Review of Doctoral Thesis

Author: Jan Bok
Title: Structure and complexity of homomorphisms
Reviewer: Petr A. Golovach

The doctoral thesis belongs to the fields of Theoretical Computer Science and Graph Theory. More precisely, the author is interested in classifying the computational complexity (P vs. NP-hard) of various variants of graph homomorphism problems and related questions. A *homomorphism* of a graph G to a graph H is a mapping of the set of vertices of G into the set of vertices of H preserving adjacencies, that is, adjacent vertices of G are mapped into adjacent vertices of H. The task of classical H-HOMOMORPHISM problem, where H is a fixed graph, is to decide whether a given graph G admits a homomorphism to H. By the well-known result of Hell and Nešetřil from 1990, if H is a simple graph, then H-HOMOMORPHISM is in P when H is bipartite and NP-complete otherwise. Motivated by this classical result and recent breakthrough results of Bulatov and Zhuk about the complexity of constraint satisfaction problems, the author aims to classify computational complexity of several natural variants of H-HOMOMORPHISM with additional constraints on considered homomorphisms for generalizations of standard graphs. The thesis consists of an introduction and four parts.

In the introduction, the author gives a very brief introduction to the area of graph homomorphism problems and provides a high-level overview of his results. The detailed surveys of known results for specific problems considered in the subsequent parts are postponed and are given in the corresponding sections.

Part I is devoted to list homomorphisms of signed graphs and consists of three chapters. In a signed (multi)graph, every edge is labeled by either "+" or "−". Such a labeling forms a *signature*, and two signatures are assumed to be equivalent if one can be obtained from the other by a so-named *switching* operation. This leads to the notion of homomorphism of signed multigraphs with respect to the equivalence relation. In the list variant of H-HOMOMORPHISM, every vertex v of G is supplied with a list of vertices of H where v can be mapped. First, the author gives a survey of known results about (list) homomorphism problems for signed graphs and provides auxiliary results. Then the author demonstrates the computational complexity dichotomy for the case when H is a tree (with possible loops and multiple edges). Finally, the author explicitly classifies the complexity for the special case of (path, cycle)-separable irreflexive signed multigraphs.

In Part II, the author considers covering problems. This part consists of four chapters. A homomorphism of G into H is said to be a *covering* or *locally-bijective* homomorphism if the mapping is surjective and for every vertex v of G, the edges incident to v are bijectively mapped to the edges incident to the image of v by the natural extension of the homomorphism on the edges. For covering problems, it is natural to consider the generalization of (multi)graphs which allows so-named *semi-edges*. Similarly to the first part, the author first surveys the literature about graph covering problems. Then he introduces specific notation and gives auxiliary results. Further, the full complexity dichotomy for one-vertex and two-vertex target graphs H is provided. In the next chapter, the author discusses the covering problems for the case when the target graph is disconnected. Finally in this section, the author considers list covering problems. As could be expected, this variant is harder, and it is shown that the problem is NP-hard for very restricted cases of target graphs. Based on these hardness results, the author provides the computational complexity dichotomy for the case when H is a cubic multigraph.

Part III deals with problems of a different nature and the main results are about acyclic colorings. A proper coloring of a graph G is *acyclic* if the union of any two color classes induces a forest, that is, an acyclic graph. The task of Acyclic Coloring is to decide, given a graph G and a positive integer k, whether G admits an acyclic k-coloring. In Acyclic k-Coloring, k is a fixed constant. The author is interested in the classification of the computational complexity for these and related coloring problems for hereditary graph classes defined by a forbidden induced subgraph H. The author surveys the known results and gives the computational complexity dichotomy for Acyclic k-Coloring. Besides it, a partial classification is provided for Acyclic Coloring that excludes a few open cases.

The author concludes in Part IV by stating some additional results and discussing future research directions.

My overall view of the thesis is that it is substantial and well-presented. The topics of the thesis are interesting and relevant in the context of the area of graph homomorphisms. The author demonstrates a good knowledge and clear understanding of the field. The results and techniques are, definitely, non-trivial. The majority of the proofs are quite technical and demanded a lot of work and attention to the details, and this shows the qualification of the author. I would like to underline the following strong sides of the thesis that make it noteworthy. Graph homomorphism problems are well-known and thoroughly investigated. Thus, the open problems in the area tend to be hard to deal with. Furthermore, it is difficult to find new interesting directions of research and discover non-trivial novel results. In my opinion, the author of the thesis overcame these difficulties and his results fit very well into the established field. I believe that the author proved his ability for a creative scientific work. Due to a short reviewing time and the volume of the thesis, I did not verify all the technical details. However, I find that all the results are supplied with full detailed proofs and I feel sufficiently convinced that the results are correct.

My main criticism of the thesis is that I would like to have it a bit more focused. In particular, I believe that the most interesting results are presented in Parts I and II, and the results from Part III are rather supplementary. Furthermore, Part III concerns the classification of the computational complexity with respect to the structure of input graphs while the focus of the main part is on the structure of the target. It could be better if the author would omit the results from Part III. Also because the majority of the results, and especially hardness proofs, are very technical, the thesis would benefit from informal explanations and comments improving the readability.

Nevertheless, the above remarks do not change my general evaluation and I believe that the thesis of Jan Bok entitled “Structure and complexity of homomorphisms” satisfies all the demands for a doctoral thesis and the doctoral degree can be awarded to the author after a successful defense.

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