properties of finite-dimensional decompositions, see the discussion after [20, Definition 1.g.1.]. The adjoint P_n^* is then a projection of W^* onto

$$[\widetilde{x}_1, \widetilde{x}_2, \widetilde{u}_1, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{u}_2, \dots, \widetilde{x}_{n_n}, \widetilde{x}_{n_n+1} \dots, \widetilde{x}_{n_n+n}, \widetilde{u}_n].$$

   These projections satisfy $P_n^*(x^*) \xrightarrow{w^*} x^*$ for each $x^* \in W^*$ (again, see the discussion after [20, Definition 1.g.1.]).

5. Let $i, k \in \mathbb{N}$. We say that $g \in W^{**}$ is of type $t(i,k)$, if either

   - $i \neq k$ and $g = x_j^i + af_k$ for some $a \in \mathbb{F}$ and $j \in \mathbb{N}$, or
   - $g = u_i + af_k$ for some $a \in \mathbb{F}$.

   Let $A \subseteq N$. We say that a vector g of type $t(i,k)$ is compatible with $A \subseteq \mathbb{N}$ if

   - $g = x_j^i + af_k$ and $i, k \notin A$ or
   - $g = u_i + af_k$ and $k \notin A$.

6. We denote the closed span of $(u_n)_{n\in\mathbb{N}}$ by U. Then by Lemma 2.4 *(ii)* the sequence $(\widetilde{u}_n \restriction U)_{n=1}^\infty$ is a basis of U^*.

7. We will say that a vector $z \in W$ is finitely supported if $z \in \operatorname{span} \mathcal{W}$. Similarly, we will say that a vector $z^* \in W^*$ is finitely supported if $z^* \in \operatorname{span}\{\widetilde{w} :\ w \in \mathcal{W}\}$.

Construction 2.6. For $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ infinite with infinite complement and a vector $z + af_k$ of type $t(i,k)$ compatible with A, we define sets $\Omega(\alpha, A, z + af_k)$ in the following recursive way:

- $\Omega(0, A, z + af_k) = \{z + af_k\}$;
- If $\alpha > 0$ is a successor ordinal, we split A into infinitely many infinite subsets $(A_n)_{n=0}^{\infty}$ and take a summable sequence $(a_n)_{n=1}^{\infty}$ of positive numbers. Let $(p(n))_{n=1}^{\infty}$ be the increasing enumeration of A_0. We set

$$\Omega(\alpha, A, z + af_k) = \{z + af_k\} \cup \bigcup_{n=1}^{\infty} \Omega(\alpha - 1, A_n, x_n^k + a_n f_{p(n)}).$$

- If $\alpha > 0$ is a countable limit ordinal, we fix an increasing sequence $(\alpha_n)_{n=1}^{\infty}$ of ordinals such that $\alpha_n \to \alpha$ and again split A into infinitely many infinite subsets $(A_n)_{n=0}^{\infty}$ and take a summable sequence $(a_n)_{n=1}^{\infty}$ of positive numbers. Let $(p(n))_{n=1}^{\infty}$ be the increasing enumeration of A_0. We set

$$\Omega(\alpha, A, z + af_k) = \{z + af_k\} \cup \bigcup_{n=1}^{\infty} \Omega(\alpha_n, A_n, x_n^k + a_n f_{p(n)}).$$

- We set $K(\alpha, A, z + af_k) = (\Omega(\alpha, A, z + af_k))_{\perp}$.

Remark. Note that if we set $\alpha_n = \alpha - 1$ for a successor ordinal $\alpha > 0$ and $n \in \mathbb{N}$, we can cover both cases of successor or limit α by the definition

$$\Omega(\alpha, A, z + af_k) = \{z + af_k\} \cup \bigcup_{n=1}^{\infty} \Omega(\alpha_n, A_n, x_n^k + a_n f_{p(n)}). \tag{$*$}$$

We will use this notation later, if the proofs do not depend on whether α is a successor or limit ordinal.

Lemma 2.7. *Let $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ infinite with infinite complement and a vector $z + af_k$ of type $t(i,k)$ compatible with A. Then every element of $\Omega(\alpha, A, z + af_k)$ is either $z + af_k$ or an element of type $t(l,m)$ for $l \in A \cup \{k\}$ and $m \in A$.*

Proof. We shall proceed by induction over α. If $\alpha = 0$, the only element of $\Omega(0, A, z + af_k)$ is $z + af_k$ and the claim holds. Suppose the claim holds for all $\beta < \alpha$. By Construction 2.6 and the remark following it

$$\Omega(\alpha, A, z + af_k) = \{z + af_k\} \cup \bigcup_{n=1}^{\infty} \Omega(\alpha_n, A_n, x_n^k + a_n f_{p(n)}).$$

It follows that any element of $\Omega(\alpha, A, z + af_k)$ is either $z + af_k$ or an element of $\Omega(\alpha_n, A_n, x_n^k + a_n f_{p(n)})$ for some $n \in \mathbb{N}$. By the induction hypothesis, for all $n \in \mathbb{N}$, all elements of $\Omega(\alpha_n, A_n, x_n^k + a_n f_{p(n)})$ are either $x_n^k + a_n f_{p(n)}$ or of type $t(l,m)$ where $l \in A_n \cup \{p(n)\} \subseteq A$ and $m \in A_n \subseteq A$. Hence, in either case, they are of type $t(l,m)$ where $l \in A \cup \{k\}$ and $m \in A$. □

Proposition 2.8. *Let $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ infinite with infinite complement and let $z + af_k$ be of type $t(i,k)$ compatible with A. Then $(K(\alpha, A, z + af_k))^{(\alpha)} \subseteq \operatorname{Ker}(z + af_k)$.*

Proof. We will proceed by induction over α. It obviously holds for $\alpha = 0$ as $z + af_k \in \Omega(\alpha, A, z + af_k)$. Let us suppose that the claim holds for all ordinals smaller than α. We will prove that $(K(\alpha, A, z + af_k))^{(\beta)} \subseteq \mathrm{Ker}(z + af_k)$ for all $\beta \leq \alpha$ by induction over β. Again, the case $\beta = 0$ is clear. Now, suppose it holds for some $\beta < \alpha$. Take $y^* \in (K(\alpha, A, z + af_k))^{(\beta+1)}$. Then there is a sequence $(y_n^*)_{n=1}^\infty$ of elements in $(K(\alpha, A, z + af_k))^{(\beta)}$ which weak* converges to y^*. Recall that by Construction 2.6 there is $n_0 \in \mathbb{N}$ such that $\beta \leq \alpha_n$ for all $n \geq n_0$. Hence,

$$\begin{aligned}(K(\alpha, A, z + af_k))^{(\beta)} &= \left(\mathrm{Ker}(z + af_k) \cap \bigcap_{n=1}^{\infty} K(\alpha_n, A_n, x_n^k + a_n f_{p(n)})\right)^{(\beta)} \\ &\subseteq \bigcap_{n=n_0}^{\infty} \left(K(\alpha_n, A_n, x_n^k + a_n f_{p(n)})\right)^{(\beta)} \\ &\subseteq \bigcap_{n=n_0}^{\infty} \mathrm{Ker}(x_n^k + a_n f_{p(n)}),\end{aligned}$$

where the first equality follows from $(*)$ and the last inclusion follows from the induction hypothesis for $\beta \leq \alpha_n$ and for $n \geq n_0$. Hence, there is $C > 0$ such that for every $n \in \mathbb{N}$ and $j \geq n_0$ we have

$$|y_n^*(x_j^k)| = a_j |f_{p(j)}(y_n^*)| \leq C\, a_j. \tag{2.1}$$

Indeed, the sequence $(y_n^*)_{n=1}^\infty$ is weak* convergent and thus bounded, and $(f_{p(j)})_{j=1}^\infty$ is also bounded, see point 3. of Notation 2.5.

We will show that $f_k(y_n^*) \xrightarrow{n} f_k(y^*)$. Fix $\epsilon > 0$. By inequality (2.1) there is $m_0 \geq n_0$ such that

$$\sum_{j=m_0}^{\infty} |y_n^*(x_j^k)| < \epsilon/8. \tag{2.2}$$

As f_k is a weak* cluster point of $\left(\sum_{j=1}^m x_j^k\right)_{m=1}^\infty$, there exists $m_1 > m_0$ such that

$$|f_k(y^*) - \sum_{j=1}^{m_1} y^*(x_j^k)| < \epsilon/4. \tag{2.3}$$

Let us now show that for any $n \in \mathbb{N}$ it holds that

$$|f_k(y_n^*) - \sum_{j=1}^{m_1} y_n^*(x_j^k)| < \epsilon/4. \tag{2.4}$$

Indeed, for any $n \in \mathbb{N}$ there is $r_n > m_1$ such that $|f_k(y_n^*) - \sum_{j=1}^{r_n} y_n^*(x_j^k)| < \epsilon/8$. Since $m_1 > m_0$, it follows from (2.2) that

$$\begin{aligned}|f_k(y_n^*) - \sum_{j=1}^{m_1} y_n^*(x_j^k)| &\leq |f_k(y_n^*) - \sum_{j=1}^{r_n} y_n^*(x_j^k)| + |\sum_{j=m_1+1}^{r_n} y_n^*(x_j^k)| \\ &\leq \epsilon/8 + \sum_{j=m_1+1}^{\infty} |y_n^*(x_j^k)| < \epsilon/8 + \epsilon/8 < \epsilon/4.\end{aligned}$$

It follows easily from the fact that $(y_n^*)_{n=1}^\infty$ is weak* convergent to y^* that there is $n' \in \mathbb{N}$ such that for all $n > n'$ it holds that

$$|\sum_{j=1}^{m_1} y_n^*(x_j^k) - \sum_{j=1}^{m_1} y^*(x_j^k)| < \epsilon/4. \tag{2.5}$$

By applying the triangle inequality and (2.3), (2.4) and (2.5), we finally get that for all $n > n'$

$$|f_k(y_n^*) - f_k(y^*)| \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon.$$

But this precisely means that indeed $f_k(y_n^*) \xrightarrow{n} f_k(y^*)$.

Since $z \in W$ and thus $y_n^*(z) \to y^*(z)$, it follows that

$$(z + af_k)(y^*) = \lim_n (z + af_k)(y_n^*) = 0$$

as $y_n^* \in \operatorname{Ker}(z + af_k)$ by the induction hypothesis. Hence, $y^* \in \operatorname{Ker}(z + af_k)$. What is left is the induction step for a limit ordinal β which follows easily from the definition of weak* derived sets for limit ordinals. □

Definition. Let $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ infinite with infinite complement, $z + af_k$ a vector of type $t(i,k)$ compatible with A. For $d \in \mathbb{F}$ define

$$Q_d(\alpha, A, z + af_k) = K(\alpha, A, z + af_k) \cap \left(d\widetilde{z} + \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ t \in A \cup \{k\}\}\right).$$

That is, $Q_d(\alpha, A, z + af_k)$ are those elements from $K(\alpha, A, z + af_k)$ which have finite support in the relevant set and have d as the coefficient at $\widetilde{z}$.

Lemma 2.9. *Let $k, i, j \in \mathbb{N}$. Then $f_k(\widetilde{x}_j^i) = \delta_{k,i}$ and $f_k(\widetilde{u}_i) = 0$.*

Proof. Note that $f_k(\widetilde{u}_i)$ is a cluster point of the sequence $\left(\widetilde{u}_i\left(\sum_{l=1}^m x_l^k\right)\right)_{m=1}^\infty$. It thus follows from biorthogonality that $f_k(\widetilde{u}_i)$ is a cluster point of a sequence of zeroes, and thus $f_k(\widetilde{u}_i) = 0$. Further, $f_k(\widetilde{x}_j^i)$ is a cluster point of the sequence $\left(\widetilde{x}_j^i\left(\sum_{l=1}^m x_l^k\right)\right)_{m=1}^\infty$, and again by biorthogonality, $\widetilde{x}_j^i\left(\sum_{l=1}^m x_l^k\right) = 1$ if $i = k$ and $m \geq l$, and $\widetilde{x}_j^i\left(\sum_{l=1}^m x_l^k\right) = 0$ otherwise. Hence, if $i \neq k$, $f_k(\widetilde{x}_j^i) = 0$ as it is a cluster point of a sequence of zeroes, and if $i = k$, $f_k(\widetilde{x}_j^i) = 1$ as is is a cluster point of a sequence which eventually equals one. □

Lemma 2.10. *Let $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ infinite with infinite complement, $z + af_k$ a vector of type $t(i,k)$ compatible with A. Let $b \in \mathbb{F}$ and $y^* \in b\widetilde{z} + \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ t \in A \cup \{k\}\}$. Then*

(a) $y^* \in (Q_b(\alpha, A, z + af_k))^{(\alpha+1)}$;

(b) *If moreover $y^* \in \operatorname{Ker}(z + af_k)$, then $y^* \in (Q_b(\alpha, A, z + af_k))^{(\alpha)}$.*

Proof. First we will show that *(b)* implies *(a)*. Indeed, if we set for $n \in \mathbb{N}$

$$y_n^* = y^* - \frac{1}{a}(z + af_k)(y^*)\widetilde{x}_n^k,$$

then for each $n \in \mathbb{N}$ clearly $y_n^* \in b\widetilde{z} + \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ t \in A \cup \{k\}\}$. Also, by Lemma 2.9, $(z + af_k)(\widetilde{x}_n^k) = a$, and thus $y_n^* \in \operatorname{Ker}(z + af_k)$. It thus follows from *(b)* that $y_n^* \in (Q_b(\alpha, A, z + af_k))^{(\alpha)}$. Further, the sequence $(y_n^*)_{n=1}^\infty$ weak* converges to y^* as the sequence $(\widetilde{x}_n^k)_{n=1}^\infty$ is weak* null. Indeed, $(\widetilde{x}_n^k)_{n=1}^\infty$ is bounded and pointwise null (that is converging to zero in the topology generated by $\mathcal{W}$), and hence it is also weak* null as these topologies coincide on bounded sets. Thus, $y^* \in (Q_b(\alpha, A, z + af_k))^{(\alpha+1)}$ and *(a)* is true.

We will prove *(b)* by induction over α. The case $\alpha = 0$ is clear as $K(0, A, z + af_k) = \operatorname{Ker}(z + af_k)$, see Construction 2.6. Let us suppose that both *(a)* and *(b)* hold for all $\beta < \alpha$ and take $y^* \in \operatorname{Ker}(z + af_k)$ as in the statement of the lemma, that is

$$y^* = b\widetilde{z} + \sum_{j=1}^{m} \left(c_j \widetilde{x}_j^k + v_j^* \right),$$

where $c_j \in \mathbb{F}$ and $v_j^* \in \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ s \in A_j \cup \{p(j)\}\}$ (recall that A_j and $p(j)$ are defined in Construction 2.6). It follows from Lemma 2.9 and the fact that $y^* \in \operatorname{Ker}(z + af_k)$ that $\sum_{j=1}^m c_j = -\frac{b}{a}$. Indeed, $y^*(z) = b$ and $af_k(y^*) = a\sum_{j=1}^m c_j$. Recall that by Construction 2.6 and $(*)$

$$K(\alpha, A, z + af_k) = \operatorname{Ker}(z + af_k) \cap \bigcap_{n=1}^{\infty} K(\alpha_n, A_n, x_n^k + a_n f_{p(n)}).$$

It follows from *(a)* of the induction hypothesis for $\alpha_j < \alpha$, $j = 1, \dots, m$, that

$$\begin{aligned} c_j \widetilde{x}_j^k + v_j^* &\in \left(Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)}) \right)^{(\alpha_j+1)} \\ &\subseteq \left(Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)} \right)^{(\alpha)}. \end{aligned}$$

Let us now show that

$$b\widetilde{z} + \sum_{j=1}^{m} Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)}) \subseteq Q_b(\alpha, A, z + af_k). \tag{2.6}$$

Indeed, let us fix an element of the set on the left-hand side of (2.6),

$$w^* = b\widetilde{z} + \sum_{j=1}^{m} w_j^*,$$

where $w_j^* \in Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)})$. For later convenience set $w_j^* = 0$ for $j > m$. As $A_0 = \{p(j)\}_{j=1}^\infty$, and thus

$$A \cup \{k\} = \{k\} \cup \bigcup_{j=1}^{\infty} (A_j \cup \{p(j)\}),$$

we get that $w^* \in b\widetilde{z} + \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ t \in A \cup \{k\}\}$. What is left is to show is that $w^* \in K(\alpha, A, z + af_k) = (\Omega(\alpha, A, z + af_k))_\perp$. Take any $g \in \Omega(\alpha, A, z + af_k)$. Then by Construction 2.6 either $g = z + af_k$ or $g \in \Omega(\alpha_j, A_j, x_j^k + a_j f_{p(j)})$ for some $j \in \mathbb{N}$. We shall first deal with the second case. By Lemma 2.7, either g is of type $t(l_1, l_2)$, where $l_1 \in A_j \cup \{p(j)\}$ and $l_2 \in A_j$, or $g = x_j^k + a_j f_{p(j)}$. In both cases, as the sets $A_l \cup \{p(l)\}$, $l \in \mathbb{N}$, are disjoint, we get, using Lemma 2.9, that $g(w_l^*) = 0$ for $l \neq j$. Hence, as $w_j^* \in K(\alpha_j, A_j, x_j^k + a_j f_{p(j)})$,

$$g(w^*) = g(w_j^*) = 0.$$

Now we deal with the case $g = z + af_k$. We have that

$$g(w^*) = (z + af_k)(w^*) = b + \sum_{j=1}^{m} ac_j = 0$$

as $w_j^* \in Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)})$ and $\sum_{j=1}^{m} c_j = -\frac{b}{a}$. We have shown that for any $g \in \Omega(\alpha, A, z + af_k)$, $g(w^*) = 0$, and thus $w^* \in K(\alpha, A, z + af_k)$. Hence, (2.6) is proved.

Finally,

$$\begin{aligned} y^* = b\widetilde{z} + \sum_{j=1}^{m} \left(c_j \widetilde{x}_j^k + v_j^*\right) &\in b\widetilde{z} + \sum_{j=1}^{m} \left(Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)})\right)^{(\alpha)} \\ &\subseteq \left(b\widetilde{z} + \sum_{j=1}^{m} Q_{c_j}(\alpha_j, A_j, x_j^k + a_j f_{p(j)})\right)^{(\alpha)} \subseteq (Q_b(\alpha, A, z + af_k))^{(\alpha)} \end{aligned}$$

and *(b)* is proved. □

Corollary 2.11. *Let $\alpha < \omega_1$, $A \subseteq \mathbb{N}$ be infinite with infinite complement and let $z + af_k$ be a vector of type $t(i,k)$ which is compatible with A. Then $W^* = (K(\alpha, A, z + af_k))^{(\alpha+1)}$. Moreover, any finitely supported $y^* \in \mathrm{Ker}(z + af_k)$ is an element of $(K(\alpha, A, z + af_k))^{(\alpha)}$.*

Proof. Let us first prove the second statement. Take a finitely supported vector $y^* \in \mathrm{Ker}(z + af_k)$. As y^* is finitely supported, we have

$$y^* = \sum_{w \in \mathcal{W}} y^*(w)\widetilde{w},$$

where only finitely many of the summands are nonzero. We can thus set

$$z^* = \sum_{w \in \mathcal{W} \setminus (\{z\} \cup \{x_s^t :\, s \in \mathbb{N},\, t \in A \cup \{k\}\})} y^*(w)\widetilde{w}.$$

Then $y^* - z^* \in y^*(z)\widetilde{z} + \mathrm{span}\{\widetilde{x}_s^t :\ s \in \mathbb{N}, t \in A \cup \{k\}\}$ and $y^* - z^* \in \mathrm{Ker}(z + af_k)$ by Lemma 2.9. It follows from Lemma 2.10 that $y^* - z^* \in (K(\alpha, A, z + af_k))^{(\alpha)}$. Hence, also $y^* \in (K(\alpha, A, z + af_k))^{(\alpha)}$, as $z^* \in K(\alpha, A, z + af_k)$ by Lemmata 2.7 and 2.9.

Now let us prove the first statement. Take any $y^* \in W^*$ and define for $n \in \mathbb{N}$

$$y_n^* = P_n^*(y^*) - \frac{1}{a}(z + af_k)(P_n^*(y^*))\widetilde{x}_n^k.$$

Then each y_n^* is finitely supported and is also an element of $\mathrm{Ker}(z + af_k)$ as by Lemma 2.9 we have that $(z + af_k)(\widetilde{x}_n^k) = a$. Hence, y_n^* is an element of $(K(\alpha, A, z + af_k))^{(\alpha)}$ by the already proved part of the corollary. It thus follows that y^*, which is the weak* limit of the sequence $(y_n^*)_{n=1}^{\infty}$ as the sequence $(\widetilde{x}_n^k)_{n=1}^{\infty}$ is weak* null, is an element of $(K(\alpha, A, z + af_k))^{(\alpha+1)}$. □

Theorem 2.12. *Let $0 < \alpha < \omega_1$ be a successor ordinal and $(a_n)_{n=1}^{\infty}$ be a summable sequence of positive numbers. Let $(A_n)_{n=0}^{\infty}$ be a partition of $\mathbb{N}$ into countably many infinite subsets and let $(q(n))_{n=1}^{\infty}$ be the increasing enumeration of A_0. Set*

$$K = \bigcap_{n=1}^{\infty} K(\alpha - 1, A_n, u_n + a_n f_{q(n)}).$$

Then $K^{(\alpha)} \subsetneq \overline{K^{(\alpha)}} = W^*$.

Proof. We will prove the following claims:

Claim 1 $K^{(\alpha)} \neq W^*$. Indeed, by Proposition 2.8

$$K^{(\alpha-1)} \subseteq \bigcap_{n=1}^{\infty} \operatorname{Ker}(u_n + a_n f_{q(n)}).$$

Hence, there is $C > 0$ such that any functional $y^* \in K^{(\alpha-1)}$ of norm at most one satisfies for each $n \in \mathbb{N}$

$$|y^*(u_n)| = a_n |f_{q(n)}(y^*)| \leq C a_n$$

(recall that $(f_{q(n)})_{n=1}^{\infty}$ is bounded). It follows that $K^{(\alpha-1)}$ is not norming. Indeed, if $K^{(\alpha-1)}$ was norming, the sequence $(u_n)_{n=1}^{\infty}$ would be norm null. But this cannot happen as the biorthogonal functionals $(\widetilde{u}_n)_{n=1}^{\infty}$ are bounded. Hence, $K^{(\alpha)} = \left(K^{(\alpha-1)}\right)^{(1)} \neq W^*$.

Claim 2 $\widetilde{u}_n \in K^{(\alpha)}$ for each $n \in \mathbb{N}$. It follows from Lemma 2.10 *(a)* that $\widetilde{u}_n \in \left(Q_1(\alpha - 1, A_n, u_n + a_n f_{q(n)})\right)^{(\alpha)}$. Further, it follows from Lemma 2.7 and Lemma 2.9 that for any $j \neq n$ we have that $Q_1(\alpha - 1, A_n, u_n + a_i f_{q(n)}) \subseteq K(\alpha - 1, A_j, u_j + a_j f_{q(j)})$. Moreover, $Q_1(\alpha - 1, A_n, u_n + a_i f_{q(n)}) \subseteq K(\alpha - 1, A_n, u_n + a_n f_{q(n)})$ by definition. Hence,

$$\begin{aligned} \widetilde{u}_n &\in \left(Q_1(\alpha - 1, A_n, u_n + a_n f_{q(n)})\right)^{(\alpha)} \\ &\subseteq \left(\bigcap_{j=1}^{\infty} K(\alpha - 1, A_j, u_j + a_j f_{q(j)})\right)^{(\alpha)} = K^{(\alpha)}. \end{aligned}$$

Claim 3 $U^{\perp} \subseteq \overline{K^{(\alpha)}}$. Recall that $U = [u_n]_{n=1}^{\infty}$. Let $y^* \in U^{\perp}$ and for $n \in \mathbb{N}$ set $y_n^* = P_n^*(y^*)$ and

$$z_n^* = y_n^* - \sum_{m=1}^{\infty} a_m f_{q(m)}(y_n^*) \widetilde{u}_m.$$

Then, for each $n \in \mathbb{N}$, the element y_n^* is finitely supported. It follows from Lemma 2.9 that for each $n \in \mathbb{N}$ the number $f_{q(m)}(y_n^*)$ is nonzero for only finitely many $m \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, z_n^* is also finitely supported. Further, as

$$\mathcal{W} = \bigcup_{i \in \mathbb{N}} \{u_i\} \cup \{x_s^t : \ s \in \mathbb{N}, \ t \in A_i \cup \{q(i)\}\},$$

each z_n^* can be decomposed as $z_n^* = \sum_{i=1}^m w_{n,i}^*$ for some $m \in \mathbb{N}$, where

$$w_{n,i}^* \in a_{n,i}\widetilde{u}_i + \operatorname{span}\{\widetilde{x}_s^t : \ s \in \mathbb{N},\ t \in A_i \cup \{q(i)\}\}$$

for $a_{n,i} = z_n^*(u_i) = y_n^*(u_i) - a_i f_{q(i)}(y_n^*) = -a_i f_{q(i)}(y_n^*)$. Indeed,

$$y_n^*(u_i) = (P_n^*\, y^*)(u_i) = y^*(P_n\, u_i) = 0$$

as either $P_n u_i = u_i$ if $i \leq n$, or $P_n u_i = 0$ if $i > n$ – in both cases $y^*(P_n u_i) = 0$ as $y^* \in U^\perp$. Moreover, for $i = 1, \dots, m$, we have that $w_{n,i}^*$ is an element of $\operatorname{Ker}(u_i + a_i f_{q(i)})$ as

$$\begin{aligned}(u_i + a_i f_{q(i)})(w_{n,i}^*) &= (u_i + a_i f_{q(i)})(z_n^*) = (u_i + a_i f_{q(i)})(y_n^*) - a_i f_{q(i)}(y_n^*)\\ &= y_n^*(u_i) = 0,\end{aligned}$$

where the first equality follows from Lemma 2.9. Recall that for $i, j \in \mathbb{N}$ and $c \in \mathbb{F}$ it holds that $Q_c(\alpha - 1, A_i, u_i + a_i f_{q(i)}) \subseteq K(\alpha - 1, A_j, u_j + a_j f_{q(j)})$ by Lemma 2.7 and Lemma 2.9. It then follows from Lemma 2.10 *(b)* that for $i = 1, \dots, m$

$$\begin{aligned} w_{n,i}^* &\in \Big(Q_{a_{n,i}}(\alpha - 1, A_i, u_i + a_i f_{q(i)})\Big)^{(\alpha-1)} \\ &\subseteq \left(\bigcap_{j=1}^{\infty} K(\alpha - 1, A_j, u_j + a_j f_{q(j)})\right)^{(\alpha-1)} = K^{(\alpha-1)}. \end{aligned}$$

Hence, $z_n^* \in K^{(\alpha-1)}$ for each $n \in \mathbb{N}$. It follows from boundedness and a diagonal argument that there is an increasing sequence $(n_k)_{k=1}^\infty$ of integers and a sequence of scalars $(c_m)_{m\in\mathbb{N}}$ such that $a_m f_{q(m)}(y_{n_k}^*) \to c_m$ for each $m \in \mathbb{N}$. Further, $(c_m)_{m=1}^\infty$ is absolutely summable as $|c_m| \leq \sup_j \|f_{q(j)}\| \sup_k \|y_{n_k}^*\| a_m \leq C a_m$ for some $C > 0$ which does not depend on m. Then

$$z_{n_k}^* \xrightarrow[k\to\infty]{w^*} y^* - \sum_{m=1}^{\infty} c_m \widetilde{u}_m =: z^*.$$

Hence, $z^* \in K^{(\alpha)}$ and since

$$y^* = \lim_{n\to\infty}\left(z^* + \sum_{m=1}^{n} c_m \widetilde{u}_m\right)$$

and $z^* + \sum_{m=1}^n c_m \widetilde{u}_m \in K^{(\alpha)}$ by Claim 2, it follows that $y^* \in \overline{K^{(\alpha)}}$.

Claim 4 $\overline{K^{(\alpha)}} = W^*$. Take any $y^* \in W^*$ and $\epsilon > 0$. Since $(u_n)_{n=1}^\infty$ is shrinking (by Lemma 2.4 *(ii)*) there is a finite linear combination $u^* = \sum_{j=1}^m \lambda_j \widetilde{u}_j \restriction U \in U^*$, such that $\|y^* \restriction U - u^*\| \leq \epsilon$. Let w^* be a Hahn-Banach extension of $y^* \restriction U - u^*$ to W. That is $\|w^*\| \leq \epsilon$ and $w^* \restriction U = y^* \restriction U - u^*$. Then

$$y^* - w^* - \sum_{j=1}^{m} \lambda_j \widetilde{u}_j \in U^\perp \subseteq \overline{K^{(\alpha)}}.$$

by Claim 3. Claim 2 thus yields that $y^* - w^* \in \overline{K^{(\alpha)}}$. Moreover, as $\|w^*\| \leq \epsilon$, we get $\operatorname{dist}(y^*, \overline{K^{(\alpha)}}) \leq \epsilon$. As ϵ was arbitrary, we get $y^* \in \overline{K^{(\alpha)}}$. □

Now we are all prepared to prove Theorem 2.3.

Proof of Theorem 2.3. The equivalence *(1)* $\iff$ *(2)* follows from [27, Theorem 1] and the implication *(3)* $\implies$ *(2)* is clear. To prove the implication *(1)* $\implies$ *(3)* we fix a successor ordinal $\alpha < \omega_1$ and use Lemma 2.4 and Theorem 2.12 to find a subspace W of X and a subspace K of W^*, such that $K^{(\alpha)} \subsetneq \overline{K^{(\alpha)}} = W^*$. Let $E : W \to X$ be the identity embedding. Then $E^* : X^* \to W^*$ is the restriction map. Set $A = (E^*)^{-1}(K)$. It follows from [25, Lemma 1] and the fact that E^* is onto that $A^{(\alpha)} = (E^*)^{-1}(K^{(\alpha)}) \subsetneq X^*$. As E^* is an open mapping, the preimage of a dense set is dense. Thus $\overline{A^{(\alpha)}} = X^*$. ☐

Note that in Theorem 2.3 *(3)* we restrict ourselves only to successor ordinals. The reason lies in the proof of Claim 3 of Theorem 2.12. More specifically, we needed to pass from a general $y^* \in U^\perp$ to elements $y^*_n = P^*_n(y^*)$ with finite support. Following the proof of Claim 3 for limit ordinal α, with considering $(\alpha_n)_{n=1}^\infty$ instead of $\alpha - 1$ (see Construction 2.6), we would end with $z^*_{n_k} \in K^{(\alpha_{n_k})}$, and thus $z^* \in \left(\bigcup_{k=1}^\infty K^{(\alpha_{n_k})}\right)^{(1)} \subseteq K^{(\alpha+1)}$. We would, however, need z^* to be in $K^{(\alpha)}$. The problem for limit ordinals thus remains open:

Question 3. *Let α be a limit ordinal and X be a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual. Is there a subspace A of X^* such that $A^{(\alpha)} \subsetneq \overline{A^{(\alpha)}} = X^*$?*[2]

[2]It can be shown that if the dual projections P^*_n from Notation 2.5 4. satisfy a stronger property $P^*_n(x^*) \to x^*$ for each $x^* \in W^*$ (that is if the FDD is shrinking), then the answer is positive. Hence, for example if $X = c_0$ and $\mathcal{W}$ is just the cannonical basis, we can find a subspace A of X^* such that $A^{(\alpha)} \subsetneq \overline{A^{(\alpha)}} = X^*$ even for limit ordinals α.

3. Quantification of Banach-Saks properties of higher orders

Abstract: We investigate possible quantifications of Banach-Saks sets and weak Banach-Saks sets of higher orders and their relations to other quantities. We prove a quantitative version of the characterization of weak ξ-Banach-Saks sets using $\ell_1^{\xi+1}$-spreading models and a quantitative version of the relation of ξ-Banach-Saks sets, weak ξ-Banach-Saks sets, norm compactness and weak compactness. We further introduce a new measure of weak compactness. Finally, we provide some examples showing the limitations of these quantifications.

3.1 Introduction

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence in X admits a Cesàro summable subsequence. This property was first investigated by Banach and Saks in [6], where they showed that the spaces L_p for $1 < p < \infty$ enjoy this property. Every space with the Banach-Saks property is reflexive [23] but there are reflexive spaces which do not have the Banach-Saks property, see [4] or Example 3.27 below. However, every uniformly convex space (or more generally every super-reflexive space, as super-reflexive spaces admit a uniformly convex renorming [13]) has the Banach-Saks property [19].

A Banach space X has the *weak Banach-Saks property* if every weakly convergent sequence in X admits a Cesàro summable subsequence. For reflexive spaces the weak Banach-Saks property and Banach-Saks property are equivalent but there are non-reflexive spaces that have the weak Banach-Saks property, like c_0 or L_1, see [14] and [35].

There is a localized version of these properties – a bounded set A in a Banach space X is said to be a *Banach-Saks set*, if every sequence in A admits a Cesàro summable subsequence, and is called a *weak Banach-Saks set*, if every weakly convergent sequence in A admits a Cesàro summable subsequence. It follows that a Banach space X has the Banach-Saks property, resp. the weak Banach-Saks property, if and only if its closed unit ball B_X is a Banach-Saks set, resp. a weak Banach-Saks set. It is easy to see that a relatively weakly compact weak Banach-Saks set is a Banach-Saks set. The other implication also holds. Indeed, a Banach-Saks set is obviously a weak Banach-Saks set and the fact that it is also relatively weakly compact follows from [21, Proposition 2.3.]. A quantitative version of this statement was investigated in [7].

The property of being a weak Banach-Saks set is closely tied to two other notions, which will be explained in detail in Section 3.2 below. First of them is the notion of an ℓ_1-*spreading model*. Recall that a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is said to generate an ℓ_1-spreading model if there is a positive constant c such that for all finite subsets F of $\mathbb{N}$ satisfying $|F| \leq \min F$, where $|F|$ is the cardinality of the set F, and all sequences $(a_i)_{i\in F}$ of scalars we have

$$\left\| \sum_{i\in F} a_i x_i \right\| \geq c \sum_{i\in F} |a_i|.$$

The second related notion is *uniform weak convergence*. A sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly weakly convergent to x if for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $x^* \in B_{X^*}$ we have

$$\#\{k \in \mathbb{N} : \ |x^*(x_k - x)| \geq \epsilon\} \leq n,$$

where $\#A$ is another notation for the cardinality of a set A. It follows from [21, Section 2] that a bounded set A in a Banach space X is a weak Banach-Saks set, if and only if no weakly convergent sequence in A generates an ℓ_1-spreading model, if and only if every weakly convergent sequence in A admits a uniformly weakly convergent subsequence. Quantitative version of this result was also provided in [7].

It follows from the Mazur's theorem that if we have a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X, then there is a sequence of convex combinations which converges to zero in norm. The weak Banach-Saks property of X then means that these convex combinations can be chosen to be the Cesàro sums. In [3] the authors investigated how regular these convex combinations can be in spaces failing the weak Banach-Saks property and defined the *ξ-Banach-Saks property* and the *weak ξ-Banach-Saks property* for a countable ordinal ξ (see Section 3.2). The main goal of this paper is to provide quantifications, analogous to those provided in [7], for the properties of higher orders. The investigation of these properties also led to a new measure of weak non-compactness.

3.2 Preparation

3.2.1 Notation

For an infinite subset M of $\mathbb{N}$ we will denote by $[M]$ the set of all infinite subsets of M. On the other hand, for any subset M of $\mathbb{N}$ we will denote by $[M]^{<\infty}$ the set of all finite subsets of M. If $n \in \mathbb{N}$ we will denote by $[M]^{<n}$ the sets of all subsets of M of cardinality less than n.

If M is an infinite subset of $\mathbb{N}$ and we write $M = (m_n)_{n\in\mathbb{N}}$, then we always mean that $M = \{m_n : \ n \in \mathbb{N}\}$ and $m_1 < m_2 < \dots$. We also use an analogous convention for finite subsets of $\mathbb{N}$.

For a Banach space X we will denote by B_X the closed unit ball of X and by S_X the unit sphere of X. In the special case where $X = \ell_1$, we will denote by $S^+_{\ell_1}$ the set of those elements of S_{ℓ_1} which have non-negative coordinates.

If $(a_n)_{n\in\mathbb{N}} \in \ell_1$, F is a subset of integers and $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in a Banach space X, we set

- $\langle (a_n)_{n\in\mathbb{N}}, F\rangle = \sum\limits_{n\in F} a_n$;
- $(a_n)_{n\in\mathbb{N}} \cdot (x_n)_{n\in\mathbb{N}} = \sum\limits_{n\in\mathbb{N}} a_n x_n$.

We denote the canonical basis of the space c_{00} of eventually zero sequences by $(e_n)_{n\in\mathbb{N}}$.

Let A, B be subsets of $\mathbb{N}$. If we write $A < B$, then we mean that $\max A < \min B$. Analogously, $A \leq B$ means that $\max A \leq \min B$. We write $n \leq A$, resp. $n < A$, instead of $\{n\} \leq A$, resp. $\{n\} < A$.

If F is a finite set, we will write $|F|$ or $\#F$ for the cardinality of F.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence and $M = (m_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$, then we denote the subsequence $(x_{m_n})_{n\in\mathbb{N}}$ by $(x_n)_{n\in M}$.

3.2.2 Families of subsets of integers

We identify subsets of $\mathbb{N}$ with their characteristic functions, and thus with elements of the Cantor set $\{0,1\}^{\mathbb{N}}$. This characterization provides us with a metrizable topology on the power set of $\mathbb{N}$.

Definition. Let $\mathcal{F}$ be a family of finite sets of integers. We say that $\mathcal{F}$ is

- *Hereditary*, if $A \in \mathcal{F}$ and $B \subseteq A$ implies $B \in F$;
- *Precompact*, if the closure of $\mathcal{F}$ consists only of finite sets;
- *Adequate*, if it is both hereditary and precompact.

If $M \in [\mathbb{N}]$, we define the *trace* of $\mathcal{F}$ on M by

$$\mathcal{F}[M] = \{F \cap M : \ F \in \mathcal{F}\}.$$

Note that the trace of an adequate family is also adequate. If $\mathcal{F}$ is hereditary, then $\mathcal{F}[M] = \{F \in \mathcal{F} : \ F \subseteq M\}$.

3.2.3 Schreier families and Repeated Averages

In this subsection we will define the Schreier families and the Repeated Averages. For a countable limit ordinal ξ we fix an increasing sequence $(\xi_n)_{n\in\mathbb{N}}$ of successor ordinals with $\xi = \sup \xi_n$. This choice is necessary for us to define the Schreier families and Repeated Averages for limit ordinals. While these definitions certainly depend on this choice, some of the quantities defined in the following subsection do not. The independence on this choice will be discussed in detail in Section 3.7 below.

Definition. The *Schreier families* $(\mathcal{S}_\xi)_{\xi<\omega_1}$ are defined recursively. First we define the family $\mathcal{S}_0$ as

$$\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$

For a successor ordinal $\xi + 1 < \omega_1$ we define

$$\mathcal{S}_{\xi+1} = \left\{ \bigcup_{i=1}^{n} F_i : \ n \leq F_1 < F_2 < \cdots < F_n, \ F_i \in \mathcal{S}_\xi, \ n \in \mathbb{N} \right\} \cup \{\emptyset\}$$

and for a limit ordinal $\xi < \omega_1$ we take the fixed increasing sequence of successor ordinals $(\xi_n)_{n\in\mathbb{N}}$ with $\xi = \sup \xi_n$ and define

$$\mathcal{S}_\xi = \{F \in \mathcal{S}_{\xi_n} : \ n \leq F, \ n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Note that the family $\mathcal{S}_{\xi+1}$ always contains the family $\mathcal{S}_\xi$. On the other hand, it is not generally true that the family $\mathcal{S}_\zeta$ contains the family $\mathcal{S}_\xi$ for $\zeta > \xi$. It does, however, contain all the sets from $\mathcal{S}_\xi$ with sufficiently large minimal element, see [3, Lemma 2.1.8.(a)]. The family $\mathcal{S}_1$ is the classical Schreier family

$$\mathcal{S}_1 = \{F \in [\mathbb{N}]^{<\infty} : \ |F| \leq \min F\} \cup \{\emptyset\}.$$

It is readily proved by induction that the families $\mathcal{S}_\xi$, $\xi < \omega_1$, are adequate and have the following spreading property:

> If $F = (f_1, \ldots, f_n) \in \mathcal{S}_\xi$ and $G = (g_1, \ldots, g_n)$ is such that $f_i \leq g_i, \ i = 1, \ldots, n,$ then $G \in \mathcal{S}_\xi$.

Definition. Let $\xi < \omega_1$ and $M = (m_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$. We define

$$\mathcal{S}_\xi^M = \{(m_i)_{i\in F} : \ F \in \mathcal{S}_\xi\}.$$

In the case that $\xi = 0$, we have that

$$\mathcal{S}_0^M = \mathcal{S}_0[M] = \{\{m_n\} : \ n \in \mathbb{N}\} \cup \{\emptyset\}.$$

However, if $\xi > 0$, then $\mathcal{S}_\xi^M \subsetneq \mathcal{S}_\xi[M]$. Indeed $\mathcal{S}_\xi^M$ is a subset of $\mathcal{S}_\xi$ by the spreading property of the family $\mathcal{S}_\xi$ as $i \leq m_i$ for each $i \in \mathbb{N}$, and the sets from $\mathcal{S}_\xi^M$ are obviously subsets of M. The fact that the inclusion is strict can be proved by induction and is illustrated by the following example: If we set $m_n = n + 1$ and $M = (m_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$, then the set $\{2, 3\} \in \mathcal{S}_1[M] \setminus \mathcal{S}_1^M$. For more information about the relation of the families $\mathcal{S}_\xi^M$ and $\mathcal{S}_\xi[M]$ see [3, Remark 2.1.12].

Definition. Let $M \in [\mathbb{N}]$. An M-summability method is a sequence $(A_n)_{n\in\mathbb{N}}$ where $A_n \in S_{\ell_1}^+$ are such that $\operatorname{supp} A_n < \operatorname{supp} A_{n+1}$ for all $n \in \mathbb{N}$ and $M = \bigcup_{n=1}^\infty \operatorname{supp} A_n$, where $\operatorname{supp} F$ denotes the support of an element F of ℓ_1, that is the set of coordinates where F is nonzero.

We say that a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in some Banach space X is $(A_n)_{n\in\mathbb{N}}$-summable if the sequence $(A_n \cdot (x_k)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ is Cesàro summable.

Note that if $A_n = e_{m_n}$ for some increasing sequence $M = (m_n)_{n\in\mathbb{N}}$ of integers, then the $(A_n)_{n\in\mathbb{N}}$-summability of a sequence $(x_n)_{n\in\mathbb{N}}$ is just the Cesàro summability of the subsequence $(x_{m_n})_{n\in\mathbb{N}}$. One important fact we will need later is the simple observation that the summability methods preserve convergence. We will specifically use that if a sequence $(x_n)_{n\in\mathbb{N}}$ is weakly null, then $(A_n \cdot (x_k)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ is also weakly null for any M-summability method $(A_n)_{n\in\mathbb{N}}$.

The Repeated Averages are a special type of summability methods that arise by iterating consecutive averages.

Definition. Let $M = (m_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$. The *Repeated Averages* are the M-summability methods $(\xi_n^M)_{n\in\mathbb{N}}$, $\xi < \omega_1$, which are defined recursively in the following way.

1. If $\xi = 0$, we set $\xi_n^M = e_{m_n}$, $n \in \mathbb{N}$.

2. If $\xi = \zeta + 1$ and $(\zeta_n^M)_{n\in\mathbb{N}}$ have already been defined, we recursively define ξ_n^M in the following way

$$k_1 = 0,\ s_1 = \min\operatorname{supp}\zeta_1^M = \min M, \qquad \xi_1^M = \frac{1}{s_1}\sum_{i=1}^{s_1}\zeta_i^M$$

$$\vdots$$

$$k_n = k_{n-1} + s_{n-1},\ s_n = \min\operatorname{supp}\zeta_{k_n+1}^M, \qquad \xi_n^M = \frac{1}{s_n}\sum_{i=k_n+1}^{k_n+s_n}\zeta_i^M$$

$$\vdots$$

3. If ξ is a limit ordinal and $(\zeta_n^N)_{n\in\mathbb{N}}$ have already been defined for all $\zeta < \xi$ and $N \in [\mathbb{N}]$, we take the increasing sequence of successor ordinals $(\xi_n)_{n\in\mathbb{N}}$ which was used to define the Schreier family $\mathcal{S}_\xi$. We use the notation $[\xi_n]_j^N$ for the already defined summability method ζ_j^N for $\zeta = \xi_n$. Set

$$\begin{aligned}
M_1 &= M, & n_1 &= m_1,\\
M_2 &= M_1 \setminus \operatorname{supp}[\xi_{n_1}]_1^{M_1}, & n_2 &= \min M_2,\\
&\vdots\\
M_j &= M_{j-1} \setminus \operatorname{supp}[\xi_{n_{j-1}}]_1^{M_{j-1}}, & n_j &= \min M_j,\\
&\vdots
\end{aligned}$$

Finally we set for $j \in \mathbb{N}$

$$\xi_j^M = [\xi_{n_j}]_1^{M_j}.$$

It is readily proved by induction that for each $\xi < \omega_1$ and $M \in [\mathbb{N}]$ the sequence $(\xi_n^M)_{n\in\mathbb{N}}$ is an M-summability method. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is (ξ, M)-summable instead of $(\xi_n^M)_{n\in\mathbb{N}}$-summable.

A nice property of the Repeated Averages is that their supports are elements of the corresponding Schreier family, that is $\operatorname{supp}\xi_n^M \in \mathcal{S}_\xi[M]$ for all $M \in [\mathbb{N}]$ and $n \in \mathbb{N}$.

3.2.4 ℓ_1^ξ-spreading models and (weak) ξ-Banach-Saks sets

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X, $\xi < \omega_1$ and $c > 0$. We say that $(x_n)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model with constant c if

$$\forall F \in \mathcal{S}_\xi\ \forall(\alpha_i)_{i\in F} \in \mathbb{R}^F :\ \left\|\sum_{i\in F}\alpha_i x_i\right\| \geq c\sum_{i\in F}|\alpha_i|.$$

We say that $(x_n)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model if it generates an ℓ_1^ξ-spreading model with some constant $c > 0$.

This definition generalises the classical notion of an ℓ_1-spreading model, which corresponds to the case $\xi = 1$.

Definition. Let A be a bounded subset of a Banach space X and $\xi < \omega_1$. We say that A is a ξ-Banach-Saks set if for every sequence $(x_n)_{n\in\mathbb{N}}$ in A there is some $M \in [\mathbb{N}]$ such that $(x_n)_{n\in\mathbb{N}}$ is (M, ξ)-summable.

A is called a weak ξ-Banach-Saks set if the same property holds for weakly convergent sequences in A, that is, if for every sequence $(x_n)_{n\in\mathbb{N}}$ in A weakly convergent to some $x \in X$ there is some $M \in [\mathbb{N}]$ such that $(x_n)_{n\in\mathbb{N}}$ is (M, ξ)-summable.

These definitions generalise the notion of a Banach-Saks set and a weak Banach-Saks set, which correspond to the case $\xi = 0$. Following [7], we will now define some quantities that we will later use to quantify the notions of (weak) ξ-Banach-Saks sets and ℓ_1^ξ-spreading models.

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X. We define the following two quantities

- $\operatorname{ca}(x_n) = \inf\limits_{n\in\mathbb{N}} \sup\{\|x_k - x_l\| : \ k, l \geq n\}$;
- $\operatorname{cca}(x_n) = \operatorname{ca}\left(\frac{1}{n}\sum_{i=1}^n x_i\right)$.

The quantity ca measures how far a given sequence is from being norm Cauchy. Indeed, $\operatorname{ca}(x_n) = 0$ if and only if the sequence $(x_n)_{n\in\mathbb{N}}$ is norm Cauchy. The quantity cca then measures how far are the Cesàro sums of a given sequence from being norm Cauchy.

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X and $\xi < \omega_1$. We define

$$\widetilde{\operatorname{cca}}_\xi((x_n)_{n\in\mathbb{N}}) = \inf_{M\in[\mathbb{N}]} \left(\operatorname{cca}(\xi_n^M \cdot (x_k)_{k\in\mathbb{N}})\right)$$

$$\widetilde{\operatorname{cca}}_\xi^s((x_n)_{n\in\mathbb{N}}) = \sup_{M\in[\mathbb{N}]} \left(\inf_{N\in[M]} \operatorname{cca}(\xi_n^N \cdot (x_k)_{k\in\mathbb{N}})\right).$$

The quantity $\widetilde{\operatorname{cca}}_0$ is the quantity $\widetilde{\operatorname{cca}}$ used in [7] and measures how far a given sequence is from containing a Cesàro summable subsequence. We will, however, mostly work with the quantity $\widetilde{\operatorname{cca}}_0^s$, which measures if all subsequences of a given sequence contain a further subsequence which is Cesàro summable, and with its generalizations for $\xi > 0$. The precise correspondence between these quantities for $\xi = 0$ is

$$\widetilde{\operatorname{cca}}_0^s((x_n)_{n\in\mathbb{N}}) = \sup\{\widetilde{\operatorname{cca}}_0((y_n)_{n\in\mathbb{N}}) : \ (y_n)_{n\in\mathbb{N}} \text{ is a subsequence of } (x_n)_{n\in\mathbb{N}}\},$$

for larger ξ the correspondence is not so clear.

Now we can define the quantifications of the notions of (weak) ξ-Banach-Saks sets and ℓ_1^ξ-spreading models.

Definition. Let A be a bounded subset of a Banach space X and $\xi < \omega_1$. We define the following quantities:

$$\begin{aligned}\operatorname{sm}_\xi(A) = \sup\{c > 0 :\ & \text{there is a sequence } (x_n)_{n\in\mathbb{N}} \text{ in } A \text{ weakly convergent} \\ & \text{to some } x \in X \text{ such that } (x_n - x)_{n\in\mathbb{N}} \text{ generates} \\ & \text{an } \ell_1^\xi\text{-spreading model with constant } c\},\end{aligned}$$

where we set the supremum of the empty set to be zero, and

$$\begin{aligned}
\mathrm{bs}_\xi(A) &= \sup\{\widetilde{\mathrm{cca}}_\xi(x_n) : \ (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } A\} \\
\mathrm{wbs}_\xi(A) &= \sup\{\widetilde{\mathrm{cca}}_\xi(x_n) : \ (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } A\} \\
\mathrm{bs}^s_\xi(A) &= \sup\{\widetilde{\mathrm{cca}}^s_\xi(x_n) : \ (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } A\} \\
\mathrm{wbs}^s_\xi(A) &= \sup\{\widetilde{\mathrm{cca}}^s_\xi(x_n) : \ (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } A\}.
\end{aligned}$$

It follows from the definition that $\mathrm{sm}_\xi(A) = 0$ if and only if A contains no sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to some $x \in X$ such that $(x_n - x)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model. The fact that $\mathrm{bs}_\xi(A) = 0$ (resp. $\mathrm{wbs}_\xi(A) = 0$) if and only if A is a ξ-Banach-Saks set (resp. weak ξ-Banach-Saks set) will be shown later in Proposition 3.17 for ξ-Banach-Saks sets and Propostion 3.10 for weak ξ-Banach-Saks sets. For $\xi = 0$ we have $\mathrm{bs}_0(A) = \mathrm{bs}^s_0(A)$ and $\mathrm{wbs}_0(A) = \mathrm{wbs}^s_0(A)$ as any subsequence of a sequence in A is also a sequence in A. For larger ξ we trivially have $\mathrm{bs}_\xi(A) \le \mathrm{bs}^s_\xi(A)$ and $\mathrm{wbs}_\xi(A) \le \mathrm{wbs}^s_\xi(A)$. We will show later in Theorem 3.3 that the quantities wbs_ξ and wbs^s_ξ are equivalent for $\xi > 0$.

3.2.5 (ξ, c)-large sets and uniformly weakly converging sequences

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in a Banach space X and $\delta > 0$. We define the family

$$\mathcal{F}_\delta((x_n)_{n\in\mathbb{N}}) = \{F \in [\mathbb{N}]^{<\infty} : \text{ there is } x^* \in B_{X^*} \text{ with } x^*(x_n) \ge \delta,\ n \in F\}.$$

We will usually write only $\mathcal{F}_\delta$ instead of $\mathcal{F}_\delta((x_n)_{n\in\mathbb{N}})$ if it causes no confusion.

Note that the family $\mathcal{F}_\delta$ is obviously hereditary and is also precompact as the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly null. Indeed, suppose there is a sequence $(F_n)_{n=1}^\infty$ of sets from the family $\mathcal{F}_\delta$ that converges to an infinite set $F \in [\mathbb{N}]$. For $n \in \mathbb{N}$ let x_n^* be an element of B_{X^*} witnessing that F_n belongs to $\mathcal{F}_\delta$ and let x^* be any weak* cluster point of $(x_n^*)_{n=1}^\infty$. Then, for a fixed $k \in F$, we have $x_n^*(x_k) \ge \delta$ for all but finitely many $n \in \mathbb{N}$. Hence, $x^*(x_k) \ge \delta > 0$ for infinitely many $k \in \mathbb{N}$ and $(x_n)_{n=1}^\infty$ is not weakly null – a contradiction. Hence, $\mathcal{F}_\delta$ is an adequate family of subsets of $\mathbb{N}$.

To use some of the results of [3], we need to present an alternative definition of $\mathcal{F}$ using weakly compact subsets of c_0. Let us define $D = \{(x^*(x_n))_{n=1}^\infty : \ x^* \in B_{X^*}\}$. Then D is a weakly compact subset of c_0. Indeed, D is the image of B_{X^*} under the weak*-to-weak continuous mapping $x^* \mapsto (x^*(x_n))_{n=1}^\infty$. It follows that

$$\mathcal{F}_\delta = \{F \in [\mathbb{N}]^{<\infty} : \text{ there is } f \in D \text{ with } f(n) \ge \delta,\ n \in F\}.$$

We now recall the definition of uniformly weakly convergent sequences.

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is said to be uniformly weakly convergent to some x in X if for each $\epsilon > 0$

$$\exists n \in \mathbb{N}\ \forall x^* \in B_{X^*} : \ \#\{k \in \mathbb{N} : \ |x^*(x_k - x)| \ge \epsilon\} \le n.$$

Note that the absolute value in this definition can be omitted, that is $(x_n)_{n\in\mathbb{N}}$ is uniformly weakly convergent to $x \in X$ if and only if for all $\epsilon > 0$

$$\exists n \in \mathbb{N}\ \forall x^* \in B_{X^*} : \ \#\{k \in \mathbb{N} : \ x^*(x_k - x) \ge \epsilon\} \le n,$$

Uniform weak convergence can be used to characterize Banach-Saks (resp. weak Banach-Saks) sets. Precisely, a bounded set A in a Banach space X is a Banach-Saks (resp. weak Banach-Saks) set, if and only if every (resp. every weakly convergent) sequence in A has a uniformly weakly convergent subsequence, see [21, Theorem 2.4.].

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X weakly converging to $x \in X$, $c > 0$ and $\xi < \omega_1$. We say that $(x_n)_{n\in\mathbb{N}}$ is (ξ, c)-large if there is $M \in [\mathbb{N}]$ such that $\mathcal{S}_\xi^M \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$.

Definition. Let A be a bounded subset of a Banach space X and $\xi < \omega_1$. We define the quantity

$$\begin{aligned}\operatorname{wus}_\xi(A) = \sup\{c > 0 : {} & \text{there is a sequence } (x_n)_{n\in\mathbb{N}} \text{ in } A \\ & \text{weakly convergent to some } x \in X \\ & \text{such that } (x_n)_{n\in\mathbb{N}} \text{ is } (\xi, c)\text{-large}\},\end{aligned}$$

where we again set the supremum of the empty set to be zero.

The quantity wus_ξ is a generalization of the quantity wus used in [7] which measures how far is A from having the property that every weakly convergent sequence in A has a uniformly weakly convergent subsequence. Indeed, for a bounded set A we have $\operatorname{wus}_1(A) = \operatorname{wus}(A)$ which will follow from the following lemma for sequences. It uses the quantity $\widetilde{\operatorname{wu}}$, which is defined in [7] and used to define the quantity wus.

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X weakly convergent to some $x \in X$ and let $c > 0$. Then*

(i) If $(x_n)_{n\in\mathbb{N}}$ is $(1, c)$-large, then there is $M \in [\mathbb{N}]$ such that $\widetilde{\operatorname{wu}}((x_n)_{n\in M}) \geq c$.

(ii) If $\widetilde{\operatorname{wu}}((x_n)_{n\in\mathbb{N}}) > c$, then $(x_n)_{n\in\mathbb{N}}$ is $(1, c)$-large.

Proof. If $(x_n)_{n\in\mathbb{N}}$ is $(1, c)$-large, then there is $M \in [\mathbb{N}]$ such that $\mathcal{S}_1^M \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$. We will show that $\widetilde{\operatorname{wu}}((x_n)_{n\in M}) \geq c$. Indeed, any subsequence of $(x_n)_{n\in M}$ is of the form $(x_n)_{n\in N}$ for some $N \in [M]$. It follows from the spreading property of $\mathcal{S}_1$ that $\mathcal{S}_1^N \subseteq \mathcal{S}_1^M$. Hence, $\mathcal{S}_1^N \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$ which is easily seen to be equivalent to saying $\mathcal{S}_1 \subseteq \mathcal{F}_c((x_n - x)_{n\in N})$. Therefore, as $\mathcal{S}_1$ contains sets of arbitrarily large cardinality, there is no $n \in \mathbb{N}$ such that

$$\forall x^* \in B_{X^*} : \#\{k \in N : |x^*(x_k - x)| \geq c\} \leq n$$

and $\operatorname{wu}((x_n)_{n\in N}) \geq c$. It now follows from the definition that $\widetilde{\operatorname{wu}}((x_n)_{n\in M}) \geq c$.

The other inequality follows from the proof of [22, Lemma 1.13.] (note that we apply the lemma for $\delta = c$, $\Gamma = B_{X^*}$ and that the family $\mathcal{A}_\delta$ in this proof is nothing else than $\mathcal{F}_c$ in our notation). This lemma yields that if $\widetilde{\operatorname{wu}}((x_n)_{n\in\mathbb{N}}) > c$, then there is $M = (m_1, m_2, \dots) \in [\mathbb{N}]$ such that, if we set $M_k = (m_k, m_{k+1}, \dots)$ for $k \in \mathbb{N}$, then $[M_k]^k \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$. But

$$\bigcup_{k=1}^{\infty} [M_k]^k = \{(m_i)_{i\in F} : m_{|F|} \leq m_{\min F}\} = \{(m_i)_{i\in F} : |F| \leq \min F\} = \mathcal{S}_1^M.$$

Hence, $\mathcal{S}_1^M \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$ and $(x_n)_{n\in\mathbb{N}}$ is $(1, c)$-large. $\square$

3.3 Quantitative characterization of weak ξ-Banach-Saks sets

In this section we will prove a quantified version of the following theorem from [3], which is a natural generalization of the characterization of weakly null sequences with no Cesàro summable subsequences using ℓ_1-spreading models (see [21, Section 2]).

Theorem 3.2. *[3, Theorem 2.4.1] Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in a Banach space X and $\xi < \omega_1$. Then exactly one of the following holds.*

(a) For every $M \in [\mathbb{N}]$ there is $L \in [M]$ such that for every $P \in [L]$ the sequence $(x_n)_{n\in\mathbb{N}}$ is (P, ξ)-summable.

(b) There is $M = (m_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$ such that the sequence $(x_{m_n})_{n\in\mathbb{N}}$ generates an $\ell_1^{\xi+1}$-spreading model.

More precisely, we will prove a formulation of the above-mentioned result for a bounded subset of a Banach space instead of a weakly null sequence and we will add the quantities wbs_ξ and $\mathrm{wus}_{\xi+1}$ (qualitative version of the quantity $\mathrm{wus}_{\xi+1}$ was also used in the proof of the Theorem 3.2 from [3]).

Theorem 3.3. *Let A be a bounded set in a Banach space X and $\xi < \omega_1$. Then*

$$2\,\mathrm{sm}_{\xi+1}(A) \le \mathrm{wbs}_\xi(A) \le \mathrm{wbs}_\xi^s(A) \le 2\,\mathrm{wus}_{\xi+1}(A) \le 4\,\mathrm{sm}_{\xi+1}(A). \qquad (\star)$$

As we have already mentioned, the inequality $\mathrm{wbs}_\xi(A) \le \mathrm{wbs}_\xi^s(A)$ is trivial. Recall that $\mathcal{S}_\xi^M \subseteq \mathcal{S}_\xi[M]$ for all $M \in [\mathbb{N}]$ but in general we do not have equality. We have already noted that the support of ξ_n^M is in the family $\mathcal{S}_\xi[M]$ for every $n \in \mathbb{N}$ and $M \in [\mathbb{N}]$. The following lemma from [3] will allow us to find an infinite subset of M for which the ξ-summability methods are very close to being supported on the sets from the smaller family $\mathcal{S}_\xi^M$. Note that in the following lemma the set L does not depend on ξ.

Lemma 3.4. *[3, Proposition 2.1.10.] For every $M \in [\mathbb{N}]$ and $\epsilon > 0$ there is $L \in [M]$ such that for any $P \in [L]$, $\xi < \omega_1$ and $n \in \mathbb{N}$ there is $G \in \mathcal{S}_\xi^M$ such that*

$$\langle \xi_n^P, G\rangle > 1 - \epsilon.$$

The following lemma provides sufficient conditions on an adequate family $\mathcal{F}$ so that the family $\mathcal{S}_{\xi+1}^N$, embeds into $\mathcal{F}$ for some $N \in [\mathbb{N}]$.

Lemma 3.5. *[3, Theorem 2.2.6, Proposition 2.3.6] Let $\mathcal{F}$ be an adequate family, $\xi < \omega_1$ and $\epsilon' > 0$. Suppose that there is $L = (l_n)_{n\in\mathbb{N}} \in [\mathbb{N}]$ satisfying*

- *For all $n \in \mathbb{N}$ and $N \in [L]$ with $l_n \le \min N$ there is $F \in \mathcal{F}$ such that $\langle \xi_k^N, F\rangle > \epsilon'$ for $k = 1, \dots, n$.*

Then there is $N \in [L]$ such that $\mathcal{S}_{\xi+1}^N \subseteq \mathcal{F}$.

Note that [3, Proposition 2.3.6.] has a slightly different formulation than Lemma 3.5. More specifically, the condition on L is formulated for all $N \in [L]$ with $n \le \min N$ instead of $l_n \le \min N$. However, the proof of [3, Proposition 2.3.6.] works for our formulation as well.

The following lemma is a quantified version of [3, Lemma 2.4.8].

Lemma 3.6. *Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in B_X, $\delta > 0$ and $\epsilon \in (0,1)$. Then for every $M \in [\mathbb{N}]$ there is $N \in [M]$ satisfying the following property:*

- *If $(a_n)_{n\in\mathbb{N}} \in S_{\ell_1}$, $\operatorname{supp}((a_n)_{n\in\mathbb{N}}) \subseteq N$ and $F \in \mathcal{F}_\delta$, then*

$$\left\|\sum_{n\in\mathbb{N}} a_n x_n\right\| \geq (1-\epsilon)\,\delta \cdot \langle (a_n)_{n\in\mathbb{N}}, F\rangle - \epsilon\delta.$$

Proof. We use [3, Lemma 2.4.7] to find $N \in [M]$ such that for each $F \in \mathcal{F}_\delta[N]$ there is $x^* \in B_{X^*}$ such that

(a) $x^*(x_n) \geq (1-\epsilon)\delta$ for all $n \in F$,

(b) $\sum_{n\in N\setminus F} |x^*(x_n)| < \epsilon\delta$.

Fix $(a_n)_{n\in\mathbb{N}} \in S_{\ell_1}$ with $\operatorname{supp}((a_n)_{n\in\mathbb{N}}) \subseteq N$ and $F \in \mathcal{F}_\delta$. Take

$$F' = \{n \in F \cap N :\ a_n > 0\} \in \mathcal{F}_\delta[N].$$

We find $x^* \in B_{X^*}$ such that the properties (a), (b) are satisfied for F'. Then

$$\begin{aligned}\left\|\sum_{n\in\mathbb{N}} a_n x_n\right\| \geq \sum_{n\in N} a_n x^*(x_n) &\geq \sum_{n\in F'} a_n x^*(x_n) - \sum_{n\in N\setminus F'} |a_n x^*(x_n)| \\ &\geq (1-\epsilon)\delta \cdot \langle (a_n)_{n\in\mathbb{N}}, F'\rangle - \epsilon\delta \\ &\geq (1-\epsilon)\delta \cdot \langle (a_n)_{n\in\mathbb{N}}, F\rangle - \epsilon\delta.\end{aligned}$$

The inequality $\sum_{n\in N\setminus F'} |a_n x^*(x_n)| \leq \epsilon\delta$ follows from (b) and the fact that $|a_n| \leq 1$ for each $n \in \mathbb{N}$. The last inequality holds as $a_n \leq 0$ on $F \setminus F'$. $\square$

We are all prepared to prove the first inequality of $(\star)$. We will use the natural generalization of the idea of [7, Lemma 4.5.].

Proposition 3.7. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X which weakly converges to some x. If $(x_n - x)_{n\in\mathbb{N}}$ generates an $\ell_1^{\xi+1}$-spreading model with a constant c, then*

$$\widetilde{\operatorname{cca}}_\xi((x_{n^3})_{n\in\mathbb{N}}) \geq 2c.$$

Proof. We may without loss of generality assume that $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ and that $x = 0$. We will show that for every $P \in [\mathbb{N}]$ we have $\operatorname{cca}\left(\xi_n^P \cdot (x_{k^3})_{k\in\mathbb{N}}\right) \geq 2c$, and thus $\widetilde{\operatorname{cca}}_\xi((x_{n^3})_{n\in\mathbb{N}}) \geq 2c$.

Take $l \in \mathbb{N}$ and set $n = l^2 + l$ and $m = l^3 + l$. For the sake of brevity we will write $z_j = \xi_j^P \cdot (x_{k^3})_{k\in\mathbb{N}}$. Then

$$\begin{aligned}\left\|\frac{1}{m}\sum_{j=1}^m z_j - \frac{1}{n}\sum_{j=1}^n z_j\right\| &= \left\|\left(\frac{1}{m} - \frac{1}{n}\right)\sum_{j=1}^l z_j + \left(\frac{1}{m} - \frac{1}{n}\right)\sum_{j=l+1}^n z_j + \frac{1}{m}\sum_{j=n+1}^m z_j\right\| \\ &\geq \left\|\left(\frac{1}{m} - \frac{1}{n}\right)\sum_{j=l+1}^n z_j + \frac{1}{m}\sum_{j=n+1}^m z_j\right\| - \left\|\left(\frac{1}{m} - \frac{1}{n}\right)\sum_{j=1}^l z_j\right\|.\end{aligned}$$

It follows from the triangle inequality that

$$\left\|\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^{l} z_j\right\| \leq l\cdot\left(\frac{1}{n}-\frac{1}{m}\right) = \frac{l^3-l^2}{(l^2+l)(l^3+l)} \xrightarrow{l\to\infty} 0.$$

For $j\in\mathbb{N}$ set $\xi_j^P=(b_k^j)_{k\in\mathbb{N}}$ and $F_j=\operatorname{supp}\xi_j^P$. We define $F=\bigcup_{j=l+1}^m F_j$ and the finite sequence $(a_k)_{k\in F}$ by

$$a_k=\begin{cases}\left(\frac{1}{m}-\frac{1}{n}\right)b_k^j & \dots \text{ if } k\in F_j \text{ for } l+1\leq j\leq n,\\ \frac{1}{m}b_k^j & \dots \text{ if } k\in F_j \text{ for } n+1\leq j\leq m.\end{cases}$$

Then

$$\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=l+1}^{n} z_j+\frac{1}{m}\sum_{j=n+1}^{m} z_j=\sum_{k\in F} a_k x_k.$$

We have already observed that the sets F_j's belong to the family $\mathcal{S}_\xi$. Hence, the sets $G_j=\{k^3:\ k\in F_j\}$ are also in the family $\mathcal{S}_\xi$ by its spreading property, and the set $G=\bigcup_{k=l+1}^m G_j$ is in $\mathcal{S}_{\xi+1}$ as $\min G_{l+1}\geq(l+1)^3>l^3=m-l$. Hence,

$$\left\|\sum_{k\in F} a_k x_{k^3}\right\| = \left\|\sum_{k\in G} a_{\sqrt[3]{k}} x_k\right\| \geq c\sum_{k\in G}|a_{\sqrt[3]{k}}| = c\sum_{k\in F}|a_k|$$

as $(x_k)_{k\in\mathbb{N}}$ is an $\ell_1^{\xi+1}$-spreading model with constant c. Therefore we have

$$\begin{aligned}\left\|\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=l+1}^{n} z_j+\frac{1}{m}\sum_{j=n+1}^{m} z_j\right\| &= \left\|\sum_{k\in F} a_k x_k\right\|\\ \geq c\sum_{k\in F}|a_k| &= c\left(\sum_{j=l+1}^{n}\left(\frac{1}{n}-\frac{1}{m}\right)+\sum_{j=n+1}^{m}\frac{1}{m}\right)\\ &= c\left(\frac{(l^3-l^2)l^2}{(l^2+l)(l^3+l)}+\frac{l^3-l^2}{l^3+l}\right) \xrightarrow{l\to\infty} 2c.\end{aligned}$$

Hence,

$$\liminf_{l\to\infty}\left\|\frac{1}{m}\sum_{j=1}^{m} z_j-\frac{1}{n}\sum_{j=1}^{n} z_j\right\| \geq 2c.$$

It follows that $\operatorname{cca}(z_n)=\operatorname{cca}\left(\xi_n^P\cdot(x_{k^3})_{k\in\mathbb{N}}\right)\geq 2c$. Since $P\in[\mathbb{N}]$ was chosen arbitrarily, we get $\widetilde{\operatorname{cca}}_\xi((x_{n^3})_{n\in\mathbb{N}})\geq 2c$. $\square$

A version of the preceding proposition can be also shown using the approach of [3]. However, the best result we were able to get using this approach was $\widetilde{\operatorname{cca}}_\xi^s((x_n)_{n\in\mathbb{N}})\geq c$. The approach of [7] is more elementary and gives a better constant. Note that the first inequality of $(\star)$ from Theorem 3.3 is an immediate consequence of Proposition 3.7. We proceed with proving the third inequality, using the approach of [3].

Proposition 3.8. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X which weakly converges to some $x\in X$ and let $c>0$ and $\xi<\omega_1$. Suppose that $\widetilde{\operatorname{cca}}_\xi^s((x_n)_{n\in\mathbb{N}})>c$. Then $(x_n)_{n\in\mathbb{N}}$ is $(\xi+1,\frac{c}{2})$-large.*

Proof. We can assume that $x = 0$ and $(x_n)_{n\in\mathbb{N}} \subseteq B_X$. Take $c' > c$ such that $\widetilde{\mathrm{cca}}^s_\xi((x_n)_{n\in\mathbb{N}}) > c'$ and fix $\epsilon > 0$ small enough so that $(1-2\epsilon)c' \geq c$.

As $\widetilde{\mathrm{cca}}^s_\xi((x_n)_{n\in\mathbb{N}}) > c'$, we can find $M \in [\mathbb{N}]$ such that for all $N \in [M]$ we have $\mathrm{cca}(\xi_n^N \cdot (x_k)_{k\in\mathbb{N}}) > c'$. It follows from the triangle inequality that for all $N \in [M]$ we have

$$\limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} \xi_i^N \cdot (x_k)_{k\in\mathbb{N}} \right\| > \frac{c'}{2}.$$

Now we will recursively construct a sequence $(L_n)_{n\in\mathbb{N}}$ of infinite subsets of M such that

(a) $L_1 \in [M]$, $L_n \in [L_{n-1}]$ for $n \geq 2$.

(b) For every $n \in \mathbb{N}$ and $N \in [L_n]$ there is $x^* \in B_{X^*}$ with

$$x^*\left(\xi_j^N \cdot (x_k)_{k\in\mathbb{N}}\right) > (1-\epsilon)\frac{c'}{2}, \qquad\qquad j = 1, \ldots, n.$$

We shall proceed by induction over $n \in \mathbb{N}$. For convenience we set $L_0 = M$. Let us assume that L_{n-1} was already defined. We partition $[L_{n-1}]$ into two subsets

$$A_1 = \left\{ P \in [L_{n-1}] : \ \exists x^* \in B_{X^*} \text{ such that } x^*\left(\xi_j^P \cdot (x_k)_{k\in\mathbb{N}}\right) > (1-\epsilon)\frac{c'}{2},\ j \leq n \right\}$$
$$A_2 = [L_{n-1}] \setminus A_1.$$

The set A_1 is open. Indeed, for a fixed set $P \in A_1$ the sets $P' \in [L_{n-1}]$ for which $\chi_P(j) = \chi_{P'}(j)$ for $j = 1, \ldots, \max\operatorname{supp} \xi_n^P$ form a neighbourhood of P in $[L_{n-1}]$ which is contained in A_1 as $\xi_j^P = \xi_j^{P'}$, $j = 1, \ldots, n$, for such sets by P.3 in [3, page 171]. Hence, A_1 is a Borel set, and thus a completely Ramsey set. By the infinite Ramsey theorem [1, Theorem 10.1.3.] there is $L_n \in [L_{n-1}]$ such that either $[L_n] \subseteq A_1$ or $[L_n] \subseteq A_2$. We will show that the second case is not possible.

We recall that $\limsup_s \left\| \frac{1}{s} \sum_{i=1}^s \xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}} \right\| > \frac{c'}{2}$. Therefore, we can find large enough $s \in \mathbb{N}$ and $x^* \in B_{X^*}$ such that the following two conditions are satisfied.

$$x^*\left(\frac{1}{s}\sum_{i=1}^{s} \xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}}\right) > \frac{c'}{2}, \qquad\qquad s\epsilon\frac{c'}{2} \geq n.$$

Set

$$I_1 = \left\{ 1 \leq i \leq s : \ x^*(\xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}}) > (1-\epsilon)\frac{c'}{2} \right\}$$
$$I_2 = \{1, \ldots, s\} \setminus I_1.$$

Then

$$\begin{aligned}
\frac{c'}{2} &< \frac{1}{s}\sum_{i=1}^{s} x^*\left(\xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}}\right) \\
&= \frac{1}{s}\left(\sum_{i\in I_1} x^*\left(\xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}}\right) + \sum_{i\in I_2} x^*\left(\xi_i^{L_n} \cdot (x_k)_{k\in\mathbb{N}}\right) \right) \\
&\leq \frac{1}{s}\left(|I_1| + s\,(1-\epsilon)\frac{c'}{2} \right).
\end{aligned}$$

But that implies

$$|I_1| > s\epsilon\frac{c'}{2} \geq n.$$

Hence, we can find $i_1 < i_2 < \cdots < i_n \leq s$ satisfying $x^*(\xi_{i_j}^{L_n} \cdot (x_k)_{k\in\mathbb{N}}) > (1-\epsilon)\frac{c'}{2}$. But now we can pick $P \in [L_n]$, such that $\xi_{i_j}^{L_n} = \xi_j^P$, see P.4 in [3, page 171], and for this P we have $P \in A_1$. Hence, $[L_n] \not\subseteq A_2$, and therefore $[L_n] \subseteq A_1$.

Now we take a diagonal subsequence $L = (l_k)_{k\in\mathbb{N}}$ of the sequences $(L_k)_{k\in\mathbb{N}}$. For all $n \in \mathbb{N}$ and $P \in [L]$ with $l_n \leq \min P$ there is some $x^* \in B_{X^*}$ such that

$$x^*(\xi_i^P \cdot (x_k)_{k\in\mathbb{N}}) > (1-\epsilon)\frac{c'}{2} \qquad \text{for } i = 1, \ldots, n.$$

Indeed, as $l_n \leq \min P$, we have $P \in [L_n]$ and are done by property (b).

Let us denote for brevity $c'' = (1-2\epsilon)\frac{c'}{2}$. We take a further subset $P = (p_n)_{n\in\mathbb{N}} \in [L]$ such that the conclusion of Lemma 3.6 is satisfied on P for $\delta = c''$ and ϵ.

We will show that $\mathcal{F}_{c''}$ and P satisfy the assumptions of Lemma 3.5 for $\epsilon' = \epsilon\frac{c'}{2}$. That is, we want to show that for every $n \in \mathbb{N}$ and $P' \in [P]$ with $p_n \leq \min P'$ there is $F \in \mathcal{F}_{c''}$ with $\langle \xi_i^{P'}, F\rangle > \epsilon'$ for $i = 1, \ldots, n$. Take such $n \in \mathbb{N}$ and $P' \in [P]$. As $P' \in [L]$ and $l_n \leq p_n \leq \min P'$, we can find some $x^* \in B_{X^*}$ such that

$$x^*\left(\xi_i^{P'} \cdot (x_k)_{k\in\mathbb{N}}\right) > (1-\epsilon)\frac{c'}{2} = c'' + \epsilon', \qquad i = 1, \ldots, n.$$

But then for $F = \{n \in \mathbb{N} : \ x^*(x_n) > c''\} \in \mathcal{F}_{c''}$ we have that $\langle \xi_i^{P'}, F\rangle > \epsilon'$ for $i = 1, \ldots, n$ as otherwise, if we set $\xi_i^{P'} = (b_k)_{k\in\mathbb{N}}$, we would get the following contradiction

$$\begin{aligned} c'' + \epsilon' < \sum_{k\in\mathbb{N}} b_k x^*(x_k) &= \sum_{k\in F} b_k x^*(x_k) + \sum_{k\in\mathbb{N}\setminus F} b_k x^*(x_k) \\ &\leq \sum_{k\in F} b_k + \sum_{k\in\mathbb{N}\setminus F} b_k c'' \leq \epsilon' + c''. \end{aligned}$$

Hence, the assumptions of Lemma 3.5 are satisfied and we can find $Q = (q_i)_{i\in\mathbb{N}} \in [P]$ such that $\mathcal{S}_{\xi+1}^Q \subseteq \mathcal{F}_{c''}$. But this means that $(x_n)_{n\in\mathbb{N}}$ is $(\xi+1, c'')$-large. Recall that

$$c'' = (1-2\epsilon)\frac{c'}{2} \geq \frac{c}{2},$$

and thus $\mathcal{F}_{c''} \subseteq \mathcal{F}_{\frac{c}{2}}$ and $(x_n)_{n\in\mathbb{N}}$ is also $(\xi+1, \frac{c}{2})$-large. □

The third inequality of $(\star)$ from Theorem 3.3 follows from Proposition 3.8. We finish the proof of Theorem 3.3 by proving the last inequality, for which we also use the approach of [3].

Proposition 3.9. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space X which weakly converges to some $x \in X$. Let $c > 0$ and $\xi < \omega_1$ be such that $(x_n)_{n\in\mathbb{N}}$ is (ξ, c)-large. Then for any $d < \frac{c}{2}$ there is $N \in [\mathbb{N}]$ such that $(x_n - x)_{n\in N}$ generates an ℓ_1^ξ-spreading model with constant d.*

Proof. Without loss of generality we can assume that $x = 0$ and $(x_n)_{n\in\mathbb{N}} \subseteq B_X$. As $(x_n)_{n\in\mathbb{N}}$ is (ξ, c)-large, there is $M \in [\mathbb{N}]$ such that $\mathcal{S}_\xi^M \subseteq \mathcal{F}_c$. We can take $\epsilon > 0$ small enough such that $(1-\epsilon)\frac{c}{2} - \epsilon c \geq d$. We will use Lemma 3.6 to find $N = (n_k)_{k\in\mathbb{N}} \in [M]$ such that the conclusion of Lemma 3.6 is satisfied on N for ϵ and $\delta = c$.

Now we will show that $(x_n)_{n\in N}$ generates an ℓ_1^ξ-speading model with constant d. Fix $F \in \mathcal{S}_\xi$ and a sequence of scalars $(b_k)_{k\in F}$. We can assume without loss of generality that $\sum_{k\in F} |b_k| = 1$. Then it is enough to show that

$$\left\| \sum_{k\in F} b_k x_{n_k} \right\| \geq d.$$

We define the sequence of scalars $(a_k)_{k\in\mathbb{N}}$ by the rule $a_j = b_k$, if $j = n_k$ for some $k \in F$, and $a_j = 0$ otherwise. Then $(a_k)_{k\in\mathbb{N}} \in S_{\ell_1}$ and $\operatorname{supp}((a_k)_{k\in\mathbb{N}}) \subseteq N$. Hence, we get that the following inequality holds for any $G \in \mathcal{F}_c$.

$$\left\| \sum_{k\in\mathbb{N}} a_k x_k \right\| \geq (1-\epsilon)c \cdot \langle (a_n)_{n\in\mathbb{N}}, G\rangle - \epsilon c.$$

We can also assume, if we define $F^+ = \{k \in F : \ b_k > 0\}$ and $F^- = \{k \in F : b_k < 0\}$, that $\sum_{k\in F^+} |b_k| \geq \frac{1}{2}$. If not, we can consider $(-b_k)_{k\in F}$ instead of $(b_k)_{k\in F}$. Note that $G = \{n_k : \ k \in F^+\} \in \mathcal{S}_\xi^N \subseteq \mathcal{F}_c$ as $F^+ \in \mathcal{S}_\xi$. Then we have

$$\langle (a_n)_{n\in\mathbb{N}}, G\rangle = \sum_{k\in F^+} a_{n_k} = \sum_{k\in F^+} b_k \geq \frac{1}{2}.$$

Therefore

$$\left\| \sum_{k\in F} b_k x_{n_k} \right\| = \left\| \sum_{k\in\mathbb{N}} a_k x_k \right\| \geq (1-\epsilon)\frac{c}{2} - \epsilon c \geq d.$$

□

Remark. The proof of Theorem 3.3 combines the approach of [7], which is generalised for arbitrary $\xi < \omega_1$ and used to prove Proposition 3.7, and the approach of [3], which is used to prove Propositions 3.8 and 3.9. More precisely, the proofs of Propositions 3.8 and 3.9 mimic the proof of [3, Theorem 2.4.1] with quantitative interpretation of [3, Lemmata 2.4.3, 2.4.8]. We also needed Lemmata 3.4 and 3.5 (that is [3, Propositions 2.1.10, 2.3.6 and Theorem 2.2.6]), but these results offer no quantitative improvement, and so are presented here without proof.

Now we will prove two corollaries to Theorem 3.3. The first one is that the quantity wbs_ξ indeed characterizes weak ξ-Banach-Saks sets.

Proposition 3.10. *Let A be a bounded set in a Banach space X and $\xi < \omega_1$. Then A is a weak ξ-Banach-Saks set, if and only if* $\mathrm{wbs}_\xi(A) = 0$.

Proof. It is straightforward that if $\mathrm{wbs}_\xi(A) > 0$ then A is not a weak ξ-Banach-Saks set. On the other hand, suppose that $\mathrm{wbs}_\xi(A) = 0$ and fix a sequence $(x_n)_{n\in\mathbb{N}}$ in A which is weakly convergent to some $x \in X$. It follows from Theorem 3.3 that $\mathrm{sm}_{\xi+1}(A) = 0$, and therefore $(x_n - x)_{n\in\mathbb{N}}$ contains no subsequence that generates an $\ell_1^{\xi+1}$-spreading model. Hence, by Theorem 3.2, we get that for every $M \in [\mathbb{N}]$ there is $L \in [M]$ such that for all $P \in [L]$ the sequence $(x_n - x)_{n\in\mathbb{N}}$, and thus also the sequence $(x_n)_{n\in\mathbb{N}}$, is (P,ξ)-summable. Therefore, we can take $M = \mathbb{N}$ and $P = L$, and get that A is a weak ξ-Banach-Saks set. □

The second corollary shows that weak ξ-Banach-Saks sets enjoy a formally stronger property analogous to the fact that any weakly convergent sequence in a weak Banach-Saks set admits a subequence with every further subsequence being Cesàro summable (indeed, in this case it is enough to consider a uniformly weakly convergent subsequence). Note that the following proposition is, in essence, a qualitative version of the inequalities $\mathrm{wbs}_\xi(A) \leq \mathrm{wbs}_\xi^s(A) \leq 2\,\mathrm{wbs}_\xi(A)$ from Theorem 3.3.

Proposition 3.11. *Let A be a bounded subset of a Banach space X. Then the following are equivalent:*

(a) For every weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in A and every $M \in [\mathbb{N}]$ there is $N \in M$ such that for all $P \in [N]$ the sequence $(x_n)_{n\in\mathbb{N}}$ is (P,ξ)-summable;

(b) A is a weak ξ-Banach-Saks set.

Proof. The fact that (a) implies (b) follows immediately from the definitions. We will show the other implication. Suppose that (a) does not hold. Then by Theorem 3.2 there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A which converges weakly to some $x \in X$ such that $(x_n - x)_{n\in\mathbb{N}}$ generates an $\ell_1^{\xi+1}$-spreading model with constant c for some $c > 0$. But then $\mathrm{sm}_{\xi+1}(A) \geq c$ and thus by Theorem 3.3 $\mathrm{wbs}_\xi(A) \geq 2c$ and A cannot be a weak ξ-Banach-Saks set by Proposition 3.10. □

3.4 ξ-Banach-Saks sets and compactness

Following [7], in this section we will show the quantitative interpretation of the following implications for a bounded subset A of a Banach space X and $\xi < \omega_1$:

$$\begin{gathered} A \text{ is relatively norm compact} \\ \Downarrow \\ A \text{ is a } \xi\text{-Banach-Saks set} \\ \Downarrow \\ A \text{ is relatively weakly compact and a weak } \xi\text{-Banach-Saks set.} \end{gathered}$$

Note that the second implication can be reversed but the converse implication cannot be quantified for $\xi = 0$, as illustrated by [7, Example 3.3.].

We will first define the quantities measuring weak and norm non-compactness.

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X. We define the quantity

$$\widetilde{\mathrm{ca}}(x_n) = \inf\{\mathrm{ca}(y_n) :\ (y_n)_{n\in\mathbb{N}} \text{ is a subsequence of } (x_n)_{n\in\mathbb{N}}\}.$$

Let A be a bounded subset of a Banach space X. We define

$$\begin{aligned} \beta(A) &= \sup\{\widetilde{\mathrm{ca}}(x_n) :\ (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } A\} \\ \mathrm{wck}_X(A) &= \sup\{\mathrm{d}(\mathrm{clust}_{X^{**}}(x_n), X) :\ (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } A\}, \end{aligned}$$

where $\mathrm{d}(B,C) = \inf\{\|b-c\| : b \in B, c \in C\}$ is the standard distance of sets and $\mathrm{clust}_{X^{**}}(x_n)$ is the set of all weak* cluster points of the sequence $(x_n)_{n\in\mathbb{N}}$ in the space X^{**}.

Note that the quantity β indeed measures non-compactness and the quantity wck_X indeed measures weak non-compactness. That is, $\beta(A) = 0$ if and only if A is relatively compact and $\mathrm{wck}_X(A) = 0$ if and only if A is relatively weakly compact. For more information about these quantities and their relation to other quantities see [7]. Now we are all prepared to prove the following theorem.

Theorem 3.12. *Let A be a bounded subset of a Banach space X and $\xi < \omega_1$. Then*

$$\max\{\mathrm{wck}_X(A), \mathrm{wbs}_\xi(A)\} \leq \mathrm{bs}_\xi(A) \leq \mathrm{bs}^s_\xi(A) \leq \beta(A).$$

To prove the inequality $\mathrm{wck}_X(A) \leq \mathrm{bs}_\xi(A)$ we will need to define an auxiliary quantity γ_0. For a bounded subset A of a Banach space X we define

$$\begin{aligned}\gamma_0(A) = \sup\{|\lim_{m\to\infty}\lim_{n\to\infty} x^*_m(x_n)| :\ & \\ & (x_m)^*_{m\in\mathbb{N}} \text{ is a weak}^* \text{ null sequence in } B_{X^*}, \\ & (x_n)_{n\in\mathbb{N}} \text{ is a sequence in } A \\ & \text{and all the involved limits exist}\}.\end{aligned}$$

The quantity γ_0 was introduced in [9] as a measure of weak compactness in spaces whose duals have weak* angelic unit balls. Later, it was used [7] to prove a version of Theorem 3.12 for $\xi = 1$.

Lemma 3.13. *Let A be a bounded subset of a Banach space X and $\xi < \omega_1$. Then*

$$\gamma_0(A) \leq \mathrm{bs}_\xi(A).$$

Proof. Suppose that $\gamma_0(A) > c$ for some $c > 0$. Then there is a sequence $(x_k)_{x\in\mathbb{N}}$ in A and a weak* null sequence $(x^*_n)_{n\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\lim_{k\to\infty} x^*_j(x_k) > c.$$

We can assume without loss of generality that $\lim_{k\to\infty} x^*_j(x_k) > c$ for all $j \in \mathbb{N}$. Fix $P \in [\mathbb{N}]$ and define, for $k \in \mathbb{N}$,

$$y_k = \frac{1}{k}\sum_{j=1}^{k} \xi^P_j \cdot (x_n)_{n\in\mathbb{N}}.$$

We want to show that $\mathrm{ca}(y_n) \geq c$. Note that for each $j \in \mathbb{N}$ we have

$$\lim_{k\to\infty} x^*_j(y_k) = \lim_{k\to\infty} x^*_j(x_k) > c.$$

Now fix $\epsilon > 0$ and $k \in \mathbb{N}$. Using weak* nullness of the sequence $(x^*_j)_{j\in\mathbb{N}}$, we can find $j \in \mathbb{N}$ such that $x^*_j(y_k) < \epsilon$. Then we can find $l > k$ such that $x^*_j(y_l) > c$. But then

$$\|y_l - y_k\| \geq x^*_j(y_l - y_k) > c - \epsilon.$$

As ϵ and k were chosen arbitrarily, we get that $\mathrm{ca}(y_n) \geq c$. As $P \in [\mathbb{N}]$ was also chosen arbitrarily, we get $\widetilde{\mathrm{cca}}_\xi((x_n)_{n\in\mathbb{N}}) \geq c$, and this implies that $\mathrm{bs}_\xi(A) \geq c$. $\square$

Proof of Theorem 3.12. We first note that the inequality $\mathrm{bs}_\xi(A) \leq \mathrm{bs}^s_\xi(A)$ is trivial. We proceed with the first inequality. That $\mathrm{bs}_\xi(A) \geq \mathrm{wbs}_\xi(A)$ is clear. If X is separable, then the closed unit ball of X^* is metrizable and $\gamma_0(A) = \mathrm{wck}_X(A)$ by [9, Theorem 6.1.]. Hence, for separable X we get the inequality $\mathrm{bs}_\xi(A) \geq \gamma_0(A) = \mathrm{wck}_X(A)$.

If X is arbitrary and $\mathrm{wck}_X(A) > c$ for some $c > 0$, we can find a sequence $(x_k)_{k\in\mathbb{N}}$ in A with $\mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X) > c$. If we set $Y = \overline{\mathrm{span}}\{x_k : \ k \in \mathbb{N}\}$, then Y is a separable subspace of X and $\mathrm{d}(\mathrm{clust}_{Y^{**}}(x_k), Y) \geq \mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X)$ (see the proof of [7, Theorem 3.1.]). Therefore

$$\mathrm{d}(\mathrm{clust}_{Y^{**}}(x_k), Y) \geq \mathrm{d}(\mathrm{clust}_{X^{**}}(x_k), X) > c.$$

It follows that $\mathrm{wck}_Y(A \cap Y) > c$, and therefore

$$\mathrm{bs}_\xi(A) \geq \mathrm{bs}_\xi(A \cap Y) \geq \mathrm{wck}_Y(A \cap Y) > c$$

by the already proved separable case.

The last inequality we need to prove is $\mathrm{bs}^s_\xi(A) \leq \beta(A)$. For this we use to following lemma.

Lemma 3.14. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X and $\xi < \omega_1$. Let there be $c > 0$ and $N \in [\mathbb{N}]$ such that $\mathrm{ca}((x_n)_{n\in N}) < c$. Then for any $P \in [N]$ we have $\mathrm{cca}(\xi^P_n \cdot (x_k)_{k\in\mathbb{N}}) \leq c$.*

Proof. As $\mathrm{ca}((x_n)_{n\in N}) < c$, we can find $n_0 \in \mathbb{N}$ such that

$$\|x_n - x_m\| \leq c, \qquad \text{for } n, m \in N \text{ and } n, m > n_0.$$

We define $y_n = \xi^P_n \cdot (x_k)_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$. Note that

$$\|y_n - y_m\| \leq c, \qquad \text{for } n, m > n_0.$$

To prove it we notice that $\|x_n - y_m\| \leq c$ for each $n, m > n_0$, $n \in N$ as such y_m is a convex combination of elements x_j's for which $\|x_n - x_j\| \leq c$. Hence, for $n > n_0$ we have that y_n is a convex combination of elements x_j's for which $\|x_j - y_m\| \leq c$ for each $m > n_0$, and thus also $\|y_n - y_m\| \leq c$ for each $m > n_0$. Hence, $\mathrm{ca}((y_n)_{n\in\mathbb{N}}) \leq c$ and by [7, Lemma 3.4.] $\mathrm{cca}(\xi^P_n \cdot (x_k)_{k\in\mathbb{N}}) = \mathrm{cca}(y_n) \leq c$. ☐

The only inequality left is $\beta(A) \geq \mathrm{bs}^s_\xi(A)$. Let $\beta(A) < c$ for some $c > 0$ and take an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in A. What we want to show is $\widetilde{\mathrm{cca}}^s_\xi((x_n)_{n\in\mathbb{N}}) \leq c$. Let $M \in [\mathbb{N}]$ be arbitrary, then we can find $N \in [M]$ such that $\mathrm{ca}((x_n)_{n\in M}) < c$. It then follows from Lemma 3.14 that for any $P \in [N]$ we have $\mathrm{cca}(\xi^P_n \cdot (x_k)_{k\in\mathbb{N}}) \leq c$. In particular, $\mathrm{cca}(\xi^N_n \cdot (x_k)_{k\in\mathbb{N}}) \leq c$. As M was arbitrary, $\widetilde{\mathrm{cca}}^s_\xi((x_n)_{n\in\mathbb{N}}) \leq c$. As $(x_n)_{n\in\mathbb{N}}$ was also chosen arbitrarily, $\mathrm{bs}^s_\xi(A) \leq c$ and we are done. Thus, Theorem 3.12 is proved. ☐

In the following propositions we show the converse to the second implication mentioned at the beginning of this section, that is that a relatively weakly compact weak ξ-Banach-Saks set is a ξ-Banach-Saks set. As mentioned, this implication cannot be fully quantified.

Proposition 3.15. *Let A be relatively weakly compact subset of a Banach space X and $\xi < \omega_1$. Then* $\mathrm{wbs}_\xi^s(A) = \mathrm{bs}_\xi^s(A)$.

Proof. For any bounded set A we have $\mathrm{wbs}_\xi^s(A) \le \mathrm{bs}_\xi^s(A)$. For the converse, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A and $M \in [\mathbb{N}]$. As A is relatively weakly compact, we can use the Eberlein-Šmulyan theorem to find $N = (n_k)_{k\in\mathbb{N}} \in [M]$ such that $(x_k)_{k\in N}$ is weakly convergent to some $x \in X$. Denote by $N^c = \mathbb{N} \setminus N$. We define $y_k = x_{n_k}$, for $k \in N^c$, and $y_k = x_k$, for $k \in N$. Then $(y_k)_{k\in\mathbb{N}}$ is a sequence in A weakly converging to x. Hence, for any $\epsilon > 0$ there is $L_\epsilon \in [N]$ such that $\mathrm{cca}(\xi_n^{L_\epsilon} \cdot (y_k)_{k\in\mathbb{N}}) \le \mathrm{wbs}_\xi^s(A) + \epsilon$. But $\xi_n^{L_\epsilon} \cdot (y_k)_{k\in\mathbb{N}} = \xi_n^{L_\epsilon} \cdot (x_k)_{k\in\mathbb{N}}$, as $y_k = x_k$ for $k \in L_\epsilon \subseteq N$. Thus, as $M \in [\mathbb{N}]$ and $\epsilon > 0$ were arbitrary, we have shown that $\widetilde{\mathrm{cca}}_\xi^s((x_n)_{n\in\mathbb{N}}) \le \mathrm{wbs}_\xi^s(A)$. As $(x_n)_{n\in\mathbb{N}}$ was arbitrary, we get $\mathrm{bs}_\xi^s(A) \le \mathrm{wbs}_\xi^s(A)$. ☐

We can use the same trick (that is replacing a bounded sequence $(x_n)_{n\in\mathbb{N}}$ with a weakly convergent sequence $(y_n)_{n\in\mathbb{N}}$ as in the proof of Proposition 3.15) to prove the promised converse to the second implication mentioned at the beginning of this section as well as an analogue of Proposition 3.11 for the ξ-Banach-Saks property.

Proposition 3.16. *Let $\xi < \omega_1$ and A be a bounded set in a Banach space X. Then the following are equivalent:*

(i) A is a ξ-Banach-Saks set;

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in A and every $M \in [\mathbb{N}]$ there is $L \in [M]$ such that for all $P \in [L]$ the sequence $(x_n)_{n\in\mathbb{N}}$ is (P, ξ)-summable;

(iii) A is a relatively weakly compact weak ξ-Banach-Saks set.

Proof. If A ia a ξ-Banach-Saks set, then it is trivially a weak ξ-Banach-Saks set. Further $\mathrm{bs}_\xi(A) = 0$, and thus A is relatively weakly compact by Theorem 3.12. Hence, (i) implies (iii). Clearly, (ii) implies (i).

What is left is the implication (iii) implies (ii). Let us suppose that A is a relatively weakly compact weak ξ-Banach-Saks set. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A and $M \in [\mathbb{N}]$. It follows from the Eberlein-Šmulyan theorem that there is $N \in [M]$ such that $(x_n)_{n\in N}$ is weakly convergent. We define the sequence $(y_n)_{n\in\mathbb{N}}$ in exactly the same way as in the proof of Proposition 3.15. Then $(y_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in the weak ξ-Banach-Saks set A, and thus by Proposition 3.11 there is $L \in [N]$ such that for all $P \in [L]$ the sequence $(y_n)_{n\in\mathbb{N}}$ is (P, ξ)-summable. But then again we have $x_k = y_k$ for $k \in L$, and therefore the sequence $(x_n)_{n\in\mathbb{N}}$ is also (P, ξ)-summable. Hence, we have found for any sequence $(x_n)_{n\in\mathbb{N}}$ in A and $M \in [\mathbb{N}]$ a further subset $L \in [M]$ such that for all $P \in [L]$ the sequence $(x_n)_{n\in\mathbb{N}}$ is (P, ξ)-summable and (ii) holds. ☐

In the following proposition we prove that both of the quantities bs_ξ and bs_ξ^s quantify the ξ-Banach-Saks property.

Proposition 3.17. *Let A be a bounded set in a Banach space X and $\xi < \omega_1$. Then A is a ξ-Banach-Saks set, if and only if $\mathrm{bs}_\xi(A) = 0$, if and only if $\mathrm{bs}_\xi^s(A) = 0$.*

Proof. If $\mathrm{bs}^s_\xi(A) = 0$, then trivially $\mathrm{bs}_\xi(A) = 0$. If $\mathrm{bs}_\xi(A) = 0$, we get by Theorem 3.12 and Proposition 3.10 that A is a relatively weakly compact weak ξ-Banach-Saks set, and thus A is a ξ-Banach-Saks set by Proposition 3.16. Now suppose that A is a ξ-Banach-Saks set. Then by Proposition 3.16 A is a relatively weakly compact weak ξ-Banach-Saks set. Therefore, $\mathrm{wbs}^s_\xi(A) = 0$ by Proposition 3.10 and Theorem 3.3. Hence, $\mathrm{bs}^s_\xi(A) = 0$ by Proposition 3.15. $\square$

3.5 The quantities as functions of ξ

In this section we will analyse the functions $\mathrm{bs}^s_\xi(A)$, $\mathrm{wbs}^s_\xi(A)$, $\mathrm{wus}_\xi(A)$ and $\mathrm{sm}_\xi(A)$ for a fixed bounded subset A of a Banach space X as functions of ξ. We begin with the quantities wus_ξ and sm_ξ and prove the simple observation that they are non-increasing.

Lemma 3.18. *Let A be a bounded subset of a Banach space X and let $\zeta < \xi < \omega_1$ be ordinals. Then $\mathrm{wus}_\xi(A) \leq \mathrm{wus}_\zeta(A)$ and $\mathrm{sm}_\xi(A) \leq \mathrm{sm}_\zeta(A)$.*

Proof. It follows from [3, Lemma 2.1.8.(a)] that there is $n = n(\zeta, \xi)$, such that for all $F \in \mathcal{S}_\zeta$ with $n \leq F$, we have $F \in \mathcal{S}_\xi$. In other words, if we set $N = \{m \in \mathbb{N} : n \leq m\}$, then $\mathcal{S}_\zeta[N] \subseteq \mathcal{S}_\xi$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A which generates an ℓ_1^ξ-spreading model with constant $c > 0$ then the sequence $(x_n)_{n\in N}$ generates an ℓ_1^ζ-spreading model with constant c, which gives us the inequality for the quantity sm.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A which weakly converges to some $x \in X$ and is (ξ, c)-large for some $c > 0$. Then there is $M \in [\mathbb{N}]$ such that $\mathcal{S}_\xi^M \subseteq \mathcal{F}_c((x_n - x)_{n\in\mathbb{N}})$. It is easy to check that this is equivalent to saying that $\mathcal{S}_\xi \subseteq \mathcal{F}_c((x_n - x)_{n\in M})$. It follows that

$$\mathcal{S}_\zeta^N \subseteq \mathcal{S}_\zeta[N] \subseteq \mathcal{S}_\xi \subseteq \mathcal{F}_c((x_n - x)_{n\in M})$$

and $(x_n)_{n\in M}$ is (ζ, c)-large, which gives us the inequality for the quantity wus. $\square$

Now we turn our attention to the quantities bs^s_ξ and wbs^s_ξ. We will first need the following definition.

Definition. Let $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be two sequences in a Banach space X. We say that the sequence $(z_n)_{n\in\mathbb{N}}$ is a *non-increasing block convex combination* of the sequence $(y_n)_{n\in\mathbb{N}}$ if

$$z_n = \sum_{j=k_n+1}^{k_{n+1}} \alpha(j) y_j,$$

where $(k_n)_{n\in\mathbb{N}}$ is an increasing sequence of integers with $k_1 = 0$ and $(\alpha(j))_{j\in\mathbb{N}}$ is a non-increasing sequence of real numbers satisfying $\sum_{j=k_n+1}^{k_{n+1}} \alpha(j) = 1$ for each $n \in \mathbb{N}$.

For example, the M-summability method $([\xi+1]_n^M)_{n\in\mathbb{N}}$ is a non-increasing block convex combination of the M-summability method $(\xi_n^M)_{n\in\mathbb{N}}$ for any $\xi < \omega_1$ and $M \in [\mathbb{N}]$. It is readily proved that if a sequence $(x_n)_{n\in\mathbb{N}}$ is a non-increasing block convex combination of a sequence $(y_n)_{n\in\mathbb{N}}$, which is a non-increasing block convex combination of a sequence $(z_n)_{n\in\mathbb{N}}$, then $(x_n)_{n\in\mathbb{N}}$ is a non-increasing block convex combination of $(z_n)_{n\in\mathbb{N}}$.

Now we will prove an auxiliary lemma which shows that the quantity cca behaves well with respect to taking non-increasing block convex combinations.

Lemma 3.19. *Let $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be two sequences in a Banach space X such that $(z_n)_{n\in\mathbb{N}}$ is a non-increasing block convex combination of $(y_n)_{n\in\mathbb{N}}$. Then* $\mathrm{cca}(z_n) \leq \mathrm{cca}(y_n)$.

Proof. Let $(k_n)_{n\in\mathbb{N}}$ and $(\alpha(j))_{j\in\mathbb{N}}$ be the sequences from the definition of non-increasing block convex combination. Let $c > 0$ and suppose that $\mathrm{cca}(y_n) \leq c$. Let us define $u_n = \frac{1}{n}\sum_{j=1}^{n} y_j$. The strategy is to show that the Cesàro means of the sequence $(z_n)_{n\in\mathbb{N}}$ can be written as convex combinations of u_n's.

Fix $\epsilon > 0$ and define $c' = c + \epsilon$. As $\mathrm{ca}(u_n) = \mathrm{cca}(y_n) < c'$, we can find $N_1 \in \mathbb{N}$ such that $\|u_j - u_i\| \leq c'$ for all $i, j > N_1$. We define for $n \in \mathbb{N}$ and $j \leq k_{n+1}$

$$\beta_n(j) = \begin{cases} \alpha(k_{n+1})k_{n+1} & \dots\ j = k_{n+1} \\ (\alpha(j) - \alpha(j+1))\, j & \dots\ j < k_{n+1}. \end{cases}$$

Then we have for $n \in \mathbb{N}$

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} z_j &= \frac{1}{n}\sum_{j=1}^{n}\sum_{i=k_j+1}^{k_{j+1}} \alpha(i)y_i = \frac{1}{n}\sum_{j=1}^{k_{n+1}} \alpha(j)y_j \\ &= \frac{1}{n}\left(\alpha(k_{n+1})\sum_{j=1}^{k_{n+1}} y_j + \sum_{j=1}^{k_n}(\alpha(j) - \alpha(j+1))\sum_{i=1}^{j} y_i\right) \\ &= \frac{1}{n}\left(\alpha(k_{n+1})k_{n+1}u_{k_{n+1}} + \sum_{j=1}^{k_n}(\alpha(j) - \alpha(j+1))ju_j\right) \\ &= \frac{1}{n}\sum_{j=1}^{k_{n+1}} \beta_n(j)u_j. \end{aligned}$$

We will now prove by induction over n that $\sum_{j=1}^{k_{n+1}} \beta_n(j) = n$. If $n = 1$, we have, since $k_1 = 0$,

$$\sum_{j=1}^{k_2} \beta_1(j) = \alpha(k_2)k_2 + \sum_{j=k_1+1}^{k_2-1} (\alpha(j) - \alpha(j+1))j = \sum_{j=k_1+1}^{k_2} \alpha(j) = 1.$$

Now suppose that for $n \in \mathbb{N}$ the equality $\sum_{j=1}^{k_{n+1}} \beta_n(j) = n$ holds. Notice that if $j < k_{n+1}$, we have $\beta_n(j) = \beta_{n+1}(j)$. Hence,

$$\begin{aligned} \sum_{j=1}^{k_{n+2}} \beta_{n+1}(j) - n = \sum_{j=1}^{k_{n+2}} \beta_{n+1}(j) - \sum_{j=1}^{k_{n+1}} \beta_n(j) &= \sum_{j=k_{n+1}}^{k_{n+2}} \beta_{n+1}(j) - \beta_n(k_{n+1}) \\ &= \alpha(k_{n+2})k_{n+2} + \sum_{j=k_{n+1}}^{k_{n+2}-1} (\alpha(j) - \alpha(j+1))j - \alpha(k_{n+1})k_{n+1} \\ &= \sum_{k_{n+1}+1}^{k_{n+2}} \alpha(j) = 1 \end{aligned}$$

and the induction step follows.

We proceed with estimating

$$\frac{1}{n}\sum_{j=1}^{n} z_j - \frac{1}{m}\sum_{i=1}^{m} z_i = \frac{1}{n}\sum_{j=1}^{k_{n+1}} \beta_n(j)u_j - \frac{1}{m}\sum_{i=1}^{k_{m+1}} \beta_m(i)u_i.$$

Since $\sum_{j=1}^{k_{n+1}} \beta_n(j) = n$ and $\sum_{i=1}^{k_{m+1}} \beta_m(i) = m$, we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{k_{n+1}} \beta_n(j)u_j - \frac{1}{m}\sum_{i=1}^{k_{m+1}} \beta_m(i)u_i &= \frac{1}{nm}\sum_{j=1}^{k_{n+1}}\sum_{i=1}^{k_{m+1}} \beta_n(j)\beta_m(i)(u_j - u_i)\\ &= \frac{1}{nm}\Bigg(\sum_{j=1}^{N_1}\sum_{i=1}^{k_{m+1}} \beta_n(j)\beta_m(i)(u_j - u_i)\\ &\quad + \sum_{j=N_1+1}^{k_{n+1}}\sum_{i=1}^{N_1} \beta_n(j)\beta_m(i)(u_j - u_i)\\ &\quad + \sum_{j=N_1+1}^{k_{n+1}}\sum_{i=N_1+1}^{k_{m+1}} \beta_n(j)\beta_m(i)(u_j - u_i)\Bigg).\end{aligned}$$

It follows from boundedness of the sequence $(y_n)_{n\in\mathbb{N}}$ that the sequence $(u_n)_{n\in\mathbb{N}}$ is also bounded. Let $M > 0$ be such that $\|u_n\| \leq M$ for all $n \in \mathbb{N}$. We can find $N_2 > N_1$ such that for all $k > N_2$ we have $\frac{2M(N_1+1)}{k} < \epsilon$. Fix any $m, n > N_2$. Then

$$\begin{aligned}\frac{1}{nm}\sum_{j=1}^{N_1}\sum_{i=1}^{k_{m+1}} \beta_n(j)\beta_m(i)\,\|u_j - u_i\| &\leq \frac{2M(N_1+1)m}{nm} < \epsilon\\ \frac{1}{nm}\sum_{j=N_1+1}^{k_{n+1}}\sum_{i=1}^{N_1} \beta_n(j)\beta_m(i)\,\|u_j - u_i\| &\leq \frac{2Mn(N_1+1)}{nm} < \epsilon.\end{aligned}$$

The first inequalities on each line above hold as

$$\sum_{j=1}^{N_1} \beta_n(j) = \sum_{j=1}^{N_1} \beta_{N_1+1}(j) < \sum_{j=1}^{k_{N_1+1}} \beta_{N_1+1}(j) = N_1 + 1$$

and analogically $\sum_{i=1}^{N_1} \beta_m(i) < N_1 + 1$.

What is left is the estimate of the third term, which follows easily from the choice of N_1

$$\frac{1}{nm}\sum_{j=N_1+1}^{k_{n+1}}\sum_{i=N_1+1}^{k_{m+1}} \beta_n(j)\beta_m(i)\,\|u_j - u_i\| \leq \frac{c'nm}{nm} = c'.$$

We have thus shown that for $m, n > N_2$ we have

$$\left\|\frac{1}{n}\sum_{j=1}^{n} z_j - \frac{1}{m}\sum_{i=1}^{m} z_i\right\| \leq 2\epsilon + c' = 3\epsilon + c.$$

As $\epsilon > 0$ was arbitrary, we get $\operatorname{cca}(z_n) \leq c$. □

Lemma 3.20. *For every $\xi \leq \zeta < \omega_1$ and $M \in [\mathbb{N}]$ there is $N \in [M]$ such that the following statements hold:*

(a) There is an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that we have $N = \bigcup_{k=1}^{\infty} \operatorname{supp} \xi_{n_k}^M$.

(b) $(\zeta_j^N)_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\xi_j^N)_{j\in\mathbb{N}}$.

Proof. Let us abbreviate by $S(\xi, \zeta, M)$ the statement of the lemma for $\xi \leq \zeta$ and $M \in [\mathbb{N}]$. We will prove the lemma by induction over ζ. Note that the statement $S(\xi, \zeta, M)$ is true if $\xi = \zeta$, just take $N = M$. Hence, we just need to prove the statements with strict inequality $\xi < \zeta$. If $\zeta = 0$, then the only possible choice for $\xi \leq \zeta$ is $\xi = \zeta = 0$ and we are done.

Let $\zeta + 1 > 0$ be a successor ordinal and suppose that $S(\xi, \eta, M)$ holds for any $\xi \leq \eta < \zeta + 1$ and $M \in [\mathbb{N}]$. Fix $\xi < \zeta + 1$ and $M \in [\mathbb{N}]$. By the induction hypothesis the statement $S(\xi, \zeta, M)$ is valid. Let $N \in [M]$ be witnessing that. Then the property (a) of $S(\xi, \zeta + 1, M)$ is the same as the property (a) of $S(\xi, \zeta, M)$, and so is satisfied. It follows from the definition of the N-summability method $([\zeta + 1]_n^N)_{n\in\mathbb{N}}$ that $([\zeta + 1]_j^N)_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\zeta_j^N)_{j\in\mathbb{N}}$. But $(\zeta_j^N)_{j\in\mathbb{N}}$ is in turn a non-increasing block convex combination of $(\xi_j^N)_{j\in\mathbb{N}}$. It follows that $([\zeta + 1]_j^N)_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\xi_j^N)_{j\in\mathbb{N}}$ and the property (b) of $S(\xi, \zeta + 1, M)$ also holds. Hence, the statement $S(\xi, \zeta + 1, M)$ holds.

Let $\zeta > 0$ be a limit ordinal and suppose that $S(\xi, \eta, M)$ holds for any $\xi \leq \eta < \zeta$ and $M \in [\mathbb{N}]$. Fix $\xi < \zeta$ and $M \in [\mathbb{N}]$. Let $(\zeta_n)_{n\in\mathbb{N}}$ be the sequence of successor ordinals increasing to ζ used to define the Schreier family $\mathcal{S}_\zeta$. Then $\xi < \zeta_{n_0}$ for some $n_0 \in \mathbb{N}$. Let $N_0 \in [M]$ be the set witnessing the validity of $S(\xi, \zeta_{n_0}, M)$ and set $M_0 = N_0 \setminus \left(\operatorname{supp}[\zeta_{n_0}]_1^{N_0}\right)$. We proceed recursively: suppose that for $k \geq 0$ the set M_k has already been defined. Set

- N_{k+1} to be the set witnessing the validity of $S(\zeta_{n_0+k}, \zeta_{n_0+k+1}, M_k)$;
- $M_{k+1} = N_{k+1} \setminus \left(\operatorname{supp}[\zeta_{n_0+k+1}]_1^{N_{k+1}}\right)$.

Let

$$N = \bigcup_{k=0}^{\infty} \operatorname{supp}[\zeta_{n_0+k}]_1^{N_k} \qquad \text{and} \qquad P = N \cup \bigcup_{k=1}^{n_0-1} \operatorname{supp} \zeta_k^M.$$

By the definition of the P-summability method $(\zeta_k^P)_{k\in\mathbb{N}}$ we have $\zeta_k^P = [\zeta_k]_1^{P_k}$ where $P_1 = P$ and $P_{k+1} = P_k \setminus \operatorname{supp}[\zeta_k]_1^{P_k}$. By P.3. in [3, p. 171] we get that $\zeta_k^P = \zeta_k^M$ for $k = 1, \ldots, n_0 - 1$. It follows that $\operatorname{supp}[\zeta_{n_0+k}]_1^{N_k}$ is an initial segment of P_{n_0+k} for $k \geq 0$. Hence, again by P.3. in [3, p. 171], we get $[\zeta_{n_0+k}]_1^{P_{n_0+k}} = [\zeta_{n_0+k}]_1^{N_k}$ for $k \geq 0$. Now we can use P.4. in [3, p. 171] and the fact that

$$N = \bigcup_{k=0}^{\infty} \operatorname{supp}[\zeta_{n_0+k}]_1^{N_k} = \bigcup_{k=0}^{\infty} \operatorname{supp}[\zeta_{n_0+k}]_1^{P_{n_0+k}} = \bigcup_{k=0}^{\infty} \operatorname{supp} \zeta_{n_0+k}^P = \bigcup_{k=n_0}^{\infty} \operatorname{supp} \zeta_k^P$$

to conclude that for $k \geq 0$

$$\zeta_{k+1}^N = \zeta_{n_0+k}^P = [\zeta_{n_0+k}]_1^{P_{n_0+k}} = [\zeta_{n_0+k}]_1^{N_k}.$$

We will now show that $N \in [M]$ witnesses the validity of $S(\xi, \zeta, M)$.

First, let us prove by induction that for each $n \geq 0$ the sequence $([\zeta_{n_0+n}]_j^{N_n})_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\xi_j^{N_n})_{j\in\mathbb{N}}$. The case $n = 0$ follows immediately from the choice of N_0 and property (b) of $S(\xi, \zeta_{n_0}, M)$. Suppose the claim holds for some $n \geq 0$. By the choice of N_{n+1} as the set witnessing $S(\zeta_{n_0+n}, \zeta_{n_0+n+1}, M_n)$, we can use property (b) to get that $([\zeta_{n_0+n+1}]_j^{N_{n+1}})_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $([\zeta_{n_0+n}]_j^{N_{n+1}})_{j\in\mathbb{N}}$. But by the induction hypothesis $([\zeta_{n_0+n}]_j^{N_n})_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\xi_j^{N_n})_{j\in\mathbb{N}}$, and hence, by property (a) of $S(\zeta_{n_0+n}, \zeta_{n_0+n+1}, M_n)$ and P.4. in [3, p. 171], also $([\zeta_{n_0+n}]_j^{N_{n+1}})_{j\in\mathbb{N}}$ is a non-increasing block convex combination of $(\xi_j^{N_{n+1}})_{j\in\mathbb{N}}$. Thus $([\zeta_{n_0+n+1}]_j^{N_{n+1}})_{j\in\mathbb{N}}$ is a non-increasing block convex combination of a non-increasing block convex combination of $(\xi_j^{N_{n+1}})_{j\in\mathbb{N}}$, and hence is itself a non-increasing block convex combination of $(\xi_j^{N_{n+1}})_{j\in\mathbb{N}}$. Therefore the claim is proved.

It follows that for each $n \in \mathbb{N}$ we have

$$[\zeta_{n_0+n-1}]_j^{N_{n-1}} = \sum_{i=k_j^n+1}^{k_{j+1}^n} \alpha_n(i)\xi_i^{N_{n-1}},$$

where $(k_j^n)_{j\in\mathbb{N}}$ is an increasing sequence of integers with $k_1^n = 0$ and $(\alpha_n(i))_{i\in\mathbb{N}}$ is the sequence of coefficients of non-increasing block convex combinations.

Let us recursively define an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ and a sequence of positive numbers $(\alpha(j))_{j\in\mathbb{N}}$ satisfying $\sum_{j=k_n+1}^{k_{n+1}} \alpha(j) = 1$ for each $n \in \mathbb{N}$. Set $k_1 = 0$, $k_2 = k_2^1$ and $\alpha(j) = \alpha_1(j)$ for $1 \leq j \leq k_2^1$. If for some $n \in \mathbb{N}$ the number k_n has already been defined, set $k_{n+1} = k_n + k_2^n$ and for $k_n + 1 \leq j \leq k_{n+1}$ set $\alpha(j) = \alpha_{n+1}(j - k_n)$. We will also need the fact that

$$\xi_j^{N_{n-1}} = \xi_{k_n+j}^N$$

which is readily proved by induction over j using P.3. and P.4. in [3, p. 171] and the fact that

$$N = \bigcup_{n=1}^{\infty} \operatorname{supp}[\zeta_{n_0+n-1}]_1^{N_{n-1}} = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{k_2^n} \operatorname{supp} \xi_j^{N_{n-1}}.$$

We will now show that the sequence $(\alpha(j))_{j\in\mathbb{N}}$ in non-increasing. The only part that does not follow from the choice of the sequences $(\alpha_n(j))_{j=1}^{\infty}$ is that $\alpha(k_{n+1}) \geq \alpha(k_{n+1} + 1)$, that is $\alpha_n(k_2^n) \geq \alpha_{n+1}(1)$, for every $n \in \mathbb{N}$. This follows from the property $S(\zeta_{n_0+n-1}, \zeta_{n_0+n}, M_{n-1})$. Indeed, by property (a) and P.4. in [3, p. 171] we have that for each $k \in \mathbb{N}$

$$[\zeta_{n_0+n-1}]_k^{N_n} = [\zeta_{n_0+n-1}]_{n_k}^{N_{n-1}}$$

for some increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$. Further, by property (b) of

$S(\zeta_{n_0+n-1}, \zeta_{n_0+n}, M_{n-1})$ we have

$$\begin{aligned}
\zeta_{n+1}^N = [\zeta_{n_0+n}]_1^{N_n} &= \sum_{j=1}^m \beta_j [\zeta_{n_0+n-1}]_j^{N_n} = \sum_{j=1}^m \beta_j [\zeta_{n_0+n-1}]_{n_j}^{N_{n-1}} \\
&= \sum_{j=1}^m \beta_j \sum_{i=k_{n_j}^n+1}^{k_{n_j+1}^n} \alpha_n(i) \xi_i^{N_{n-1}} \\
&= \sum_{j=1}^m \sum_{i=k_{n_j}^n+1}^{k_{n_j+1}^n} (\beta_j \alpha_n(i))\, \xi_{k_n+i}^N.
\end{aligned}$$

for some $m \in \mathbb{N}$ and a non-increasing sequence $(\beta_j)_{j=1}^m$ satisfying $\beta_1 \leq 1$. As we also have

$$\zeta_{n+1}^N = [\zeta_{n_0+n}]_1^{N_n} = \sum_{i=k_1^{n+1}+1}^{k_2^{n+1}} \alpha_{n+1}(i) \xi_i^{N_n} = \sum_{i=k_1^{n+1}+1}^{k_2^{n+1}} \alpha_{n+1}(i) \xi_{k_{n+1}+i}^N$$

and ξ_j^N, $j \in \mathbb{N}$, have disjoint supports, we get

$$\alpha_{n+1}(1) = \beta_1 \alpha_n(k_{n_1}^n + 1) \leq \alpha_n(k_{n_1}^n + 1) \leq \alpha_n(k_2^n),$$

where the last inequality holds as $n_1 \geq 2$, which in turn follows from the choice of $M_{n-1} = N_{n-1} \setminus \left(\operatorname{supp}[\zeta_{n_0+n-1}]_1^{N_{n-1}}\right)$ – the set $N_n \in [M_{n-1}]$ cannot contain $\operatorname{supp}[\zeta_{n_0+n-1}]_1^{N_{n-1}}$, and the fact that the sequence $(\alpha_n(j))_{j\in\mathbb{N}}$ in non-increasing.

Hence, for any $n \in \mathbb{N}$

$$\zeta_n^N = [\zeta_{n_0+n-1}]_1^{N_{n-1}} = \sum_{j=k_n+1}^{k_{n+1}} \alpha(j) \xi_j^N$$

and property (b) of $S(\xi, \zeta, M)$ is valid for N. Property (a) is also valid as we have already shown:

$$N = \bigcup_{n=1}^\infty \operatorname{supp}[\zeta_{n_0+n-1}]_1^{N_{n-1}} = \bigcup_{n=1}^\infty \bigcup_{j=1}^{k_2^n} \operatorname{supp} \xi_j^{N_{n-1}} = \bigcup_{n=1}^\infty \bigcup_{j=1}^{k_2^n} \operatorname{supp} \xi_{k_n+j}^N.$$

Hence, the induction step for limit ordinals is done and the lemma is proved. □

Proposition 3.21. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X and $\xi < \zeta < \omega_1$. Then $\widetilde{\operatorname{cca}}_\zeta^s((x_n)_{n\in\mathbb{N}}) \leq \widetilde{\operatorname{cca}}_\xi^s((x_n)_{n\in\mathbb{N}})$. In particular, for any bounded subset A of X we have $\operatorname{wbs}_\zeta^s(A) \leq \operatorname{wbs}_\xi^s(A)$ and $\operatorname{bs}_\zeta^s(A) \leq \operatorname{bs}_\xi^s(A)$.*

Proof. Let $\widetilde{\operatorname{cca}}_\xi^s((x_n)_{n\in\mathbb{N}}) < c$ for some $c > 0$. Then for every $M \in [\mathbb{N}]$ there is $N \in [M]$ such that $\operatorname{cca}\left(\xi_n^N \cdot (x_k)_{k\in\mathbb{N}}\right) < c$. We will show using the infinite Ramsey theorem [1, Theorem 10.1.3.] that this implies that for every $M \in [\mathbb{N}]$ there is $N \in [M]$ such that for all $L \in [N]$ we have $\operatorname{cca}\left(\xi_n^L \cdot (x_k)_{k\in\mathbb{N}}\right) < c$. Fix any $M \in [\mathbb{N}]$ and define

$$\begin{aligned}
A_1 &= \left\{P \in [M] :\ \operatorname{cca}\left(\xi_n^P \cdot (x_k)_{k\in\mathbb{N}}\right) < c\right\} \\
A_2 &= [M] \setminus A_1.
\end{aligned}$$

The set A_1 is Ramsey. Indeed, for $P \in [M]$ we have that $\operatorname{cca}\left(\xi_n^P \cdot (x_k)_{k\in\mathbb{N}}\right) < c$ if and only if

$$\exists m \in \mathbb{N} \exists n \in \mathbb{N}\, \forall i \geq m\, \forall j \geq m : \left\| \frac{1}{i} \sum_{l=1}^{i} \xi_l^P \cdot (x_k)_{k\in\mathbb{N}} - \frac{1}{j} \sum_{l=1}^{j} \xi_l^P \cdot (x_k)_{k\in\mathbb{N}} \right\| \leq c - \frac{1}{n}$$

and for any $i, j, n \in \mathbb{N}$ the set

$$A(i,j,n) = \left\{ P \in [M] : \left\| \frac{1}{i} \sum_{l=1}^{i} \xi_l^P \cdot (x_k)_{k\in\mathbb{N}} - \frac{1}{j} \sum_{l=1}^{j} \xi_l^P \cdot (x_k)_{k\in\mathbb{N}} \right\| \leq c - \frac{1}{n} \right\}$$

is open by P.3. in [3, p. 171]. Hence,

$$A_1 = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{i\geq m} \bigcap_{j\geq m} A(i,j,n)$$

is Borel, and thus Ramsey. It follows from the infinite Ramsey theorem that there is $N \in [M]$ such that either $[N] \subseteq A_1$ or $[N] \subseteq A_2$. But we have already seen that the latter case is impossible. Hence, $[N] \subseteq A_1$ which is precisely what we wanted to show.

It follows from Lemmata 3.19 and 3.20 that there is $L \in [N] \subseteq [M]$ such that $\operatorname{cca}\left(\zeta_n^L \cdot (x_k)_{k\in\mathbb{N}}\right) \leq \operatorname{cca}\left(\xi_n^L \cdot (x_k)_{k\in\mathbb{N}}\right) \leq c$. Therefore, we have found for every $M \in [\mathbb{N}]$ some $L \in [M]$ such that $\operatorname{cca}\left(\zeta_n^L \cdot (x_k)_{k\in\mathbb{N}}\right) \leq c$, which implies the desired inequality $\widetilde{\operatorname{cca}}_\zeta^s((x_n)_{n\in\mathbb{N}}) \leq c$. □

It follows from Proposition 3.21 that the quantities bs_ξ^s and wbs_ξ^s are non-increasing with respect to ξ. It is unclear if the same holds for the quantities bs_ξ and wbs_ξ. We do, however, have monotony if ζ is a finite successor of ξ.

Lemma 3.22. *Let $\xi < \omega_1$ and $\zeta = \xi + l$ for some $l \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X and $M \in [\mathbb{N}]$. Then $\operatorname{cca}(\zeta_n^M \cdot (x_k)_{k\in\mathbb{N}}) \leq \operatorname{cca}(\xi_n^M \cdot (x_k)_{k\in\mathbb{N}})$. In particular, $\widetilde{\operatorname{cca}}_\zeta((x_n)_{n\in\mathbb{N}}) \leq \widetilde{\operatorname{cca}}_\xi((x_n)_{n\in\mathbb{N}})$, and for any bounded subset A of X we have $\operatorname{bs}_\zeta(A) \leq \operatorname{bs}_\xi(A)$ and $\operatorname{wbs}_\zeta(A) \leq \operatorname{wbs}_\xi(A)$.*

Proof. This follows easily by induction over $l \in \mathbb{N}$ and the fact that for any $M \in [\mathbb{N}]$ and $l \in \mathbb{N} \cup \{0\}$ the M-summability method $([\zeta + l + 1]_n^M)_{n\in\mathbb{N}}$ is a non-increasing block convex combination of the M-summability method $([\zeta + l]_n^M)_{n\in\mathbb{N}}$. Therefore, we just need to invoke Lemma 3.19. □

We define another quantity for a bounded subset A of a Banach space X.

Definition. Let A be a bounded subset of a Banach space X. We define

$$\delta_0(A) = \min_{\xi<\omega_1} \operatorname{bs}_\xi^s(A).$$

This quantity δ_0 is a measure of weak non-compactness for separable sets. To prove this we will first need the following lemma. Notice that the assumptions of Lemma 3.23 cannot be met; it is only used to prove a contradiction in the proof of Proposition 3.24.

Lemma 3.23. *Let A be a bounded separable relatively weakly compact subset of a Banach space X which satisfies $\delta_0(A) > 0$. Then the canonical basis of ℓ_1 embeds into $\overline{A}$.*

Proof. We can suppose that $A \subseteq B_X$. Let $c > 0$ be such that $\delta_0(A) > 4c$. We define a tree $\mathcal{T}$ on X as

$$\mathcal{T} = \left\{ (x_1, \dots, x_n) \in \overline{A}^n : \left\| \sum_{j=1}^n a_j x_j \right\| \geq c \sum_{j=1}^n |a_j| \text{ for all } (a_j)_{j=1}^n \in \mathbb{R}^n \right\}.$$

Note that this is a modification of the tree $\mathcal{T}(X, c)$, used by Bourgain [8] to define the ℓ_1-index, that is made only of sequences in $\overline{A}$ instead of B_X. We will further use the terminology from [8]. If we can show that $\mathcal{T}$ is ill-founded, any infinite branch of $\mathcal{T}$ can serve as an isomorphic copy of the canonical basis of ℓ_1 and we are done. As $\mathcal{T}$ is obviously a closed tree, it is enough to show that the order of $\mathcal{T}$ is equal to ω_1 and invoke [8, Proposition 10].

Fix $\xi < \omega_1$. As $\mathrm{bs}^s_\xi(A) > 4c$, we can use Proposition 3.15 and Theorem 3.3 to find a sequence $(x_n)_{n\in\mathbb{N}}$ in A which generates an $\ell_1^{\xi+1}$-spreading model with constant c. This implies that

$$\{(x_n)_{n\in F} : \; F \in \mathcal{S}_{\xi+1}\} \subseteq \mathcal{T}.$$

It follows from [2, Lemma 4.10.] that the order of $\mathcal{S}_{\xi+1}$ (as a tree on $\mathbb{N}$) is equal to $\omega^{\xi+1}$. It is not hard to see that this implies that the order of $\mathcal{T}$ is at least $\omega^{\xi+1}$. But $\xi < \omega_1$ was arbitrary, and hence the order of $\mathcal{T}$ is ω_1. ☐

Proposition 3.24. *Let A be a bounded separable subset of a Banach space X. Then $\delta_0(A) = 0$ if and only if A is relatively weakly compact.*

Proof. If A is not relatively weakly compact, then $0 < \mathrm{wck}_X(A) \leq \mathrm{bs}^s_\xi(A)$ for all $\xi < \omega_1$ by the virtue of Theorem 3.12, and therefore $\delta_0(A) > 0$.

On the other hand, let A be relatively weakly compact. Let us assume for a contradiction that $\delta_0(A) > 0$. It follows from Lemma 3.23 that $\overline{A}$ contains a sequence equivalent to the canonical basis of ℓ_1 which contradicts the relative weak compactness of A. Hence, $\delta_0(A) = 0$ and we are done. ☐

Note that separability of A, was essential in the proof of the preceding theorem, as the result of Bourgain [8] (Lemma 3.23) relies on an argument based on trees which is valid only in separable spaces. We will illustrate the necessity of separability for δ_0 to be a measure of weak non-compactness in Example 3.28 below. However, the quantity δ_0 can be modified to be a measure of weak non-compactness.

Definition. Let A be a bounded subset of a Banach space X. We define

$$\delta(A) = \sup\{\delta_0(B) : \; B \subseteq A \text{ separable}\}.$$

Proposition 3.25. *Let A be a bounded subset of a Banach space X. Then $\delta(A) = 0$ if and only if A is relatively weakly compact.*

Proof. It follows from the Eberlein-Šmulyan theorem that A is relatively weakly compact if and only if each separable (or even countable) subset a A is relatively weakly compact. Hence, the proposition follows from Proposition 3.24. ☐

3.6 Examples

In this section we investigate, whether the inequalities of Theorem 3.3 and Theorem 3.12 are optimal and whether they can be strict. We begin with Theorem 3.12, which stated that for any $\xi < \omega_1$ and any bounded set A in some Banach space X we have

$$\max\{\mathrm{wck}_X(A), \mathrm{wbs}_\xi(A)\} \leq \mathrm{bs}_\xi(A) \leq \mathrm{bs}_\xi^s(A) \leq \beta(A).$$

We will look at the following examples of classical spaces:

- If $A = B_{C[0,1]}$, then $\mathrm{wbs}_\xi(A) = \beta(A) = 2$ as the space $C[0,1]$ contains the Schreier space of order ξ, see Example 3.26 below (in fact, it contains any separable Banach space). Hence,

$$\max\{\mathrm{wck}_{C[0,1]}(A), \mathrm{wbs}_\xi(A)\} = \mathrm{bs}_\xi(A) = \mathrm{bs}_\xi^s(A) = \beta(A).$$

- If $A = B_{\ell_1}$, then $\mathrm{wbs}_\xi(A) = 0$ as there are no nontrivial weakly null sequences in ℓ_1. Further, $\mathrm{wck}_{\ell_1}(A) = 1$, as ℓ_1 is not reflexive, and $\mathrm{bs}_\xi(A) = \beta(A) = 2$ (the fact that $\mathrm{bs}_\xi(A) = 2$ is witnessed by the canonical basis and $\mathrm{bs}_\xi(A) \leq \beta(A) \leq 2$ by Theorem 3.12 and the triangle inequality). Hence,

$$\max\{\mathrm{wck}_{\ell_1}(A), \mathrm{wbs}_\xi(A)\} < \mathrm{bs}_\xi(A) = \mathrm{bs}_\xi^s(A) = \beta(A).$$

- If $A = B_{c_0}$, then $\mathrm{wbs}_\xi(A) = 0$ as c_0 has the weak Banach-Saks property, and thus also the weak ξ-Banach-Saks property, by [14]. Further, $\mathrm{wck}_{c_0}(A) = 1$, as c_0 is not reflexive, and $\beta(A) = 2$, as witnessed by the sequence $x_n = e_1 + \cdots + e_n - e_{n-1}$. The quantity $\mathrm{bs}_\xi(A)$ is harder to compute. It follows from [7, Theorem 5.2.] that $\mathrm{bs}_0(A) = \mathrm{bs}_0^s(A) \leq 1$. Hence, by Proposition 3.21 we have $\mathrm{bs}_\xi(A) \leq \mathrm{bs}_\xi^s(A) \leq \mathrm{bs}_0^s(A) \leq 1$. On the other hand $\mathrm{bs}_\xi^s(A) \geq \mathrm{bs}_\xi(A) \geq \mathrm{wck}_{c_0}(A) = 1$, and therefore

$$\max\{\mathrm{wck}_{c_0}(A), \mathrm{wbs}_\xi(A)\} = \mathrm{bs}_\xi(A) = \mathrm{bs}_\xi^s(A) < \beta(A).$$

So, the inequalities of Theorem 3.12 are optimal and, possibly except the inequality $\mathrm{bs}_\xi(A) \leq \mathrm{bs}_\xi^s(A)$, can be strict. We proceed with Theorem 3.3, which stated that for any $\xi < \omega_1$ and any bounded subset A of some Banach space X we have

$$2\,\mathrm{sm}_{\xi+1}(A) \leq \mathrm{wbs}_\xi(A) \leq \mathrm{wbs}_\xi^s(A) \leq 2\,\mathrm{wus}_{\xi+1}(A) \leq 4\,\mathrm{sm}_{\xi+1}(A).$$

Example 3.26. Let $\xi < \omega_1$ and X_ξ denote the Schreier space of order ξ, that is the completion of c_{00} under the norm

$$\|x\| = \sup_{F \in \mathcal{S}_\xi} \|x \restriction F\|_{\ell_1}.$$

Where $x \restriction F$ denotes the sequence $(y_i)_{i\in\mathbb{N}}$ where $y_i = x_i$ for $i \in F$ and $y_i = 0$ otherwise. It can be shown using classical methods that the canonical sequence $(e_n)_{n\in\mathbb{N}}$ of c_{00} is a normalized 1-unconditional basis of X_ξ. Further, the Bourgain's

ℓ_1-index of X_ξ is countable (see [18, Remmark 5.21.]), and hence X_ξ does not contain ℓ_1 by the result of Bourgain [8]. Therefore, the basis $(e_n)_{n\in\mathbb{N}}$ is shrinking (see e.g. [1, Theorem 3.3.1.]) and in particular weakly null.

Now let us consider $A = \{e_n : n \in \mathbb{N}\}$ as a bounded subset of $X_{\xi+1}$. We will show that

(i) $\mathrm{sm}_{\xi+1}(A) = 1$,

(ii) $\mathrm{wus}_{\xi+1}(A) = 1$,

(iii) $\mathrm{wbs}_\xi(A) = \mathrm{wbs}^s_\xi(A) = 2$.

For any $F \in \mathcal{S}_{\xi+1}$ and $(a_n)_{n\in F} \in \mathbb{R}^F$ we have

$$\left\| \sum_{n\in F} a_n e_n \right\| \geq \sum_{n\in F} |a_n|$$

by the very definition of the norm of $X_{\xi+1}$. On the other hand, as A is a subset of $B_{X_{\xi+1}}$, we get that $\mathrm{sm}_{\xi+1}(A) \leq 1$ by the triangle inequality. Hence, (i) is proved.

We again notice that $A \subseteq B_{X_{\xi+1}}$, and thus $\mathrm{wus}_{\xi+1}(A) \leq 1$. On the other hand, we will show that for any $0 < c < 1$ we have $\mathcal{S}_{\xi+1} \subseteq \mathcal{F}_c = \mathcal{F}_c((e_n)_{n\in\mathbb{N}})$. Take any $F = (n_1, \dots, n_k) \in \mathcal{S}_{\xi+1}$ and define $x^* = e^*_{n_1} + \dots + e^*_{n_k}$. Then for any $x = (x_n)_{n\in\mathbb{N}} \in X_{\xi+1}$ we have

$$|x^*(x)| = \left| \sum_{j\in F} x_j \right| \leq \sum_{j\in F} |x_j| = \|x \restriction F\|_{\ell_1} \leq \|x\| .$$

Hence, $x^* \in B_{X^*_{\xi+1}}$. It follows, as $x^*(e_{n_j}) = 1$ for $j = 1, \dots, k$, that $F \in \mathcal{F}_c$. We have proved that $(e_n)_{n\in\mathbb{N}}$ is $(\xi+1, c)$-large for any $0 < c < 1$, and thus that $\mathrm{wus}_{\xi+1}(A) \geq 1$. Therefore, (ii) is proved.

(iii) now easily follows from Theorem 3.3.

Example 3.27. Let $\xi < \omega_1$. We will consider an equivalent norm on the Schreier space X_ξ of order ξ, namely

$$\|x\|_* = \max\left\{ \left\|x^+\right\|, \left\|x^-\right\| \right\},$$

where $\|\cdot\|$ is the norm defined in Example 3.26 and $x^\pm = (x_n^\pm)_{n\in\mathbb{N}}$ for $x = (x_n)_{n\in\mathbb{N}}$. Then $\|x\|_* \leq \|x\| \leq 2\|x\|_*$ for each $x \in X_\xi$ and $\|y\|_* = \|y\|$ for all y in the positive cone of X_ξ (that is y with non-negative coordinates). In particular, $(e_n)_{n\in\mathbb{N}}$ is a weakly null normalized sequence in $(X_\xi, \|\cdot\|_*)$. Consider again $A = \{e_n : n \in \mathbb{N}\}$ as a bounded subset of $(X_{\xi+1}, \|\cdot\|_*)$. We will show the following:

(i) $\mathrm{sm}_{\xi+1}(A) = \frac{1}{2}$,

(ii) $\mathrm{wus}_{\xi+1}(A) = 1$,

(iii) $\mathrm{wbs}_\xi(A) = \mathrm{wbs}^s_\xi(A) = 1$.

Fix any $F \in \mathcal{S}_{\xi+1}$ and $(a_n)_{n\in F} \in \mathbb{R}^F$. Then

$$\left\| \sum_{n\in F} a_n^+ e_n \right\| \geq \sum_{n\in F} a_n^+ \qquad \text{and} \qquad \left\| \sum_{n\in F} a_n^- e_n \right\| \geq \sum_{n\in F} a_n^-,$$

as $F \in \mathcal{S}_{\xi+1}$. But then

$$\begin{aligned} \left\| \sum_{n\in F} a_n e_n \right\|_* &= \max\left\{ \left\| \sum_{n\in F} a_n^+ e_n \right\|, \left\| \sum_{n\in F} a_n^- e_n \right\| \right\} \\ &\geq \max\left\{ \sum_{n\in F} a_n^+, \sum_{n\in F} a_n^- \right\} \geq \frac{1}{2} \sum_{n\in F} |a_n|. \end{aligned}$$

Hence, $\mathrm{sm}_{\xi+1}(A) \geq \frac{1}{2}$. To show the other inequality it is enough to show that $\mathrm{sm}_1(A) \leq \frac{1}{2}$ and use the monotony provided by Lemma 3.18. Let us have an arbitrary sequence $(f_n)_{n\in\mathbb{N}}$ in A. Note that the set $F = \{2, 3\}$ belongs to the Schreier family $\mathcal{S}_1$. We define $(a_k)_{k\in F} \in \mathbb{R}^F$ by setting $a_2 = 1$, $a_3 = -1$. If $f_2 = f_3$, then

$$\left\| \sum_{k\in F} a_k f_k \right\|_* = \|f_2 - f_3\|_* = 0 \qquad \text{but} \qquad \sum_{k\in F} |a_k| = 2$$

and $(f_n)_{n\in\mathbb{N}}$ cannot generate an ℓ_1^1-spreading model. If $f_2 \neq f_3$, then

$$\left\| \sum_{k\in F} a_k f_k \right\|_* = \|f_2 - f_3\|_* = 1 \qquad \text{but} \qquad \sum_{k\in F} |a_k| = 2$$

and $(f_n)_{n\in\mathbb{N}}$ cannot generate an ℓ_1^1-spreading model with constant greater than $\frac{1}{2}$. In any case, we have shown that $\mathrm{sm}_1(A) \leq \frac{1}{2}$ and (i) is proved.

Now we proceed with (ii). First we notice that $A \subseteq B_{X_{\xi+1}}$, and thus we have $\mathrm{wus}_{\xi+1}(A) \leq 1$. On the other hand, we will show that for $0 < c < 1$ we have $\mathcal{S}_{\xi+1} \subseteq \mathcal{F}_c = \mathcal{F}_c((e_n)_{n\in\mathbb{N}})$. Take any $F = (n_1, \dots, n_k) \in \mathcal{S}_{\xi+1}$ and define $x^* = e_{n_1}^* + \cdots + e_{n_k}^*$. Then for any $x = (x_j)_{j\in\mathbb{N}} \in X_{\xi+1}$ we have

$$|x^*(x)| = \left| \sum_{j\in F} x_j \right| \leq \max\left\{ \sum_{j\in F} x_j^+, \sum_{j\in F} x_j^- \right\} \leq \max\{\left\|x^+\right\|, \left\|x^-\right\|\} = \|x\|_*.$$

Hence, $x^* \in B_{X_{\xi+1}^*}$. But $x^*(e_{n_j}) = 1 > c$ for $j = 1, \dots, k$, and thus $F \in \mathcal{F}_c$. We have shown that $(e_n)_{n\in\mathbb{N}}$ is $(\xi + 1, c)$-large for any $0 < c < 1$, which implies that $\mathrm{wus}_{\xi+1}(A) \geq 1$. But then $\mathrm{wus}_{\xi+1}(A) = 1$ and (ii) is proved.

Finally, we prove (iii). It follows from (i) and Theorem 3.3 that $\mathrm{wbs}_\xi(A) \geq 1$. The inequality $\mathrm{wbs}_\xi^s(A) \leq 1$ follows from the fact that for any sequence $(x_n)_{n\in\mathbb{N}}$ in A, any $N \in [\mathbb{N}]$ and any $k < l \in \mathbb{N}$ we have

$$\begin{aligned} &\left\| \frac{1}{k} \sum_{j=1}^{k} \xi_j^N \cdot (x_n)_{n\in\mathbb{N}} - \frac{1}{l} \sum_{j=1}^{l} \xi_j^N \cdot (x_n)_{n\in\mathbb{N}} \right\|_* \\ &= \max\left\{ \left\| \left(\frac{1}{k} - \frac{1}{l}\right) \sum_{j=1}^{k} \xi_j^N \cdot (x_n)_{n\in\mathbb{N}} \right\|, \left\| \frac{1}{l} \sum_{j=k+1}^{l} \xi_j^N \cdot (x_n)_{n\in\mathbb{N}} \right\| \right\} \leq 1, \end{aligned}$$

where the first equality holds as the summability methods $(\xi_j^N)_{j\in\mathbb{N}}$ have non-negative coefficients and the last inequality follows from the triangle inequality.

It follows from Example 3.26 and Example 3.27 that the inequalities of Theorem 3.3 are optimal and the second and third inequalities may be strict. We note that in both of these examples we have $\mathrm{wbs}_\xi(A) = \mathrm{wbs}^s_\xi(A) = 2\,\mathrm{sm}_{\xi+1}(A)$. We do not know if these inequalities can be strict.

In [7] the authors asked, whether for a bounded set A in a Banach space X it is necessarily true that

$$\mathrm{wbs}(A) = 2\,\mathrm{sm}(A) = 2\,\mathrm{wus}(A).$$

(For the definition of these quantities see [7], note that $\mathrm{wbs}(A) = \mathrm{wbs}_0(A)$, $\mathrm{sm}(A) = \mathrm{sm}_1(A)$ and $\mathrm{wus}(A) = \mathrm{wus}_1(A)$ in our notation). Example 3.27 answers this question negatively.

In the next example we will demonstrate the need of separability in Proposition 3.24. Our non-separable space will be the ℓ_2-sum of the Schreier-Baernstein spaces, which are, in a way, reflexive versions of the Schreier spaces defined in Example 3.26.

Example 3.28. There is a non-separable reflexive Banach space X for which $\delta_0(B_X) = 2$. That is, δ_0 is not a measure of weak non-compactness on X.

Proof. For $\xi < \omega_1$ let us consider the Schreier-Baernstein space X^2_ξ, that is the completion of c_{00} under the norm

$$\|x\|_{X^2_\xi} = \sup\left\{ \left(\sum_{j=1}^{n} \Big(\sum_{i\in F_j} |x_i| \Big)^2 \right)^{\frac{1}{2}} : F_1 < F_2 < \cdots < F_n \in \mathcal{S}_\xi \right\}.$$

Then the canonical sequence $(e_n)_{n\in\mathbb{N}}$ of c_{00} is a shrinking boundedly-complete basis of X^2_ξ, see [10, Lemma 3.2.]. In particular, $(e_n)_{n\in\mathbb{N}}$ is weakly null. It also immediately follows from the definition of the norm $\|\cdot\|_{X^2_\xi}$ that $(e_n)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model with constant 1.

Let us now consider the ℓ_2-sum of the spaces X^2_ξ,

$$X = \ell_2 - \bigoplus_{\xi<\omega_1} X^2_\xi.$$

Then X is a non-separable reflexive Banach space, as the spaces X^2_ξ are reflexive by the result of James, see e.g. [1, Theorem 3.2.13.]. It follows that B_X is weakly compact. But $\mathrm{sm}_\xi(B_X) \geq 1$ for all $\xi < \omega_1$, as B_X contains isometric copies of the canonical bases of the spaces X^2_ξ. It follows from Theorem 3.3 and Proposition 3.15 that $\mathrm{bs}^s_\xi(B_X) \geq 2$. The other inequality is trivial, hence, $\mathrm{bs}^s_\xi(B_X) = 2$ for all $\xi < \omega_1$, and thus $\delta_0(B_X) = 2$. □

3.7 Remarks and open problems

First, let us show that the quantities sm_ξ and wus_ξ do not depend on the choice of successor ordinals made in the definition of the Schreier hierarchy.

Lemma 3.29. *Let $(\mathcal{S}_\xi)_{\xi<\omega_1}$ and $(\mathcal{G}_\xi)_{\xi<\omega_1}$ be two Schreier hierarchies with potentially different choices of sequences of successor ordinals defining the families $\mathcal{S}_\xi$ and $\mathcal{G}_\xi$ for limit ordinals ξ. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in a Banach space X and $c > 0$.*

- *If $(x_n)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model with respect to $\mathcal{S}_\xi$ and with constant c, then there is $M \in [\mathbb{N}]$ such that $(x_n)_{n\in M}$ generates an ℓ_1^ξ-spreading model with respect to $\mathcal{G}_\xi$ and with constant c.*
- *If $(x_n)_{n\in\mathbb{N}}$ in (ξ,c)-large with respect to $\mathcal{S}_\xi$, then there is $N \in [\mathbb{N}]$ such that $(x_n)_{n\in N}$ is (ξ,c)-large with respect to $\mathcal{G}_\xi$.*

Proof. It follows from [3, Theorem 2.2.6.] that there is $M = (m_k)_{k\in\mathbb{N}} \in [\mathbb{N}]$ such that $\mathcal{G}_\xi^M \subseteq \mathcal{S}_\xi$. For the first part, we want to show that

$$\left\|\sum_{k\in F} a_k x_{m_k}\right\| \geq c\sum_{k\in F}|a_k| \qquad \text{for all } F \in \mathcal{G}_\xi \text{ and } (a_k)_{k\in F} \in \mathbb{R}^F.$$

Fix such F and $(a_k)_{k\in F}$ and define $b_j = a_k$ if $j = m_k$ for some $k \in F$ and $b_j = 0$ otherwise. Then $F' = \{m_k :\ k \in F\} \in \mathcal{G}_\xi^M \subseteq \mathcal{S}_\xi$ and

$$\sum_{k\in F} a_k x_{m_k} = \sum_{k\in F} b_{m_k} x_{m_k} = \sum_{j\in F'} b_j x_j.$$

Hence,

$$\left\|\sum_{k\in F} a_k x_{m_k}\right\| = \left\|\sum_{j\in F'} b_j x_j\right\| \geq c\sum_{j\in F'}|b_j| = c\sum_{k\in F}|a_k|.$$

The second part is easier – if there is $N \in [\mathbb{N}]$ such that $\mathcal{S}_\xi^N \subseteq \mathcal{F}_c((x_n)_{n\in\mathbb{N}})$, then $\mathcal{S}_\xi \subseteq \mathcal{F}_c((x_n)_{n\in N})$. Hence, $\mathcal{G}_\xi^M \subseteq \mathcal{S}_\xi \subseteq \mathcal{F}_c((x_n)_{n\in N})$ and $(x_n)_{n\in N}$ is (ξ,c)-large with respect to $\mathcal{G}_\xi$. □

It easily follows from the previous lemma that the quantities sm_ξ and wus_ξ do not depend on the choice of successor ordinals made in definition of the Schreier hierarchy. We do not know if the quantities wbs_ξ and wbs_ξ^s depend on this choice, however, by Theorem 3.3, they are equivalent to the quantity $\mathrm{sm}_{\xi+1}$, which is independent on this choice. Hence, the notions of weak ξ-Banach-Saks sets are also not dependent on this choice.

As we already mentioned in Section 3.6, the inequalities of Theorem 3.12 are optimal and, possibly except for the inequality $\mathrm{bs}_\xi(A) \leq \mathrm{bs}_\xi^s(A)$, can be strict. We have also shown that the inequalities of Theorem 3.3. are optimal and the inequalities concerning the quantity $\mathrm{wus}_{\xi+1}$ can be strict. What remains open is the following question:

Question 4. *Let A be a bounded set in a Banach space X and $\xi < \omega_1$. It is necessarily true that* $\mathrm{wbs}_\xi(A) = \mathrm{wbs}_\xi^s(A) = 2\,\mathrm{sm}_{\xi+1}(A)$*?*

It follows from Theorem 3.3 that the quantities wbs_ξ and wbs_ξ^s are equivalent. The same approach, however, cannot be used for the quantities bs_ξ and bs_ξ^s.

Question 5. *Are the quantities* bs_ξ *and* bs_ξ^s *equal? Or, at least, equivalent?*

In [7, Section 5] the authors proved a dichotomy concerning the quantities applied to a unit ball. More precisely, they showed, in our notation, that for a Banach space X we have $\mathrm{wbs}_0(B_X) \in \{0,2\}$. We did not manage to use this approach to the quantities of higher orders, so the following question still remains open:

Question 6. *Let X be a Banach space and $\xi < \omega_1$. Is it necessarily true that* $\mathrm{wbs}_\xi(B_X) \in \{0, 2\}$*?*

It is known that a normalised basic sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X has a subsequence generating a spreading model, say $\mathcal{X}$ (see e.g. [1, Theorem 11.3.7.]). It is readily proved that if moreover $(x_n)_{n\in\mathbb{N}}$ generates an ℓ_1-spreading model, then $\mathcal{X}$ is isomorphic to ℓ_1. This in combination with a variation of the James' ℓ_1 distorsion theorem [1, Theorem 10.3.1.] was used in [7] to prove the dichotomy for $\xi = 0$. It could help to solve Question 6 if we could say something more about the relation of $(x_n)_{n\in\mathbb{N}}$ and $\mathcal{X}$ if we knew that $(x_n)_{n\in\mathbb{N}}$ generates an ℓ_1^ξ-spreading model for some $1 < \xi < \omega_1$.

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List of publications

- Weak* derived sets of convex sets in duals of non-reflexive spaces,
  J. Funct. Anal. 281 (2021), no. 12, Paper No. 109259, 19 pp.;
- On subspaces whose weak* derived sets are proper and norm dense,
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- Quantification of Banach-Saks properties of higher orders,
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