2.2.1 Generalized Curvature of a Planar Section

Now we describe generalized curvatures of the intersection of a set with positive reach and a plane.

Lemma 13. *Let* $X \subset \mathbb{R}^3, \operatorname{reach} X > 0$. *Let* $F \subset \mathbb{R}^3$ *be a plane such that for* $x \in X \cap F$ *and for a* regular point $(x, n) \in \operatorname{nor} X$ *we define* $\beta > 0$ *as an angle between* n *and a normal to the plane* F. *Let us denote* L *a linear subspace such that* $F = x + L$. *We define the perpendicular projection* $P_L\colon \operatorname{Nor}(X, x) \to L$ *and* $\Pi_L(n) := \frac{P_L(n)}{|P_L(n)|}$.

Then for $(u, v) \in \operatorname{Tan}(\operatorname{nor} X \cap (F \times \mathbb{R}^3), (x, n))$ *holds that*

$$\left(u, \frac{\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta}{\sin\beta} u\right) \in \operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F), (x, \Pi_L(n))),$$

where κ_1, κ_2 *are generalized principal curvatures at* (x, n), θ *is an angle between* u *and the principal direction* b_1 *and* $\operatorname{nor}^{(F)}(X \cap F) := \operatorname{nor}(X \cap F) \cap (F \times S^1)$.

Proof. From Theorem 9 we infer

$$(x, n) \in \operatorname{nor} X \Leftrightarrow (x, \Pi_L(n)) \in \operatorname{nor}^{(F)}(X \cap F). \tag{2.6}$$

If $(u, v) \in \operatorname{Tan}(\operatorname{nor} X \cap (F \times \mathbb{R}^3), (x, n))$ then from the definition of the tangent cone there exist $(x_i, n_i) \in \operatorname{nor} X$, $x_i \in X \cap F$ and $\alpha_i > 0$, $x_i \to x$, $n_i \to n$ such that $\alpha_i((x_i, n_i) - (x, n)) \to (u, v)$. Note that such u lies in $P_1(\operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F), (x, n)))$, where P_1 is a projection on the first coordinate.

From Theorem 11 we conclude

$$\begin{aligned} (u, v) &= \delta \left(\frac{1}{\sqrt{1 + \kappa_1^2}} b_1, \frac{\kappa_1}{\sqrt{1 + \kappa_1^2}} b_1 \right) + \epsilon \left(\frac{1}{\sqrt{1 + \kappa_2^2}} b_2, \frac{\kappa_2}{\sqrt{1 + \kappa_2^2}} b_2 \right) \\ &= \left(\frac{\delta}{\sqrt{1 + \kappa_1^2}} b_1 + \frac{\epsilon}{\sqrt{1 + \kappa_2^2}} b_2, \frac{\delta\kappa_1}{\sqrt{1 + \kappa_1^2}} b_1 + \frac{\epsilon\kappa_2}{\sqrt{1 + \kappa_2^2}} b_2 \right), \end{aligned}$$

for some $\delta, \epsilon \in \mathbb{R}$. Thus $\alpha_i(x_i - x) \to u = \frac{\delta b_1}{\sqrt{1+\kappa_1^2}} + \frac{\epsilon b_2}{\sqrt{1+\kappa_2^2}} \in L$ and $\alpha_i(n_i - n) \to v = \frac{\delta\kappa_1 b_1}{\sqrt{1+\kappa_1^2}} + \frac{\epsilon\kappa_2 b_2}{\sqrt{1+\kappa_2^2}}$. Since $|b_1| = |b_2| = |u| = 1$, we see that $\frac{\delta}{\sqrt{1+\kappa_1^2}} = \cos\theta$, $\frac{\epsilon}{\sqrt{1+\kappa_2^2}} = \sin\theta$, where θ is an angle between b_1 and u.

Now we want to describe v relatively in $\operatorname{nor}^{(F)}(X \cap F)$. It holds for $(x_i, \Pi_L(n_i)), (x, \Pi_L(n)) \in \operatorname{nor}^{(F)}(X \cap F)$ that

$$\begin{aligned} \alpha_i(\Pi_L(n_i) - \Pi_L(n)) &= \alpha_i \left(\frac{P_L(n_i)}{|P_L(n_i)|} - \frac{P_L(n)}{|P_L(n)|} \right) \\ &= \alpha_i \left(\frac{P_L(n_i - n)}{|P_L(n)|} + \frac{P_L(n_i)(|P_L(n)| - |P_L(n_i)|)}{|P_L(n_i)|\,|P_L(n)|} \right) \\ &= \alpha_i \left(\frac{P_L(n_i - n)}{|P_L(n)|} + q_i \frac{\Pi_L(n_i)}{|P_L(n)|} \right), \end{aligned}$$

where $q_i := |P_L(n)| - |P_L(n_i)|$. We can see that $\alpha_i q_i \le \alpha_i |P_L(n) - P_L(n_i)| = |P_L(\alpha_i(n - n_i))|$ is bounded thus there exists w such that

$$\alpha_i(\Pi_L(n_i) - \Pi_L(n)) \to w \Rightarrow w \cdot \Pi_L(n) = 0.$$

Such w must lie in a subspace $\langle u \rangle$. Hence

$$\alpha_i(\Pi_L(n_i) - \Pi_L(n)) \to \frac{P_L(v) + q\Pi_L(n)}{|P_L(n)|} = \frac{\langle u, v \rangle u}{|P_L(n)|},$$

since $P_L(v) = \langle u, v \rangle u$, where $\langle \cdot, \cdot \rangle$ denotes the dot product and since $u \in L$. Thus by 2.6 and the definition of tangent vectors

$$\left(u, \frac{\langle u, v \rangle u}{|P_L(n)|} \right) \in \operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F), (x, \Pi_L(n))).$$

Finally we describe $\langle u, v \rangle u$

$$\begin{aligned} \langle u, v \rangle u &= (\cos\theta \kappa_1 \langle u, b_1 \rangle + \sin\theta \kappa_2 \langle u, b_2 \rangle)\, u \\ &= \left(\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \right) u. \end{aligned}$$

Lastly let β be an angle as it is in the lemma statement, then $|P_L(n)| = \sin\beta$.
□

We can now conclude, from Theorem 11 and the previous lemma, a generalization of the normal curvature of a point in some direction.

Corollary. If $(x, \Pi_L(n))$ is a *regular point* of $\operatorname{nor}^{(F)}(X \cap F)$, then we describe a (generalized) normal curvature in direction u of x as $\kappa_u := \frac{\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta}{\sin\beta}$.

Note that κ_1, κ_2 could be ∞.