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Charles University

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Daniel Komárek

Sets with positive reach and their intersections

Mathematical Institute of Charles University

Supervisor of the bachelor thesis: prof. RNDr. Jan Rataj, CSc.
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Title: Sets with positive reach and their intersections

Author: Daniel Komárek

Institute: Mathematical Institute of Charles University

Supervisor: prof. RNDr. Jan Rataj, CSc., Mathematical Institute of Charles University

Abstract: The goal of this thesis is to collect various properties of sets with positive reach and to describe generalization of the directional curvatures in $\mathbb{R}^3$ as the intersection of a plane and a set with positive reach. Firstly, we define sets with positive reach, their Tangent and Normal cones, show basic properties accompanied by some characterizations of sets with positive reach. Then, we generalize principal curvatures for sets with positive reach and describe generalization of Euler's identity about normal curvature in $\mathbb{R}^3$.

Keywords: Set with positive reach, Geometry, Curvature

Contents

Introduction

Sets with positive reach were introduced in the famous paper Federer [1959] but also appear in literature as ”proximally smooth sets” or ”prox-regular sets”. Sets with positive reach are generalization of convex sets which have a unique metric projection on themselves from their positive neighbourhood. They are also generalization of C^2 sets. Initially, they were used with connection to the Curvature Measures in Federer [1959]. One can see more various applications in Colombo and Thibault [2010].

In a major part of this thesis, we collect basic properties of sets with positive reach as they are in Federer [1959] and Rataj and Zähle [2019], accompanied by some pictures and expanded proofs. In the last subsection 2.2.1, we describe a generalization of the well-known Euler's identity about the normal curvature on C^2-manifold.

In order to get to the curvatures of sets with positive reach, we need to describe basic properties of sets themselves, which is the goal of Chapter 1. In Rataj and Zähle [2019][Lemma 4.5, Corollary 4.6] a basic characterization of sets with positive reach is given and it is used to prove more advanced results throughout the thesis. We define and show duality of Tangent and Normal cones (see Rockafellar [2015][Section 16]) and show an equivalent condition for the set X being the set with positive reach in terms of the distance of a vector in a set X from the Tangent cone of X.

In Chapter 2, we show a sufficient condition for the intersection of two sets with positive reach to be the set with positive reach (Federer [1959][Theorem 4.10]). Using a distance function of a point from a set with positive reach X, we show it is continuously differentiable, compute its gradient and show that this gradient is Lipschitzian on some r-parallel neighbourhood of X (Lemma 3) – this property would be used in order to prove some properties of mappings from or onto the Normal Bundle (Federer [1959][Theorem 4.8(13)]), from which we infer that the Unit Normal Bundle $\operatorname{nor} X$ is actually a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ and the r-neighbourhood X_r of X is a closed C^1-domain with the Lipschitzian Gauss map.

From these properties of $\operatorname{nor} X$ and X_r, we define a generalization of the principal curvatures for a set with positive reach (Rataj and Zähle [2019][Proposition 4.23]) and we derive the main result of this thesis – generalization of the directional curvatures in $\mathbb{R}^3$ as the intersection of a plane and a set with positive reach (Lemma 13 and its corollary).

1. Geometric Properties

In this chapter we will define sets with positive reach and describe some basic properties. Later, we define *tangent* and *normal cones*, show their duality and convexity of *tangent cones*. In Theorem 8 we give an important characterization of sets with positive reach.

1.1 Basic Definitions and Properties

The metric projection to a nonempty set $X \subset \mathbb{R}^d$ is not defined everywhere, unless X is closed and convex. On a set X with positive reach we can define the metric projection on some open neighbourhood of X.

Definition. *Given a set $\emptyset \neq X \subset \mathbb{R}^d$, we define its* distance function *as*

$$d_X \colon x \mapsto \operatorname{dist}(x, X) = \inf\{\,|x-a| \mid a \in X\}.$$

$\operatorname{Unp} X$ *denotes the set of all $x \in \mathbb{R}^d$ for which there exists a unique point $a \in X$ nearest to x. We also define the* metric projection *to X as the mapping*

$$\Pi_X \colon \operatorname{Unp} X \to X,$$

then we write $a =: \Pi_X(x)$.

Lemma 1. Π_X *is continuous.*

Proof. Assume for contrary that Π_X is not continuous at a point $x \in \operatorname{Unp} X$. So there exist points $x_i \in \operatorname{Unp} X, x_i \to x, \inf_i |\Pi_X(x_i) - \Pi_X(x)| > 0$. Let us have $y \in X$ such that $\Pi_X(x_i) \to y$. By assumption $\Pi_X(x) \neq y$, but $|y - x| = d_X(x)$, which contradicts that $x \in \operatorname{Unp} X$.

□

Definition. *We define the* reach function *of X as*

$$\operatorname{reach}(X, a) := \sup\{r \geq 0 \mid B(a, r) \subset \operatorname{Unp} X\},$$

where $B(a, r)$ is the open ball with the centre a and the radius r, and the reach *of X is set as*

$$\operatorname{reach} X := \inf_{a \in X} \operatorname{reach}(X, a).$$

We denote by

$$X_r := \{u \in \mathbb{R}^d \mid d_X(u, X) \leq r\}$$

the r-parallel body *of the set X for $r \geq 0$.*

Note that $\operatorname{reach} \emptyset = \infty$ and any set with positive reach is closed.

Examples of sets with positive reach:

1. If X is closed and convex then (and only then) $\operatorname{reach} X = \infty$.

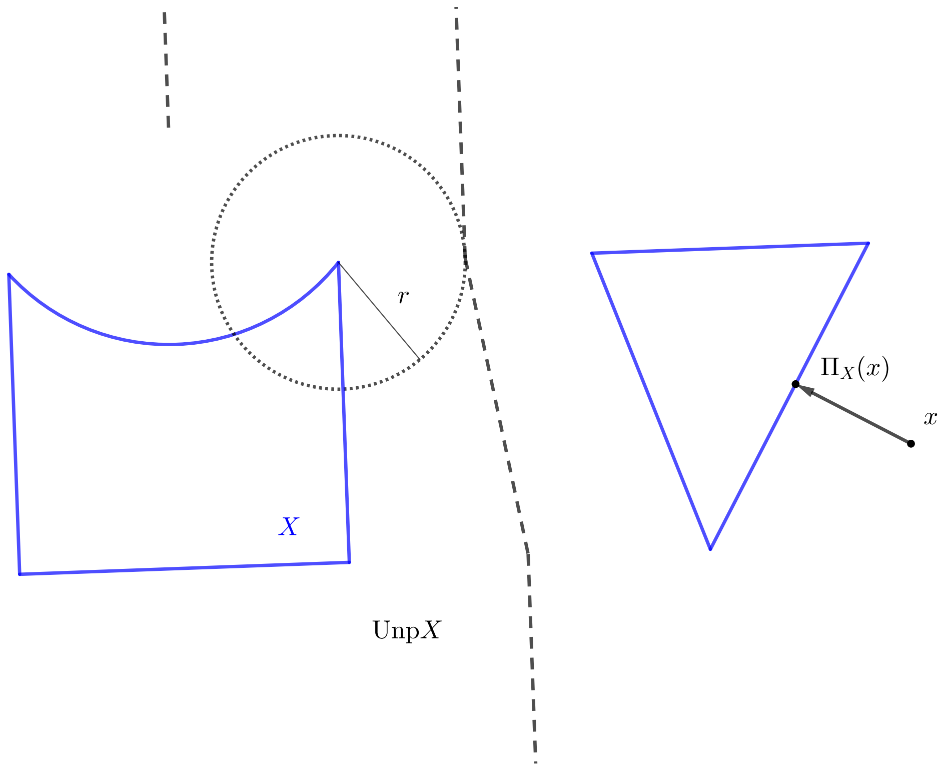

Figure 1.1: A set X with positive reach, reach $X = r$. The metric projection Π_X of a point x. Unp X, where dashed lines do not belong to the Unp X.

2. If X is finite nonempty then reach $X = \min\{\frac{1}{2}|x-y| \mid x, y \in X, x \neq y\}$.
3. The union of two disjoint compact sets with positive reach has positive reach.

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called *K-Lipschitz* if there exists a positive real constant $K \geq 0$ such that, for all $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \leq K|x-y|.$$

We show some basic differential properties of the distance function.

Lemma 2. *Let $f\colon U \to \mathbb{R}$ be Lipschitz and $U \subset \mathbb{R}^d$ open. Let $g\colon U \to \mathbb{R}$ continuous and $1 \leq i \leq d$. If*

$$\frac{\partial}{\partial x_i} f(x) = g(x) \quad \textit{whenever } f \textit{ is differentiable at } x \in U.$$

Then

$$\frac{\partial}{\partial x_i} f(x) = g(x) \quad \textit{for all } x \in U.$$

Proof. Suppose $u \in U, r > 0$ and $B(u, 2r) \subset U$. Let e_i be ith unit vector of the canonical basis of $\mathbb{R}^d$. According to the Rademacher's theorem f is differentiable almost everywhere on U and for almost all x within r of u it follows, from the absolute continuity of f, that

$$f(x+te_i) - f(x) = \int_0^t \frac{\partial}{\partial x_i} f(x+we_i)\,dw = \int_0^t g(x+we_i)\,dw \quad \text{whenever } |t| < r.$$

From the continuity of f and g it follows that

$$f(u+te_i)-f(u)=\int_0^t g(u+we_i)\,dw \quad \text{whenever } |t|<r,$$

and finally $\frac{\partial}{\partial x_i}f(u)=g(u)$.

□

Lemma 3. *If the* distance function d_X *is differentiable at some point* $x\in \operatorname{Unp} X\backslash X$ *then its gradient is*

$$\nabla d_X(x)=\frac{x-\Pi_X(x)}{|x-\Pi_X(x)|}.$$

Further, d_X *is continuously differentiable on* $\operatorname{int}(\operatorname{Unp} X\backslash X)$*.* d_X^2 *is continuously differentiable on* $\operatorname{int}(\operatorname{Unp} X)$ *and* $\nabla d_X^2(x)=2(x-\Pi_X(x))$ *for* $x\in\operatorname{int}(\operatorname{Unp} X)$*.*

Proof. We show that d_X is 1-Lipschitz. From the triangle inequality follows

$$|d_X(y)-d_X(x)|\le |d_X(x)+|x-y|-d_X(x)|=|x-y|\,,\quad x,y\in\mathbb{R}^d.$$

Thus, for every x for which d_X is differentiable, $|\nabla d_X(x)|\le 1$. For $x\in \operatorname{Unp}(X)\backslash X$ and for $u:=\dfrac{x-\Pi_X(x)}{|x-\Pi_X(x)|}$ we obtain

$$d_X(x-tu)=d_X(x)-t,\quad 0\le t\le d_X(x).$$

Consequently, $\nabla d_X(x)=u$ whenever $x\in\operatorname{Unp}(X)\backslash X$ and $d_X(x)$ is differentiable at x.

Since Π_X is continuous and because of the previous lemma we get the continuous differentiability of d_X on $W:=\operatorname{int}(\operatorname{Unp} X\backslash X)$.

For $x\in W$, the stated formula for $\nabla d_X^2(x)$ follows from the first part of this lemma. For $x\in X$, $d_X^2(x+h)\le |h|^2$ for $h\in\mathbb{R}^n$, hence $\nabla d_X^2(x)=0$, and also $\Pi_X(x)=x$. Accordingly the formula holds for all $x\in\operatorname{int}(\operatorname{Unp} X)$, and the continuity of the right side of the formula, from Lemma 1, implies the continuity of $\nabla d_X^2(x)$ on $\operatorname{int}(\operatorname{Unp} X)$.

□

Definition. *We define the* tangent cone *of* $X\subset\mathbb{R}^d$ *at a point* $a\in X$ *as the set of all vectors* $u\in\mathbb{R}^d$ *such that either* $u=0$ *or there exists a sequence of points* $a_i\in X\backslash\{a\}$ *with* $a_i\to a$ *and* $r_i(a_i-a)\to u, i\to\infty$*, for* $r_i>0$*. We denote this* tangent cone *by* $\operatorname{Tan}(X,a)$*. Note that* $\operatorname{Tan}(X,a)$ *is always closed for the general set* X*.*

Further we define the normal cone *of* X *at* $a\in X$ *as the* polar cone *of* $\operatorname{Tan}(X,a)$*, i.e.*

$$\operatorname{Nor}(X,a):=\operatorname{Tan}(X,a)^\circ=\{v\mid v\cdot u\le 0 \textit{ for any } u\in\operatorname{Tan}(X,a)\}.$$

Note that the normal cone *is always a closed convex cone.*

Lemma 4. *Let* $\operatorname{reach}(X,a) =: r > 0$*,* $a \in \partial X$ *and* $n \in S^{d-1}$*. The following statements are equivalent.*

(i) $\Pi_X(a+tn) = a$ *for some* $t > 0$*,*

(ii) $\Pi_X(a+tn) = a$ *for all* $0 < t < r$*,*

(iii) $X \cap B(a+rn, r) = \emptyset$*,*

(iv) $n \in \operatorname{Nor}(X,a)$*.*

Proof. We show $(i) \Rightarrow (ii)$. Let (i) be true and denote

$$\tau := \sup\{t > 0 \mid \Pi_X(a+tn) = a\}.$$

We know that $\tau > 0$ by assumption. Clearly $\Pi_X(a+tn) = a$ for $0 < t < \tau$. In order to show (ii), we have to verify that $\tau \geq r$. Assume, for the contrary, that $\tau < r$. Then $x_\tau := a + \tau n \in \operatorname{int} \operatorname{Unp} X$ (since $\operatorname{reach} X > \tau$). Consider differential equation

$$g'(s) = \nabla d_X \circ g(s), \quad g(0) = x_\tau. \tag{1.1}$$

By the Peano existence theorem, there exists $\delta > 0$ and a differential function $g\colon (-\delta,\delta) \to \mathbb{R}^d$ such that g from 1.1 holds for $|s| < \delta$. Since ∇d_X is always a unit vector, we have $|g'(s)| = 1$. Further,

$$(d_X \circ g)'(s) = \nabla d_X \circ g(s) \cdot g'(s) = g'(s) \cdot g'(s) = 1,$$

hence, for any $-\delta < s_1 < s_2 < \delta$,

$$\begin{aligned} s_2 - s_1 &= \int_{s_1}^{s_2} |g'(s)|\, ds = \int_{s_1}^{s_2} (d_X \circ g)'(s)\, ds \\ &= d_X(g(s_2)) - d_X(g(s_1)) \leq |g(s_2) - g(s_1)|\,. \end{aligned}$$

It follows that an image of g must be a straight segment of length 2δ. By the definition of g, any point of this segment will have its metric projection onto X in a, which is a contradiction with the definition of τ.

The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear. Now we will show that $(iv) \Rightarrow (i)$. Assume for the contrary that $n \in \operatorname{Nor}(X,a) \cap S^{d-1}$ but (i) is not true. We define the function $x(t) := a + tn, t > 0$. We know that $x(t) \notin X$ for sufficiently small $t > 0$ (otherwise, n would be a tangent vector to X at x, contradicting the assumption). Then, $a(t) := \Pi_X(x(t)) \neq a$ for all sufficiently small $t > 0$ by our assumptions. By the continuity of Π_X we see that $a(t) \to a, t \to 0$. Now we will show that the unit vectors $n(t) := \frac{x(t)-a(t)}{|x(t)-a(t)|}$ converge to n as $t \to 0$. We have

$$\limsup_{t\to 0} (a(t) - a) \cdot n \leq 0$$

since $n \in \operatorname{Nor}(X,a)$. Thus

$$\begin{aligned} \liminf_{t\to 0} n(t) \cdot n &= \liminf_{t\to 0} \frac{(x(t)-a)\cdot n + (a - a(t))\cdot n}{|x(t)-a(t)|} \\ &\geq \liminf_{t\to 0} \frac{t}{|x(t)-a(t)|} \geq 1, \end{aligned}$$

since $|x(t) - a(t)| \leq |x(t) - a| = t$ for all t. Hence $n(t) \to n, t \to 0$. By the already proven implication $(i) \Rightarrow (iii)$ applied to $a(t)$ and $n(t)$, the open balls $B(a(t) + tn(t), r)$ do not intersect X. But we have

$$B(a + rn, r) \subset \bigcup_{t>0} B(a(t) + rn(t), r),$$

hence $B(a + rn, r) \cap X = \emptyset$. We have thus shown (iii) which clearly implies (i), a contradiction.

$\square$

Corollary. If $\operatorname{reach} X > 0, a, b \in X$ and $v \in \operatorname{Nor}(X, a)$ then

$$(a - b) \cdot v \leq \frac{|a - b|^2 |v|}{2 \operatorname{reach} X}. \tag{1.2}$$

Proof. Assume that $v \neq 0$ and let $n := v/|v|$. From the previous lemma we know that $\Pi_X(a + tn) = a$ for all $0 < t < \operatorname{reach} X$. For such t we compute

$$|a + tn - b| \geq d_X(a + tn) = t, \qquad |a - b|^2 + 2tn \cdot (a - b) + t^2 \geq t^2,$$
$$2tn \cdot (a - b) \geq -|a - b|^2, \qquad v \cdot (a - b) \geq -|a - b|^2 |v| /2t.$$

$\square$

Lemma 5. *If* $0 < r < q < \infty, x, y \in X_r$ *and*

$$q \leq \operatorname{reach}(X, \Pi_X(x)), \quad q \leq \operatorname{reach}(X, \Pi_X(y)),$$

then

$$|\Pi_X(x) - \Pi_X(y)| \leq \frac{q}{q - r} |x - y|.$$

Proof. Let $a := \Pi_X(x), b := \Pi_X(y)$, one infers from Equation 1.2 that

$$(x - a) \cdot (a - b) \geq -|a - b|^2 r/2q$$

and symmetrically

$$(y - b) \cdot (b - a) \geq -|b - a|^2 r/2q.$$

Therefore

$$\begin{aligned} |x - a| \cdot |a - b| &\geq (x - y) \cdot (a - b) \\ &= ((a - b) + (x - a) + (b - y)) \cdot (a - b) \\ &\geq |a - b|^2 (1 - r/q), \\ |x - y| &\geq |a - b| (q - r)/q. \end{aligned}$$

$\square$

Now we combine this lemma, Lemma 1, Lemma 3 and conclude following corollary.

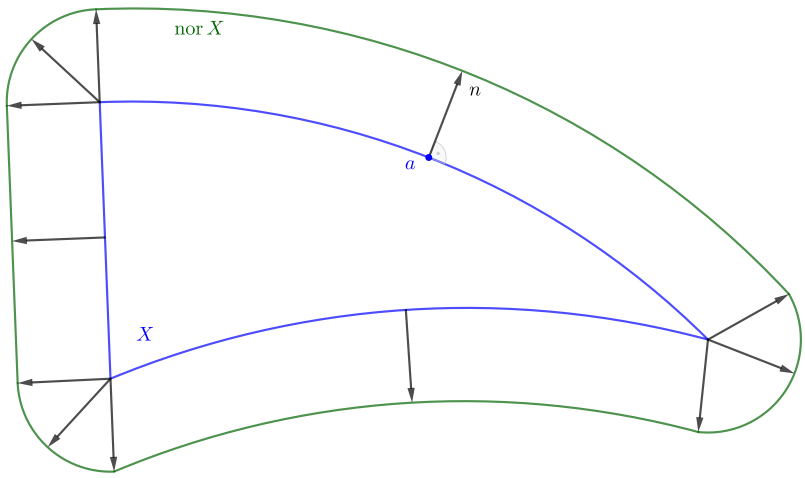

Figure 1.2: A set X with its Unit Normal Bundle nor X.

Corollary. If $0 < s < r < \operatorname{reach} X$, then ∇d_X is Lipschitzian on $\{x \mid s \leq d_X(x) \leq r\}$, and ∇d_X^2 is Lipschitzian on X_r.

Definition. *Given a set* $X \subset \mathbb{R}^d$ *with* reach > 0*, we define the* unit normal bundle *of* X *(see Figure* 1.2*) as*

$$\operatorname{nor} X := \{(a, n) \mid a \in \partial X, n \in \operatorname{Nor}(X, x) \cap S^{d-1}\}.$$

Lemma 6. nor X *is a closed subset of* $\mathbb{R}^{2d}$.

Proof. Let $(a, n) = \lim_{i \to \infty}(a_i, n_i), (a_i, n_i) \in \operatorname{nor} X$ for all i. From Lemma 4 we have $B(a_i + rn_i, r) \cap X = \emptyset$ for $r = \operatorname{reach} X$ and all i, hence also $B(a + rn, r) \cap X = \emptyset$. Thus $(a, n) \in \operatorname{nor} X$, *by* Lemma 4.

□

Corollary. Let reach $X > 0$ and $a \in X$.

(i) If $a \in \partial X$ then $\operatorname{Nor}(X, a) \neq \{0\}$.

(ii) If $u \in S^{d-1}$ belongs to the topological interior of $\operatorname{Tan}(X, a)$ then the segment $[a, a + \epsilon u]$ is included in X for some $\epsilon > 0$.

Proof. We prove (i). If $a \in \partial X$, there exist points $x_i \in \mathbb{R}^d \backslash X, x_i \to a$. Then the metric projections $\Pi_X(x_i) := a_i$ also converge to a and $n_i := \frac{x_i - a_i}{|x_i - a_i|} \in \operatorname{Nor}(X, a_i)$ by Lemma 4. We can achieve now that $n_i \to n \in S^{d-1}$. Then $(a, n) \in \operatorname{nor} X$ since nor X is closed.

Now we prove (ii). Let such u be given. Assume, for the contrary, that there exists a sequence $\epsilon_i \to 0$ such that $x_i := a + \epsilon_i u \notin X$ for all i. Now we denote $a_i := \Pi_X(a + \epsilon_i u) \to a, n_i := \frac{x_i - a_i}{|a_i - x_i|} \in \operatorname{Nor}(X, a_i)$ and we can assume that

$n_i \to n \in S^{d-1} \cap \mathrm{Nor}(X, a)$. From Equation 1.2 we see that $(a - a_i) \cdot n_i \leq (2r)^{-1} |a_i - a|^2$ where $r := \mathrm{reach}\, X$, hence

$$u \cdot n_i = \frac{1}{\epsilon_i}(x_i - a) \cdot n_i \geq \frac{1}{\epsilon_i}(a_i - a) \cdot n_i \geq -\frac{|a_i - a|^2}{2r\epsilon_i}.$$

Letting $i \to \infty$ we get $u \cdot n \geq 0$ which, however, contradicts the fact that $n \in \mathrm{Nor}(X, a) = \mathrm{Tan}(X, a)^\circ$ and $u \in \mathrm{int}\,\mathrm{Tan}(X, a)$.

□

1.2 Polar Cones and Equivalent Statements for the Positive Reach

We defined $\mathrm{Nor}(X, a)$ as the polar cone to the $\mathrm{Tan}(X, a)$, in notation $\mathrm{Nor}(X, a) := \mathrm{Tan}(X, a)^\circ$. Now we will properly define the polar cone and show some basic properties. Recall that a cone (with centre at the origin) is a set C with the property that if $x \in C$ then also $tx \in C$ for any $t > 0$.

Definition. *The* polar cone *to a set* $C \subset \mathbb{R}^d$ *is given by*

$$C^\circ := \{v \in \mathbb{R}^d \mid u \cdot v \leq 0 \textit{ for all } u \in C\}.$$

It is clear from the definition that C° is always a closed convex cone.
If C is already a cone then its second polar satisfies

$$C^{\circ\circ} = \overline{\mathrm{conv} C}.$$

If $C, D \subset \mathbb{R}^d$ are closed convex cones then

$$(C \cap D)^\circ = C^\circ + D^\circ, \quad (C + D)^\circ = C^\circ \cap D^\circ. \tag{1.3}$$

The proof of this statement could be find in Rockafellar [2015][section 16].

Lemma 7. *If* $\mathrm{reach}\, X > 0$ *and* $a \in \partial X$ *then* $\mathrm{Tan}(X, a) = \mathrm{Nor}(X, a)^\circ$, *hence* $\mathrm{Tan}(X, a)$ *is convex.*

Proof. First note that

$$\mathrm{Nor}(X, a)^\circ = \mathrm{Tan}(X, a)^{\circ\circ} \supset \mathrm{Tan}(X, a).$$

Hence it is enough to show that

$$\mathrm{Nor}(X, a)^\circ \subset \mathrm{Tan}(X, a).$$

Let $u \notin \mathrm{Tan}(X, a), |u| = 1$ be a vector. Then, by the definition of tangent vectors, there exist $\epsilon, \gamma > 0$ such that the cone

$$V := \{v \mid (v - a) \cdot u > |v - a| \cos \gamma\}$$

does not intersect $X \cap B(a, \epsilon)$. We can assume that $\epsilon \leq \mathrm{reach}\, X$. Denote $x(t) := a + tu, a(t) := \Pi_X x(t), n(t) := \frac{x(t) - a(t)}{|x(t) - a(t)|}, 0 < t < \epsilon$.

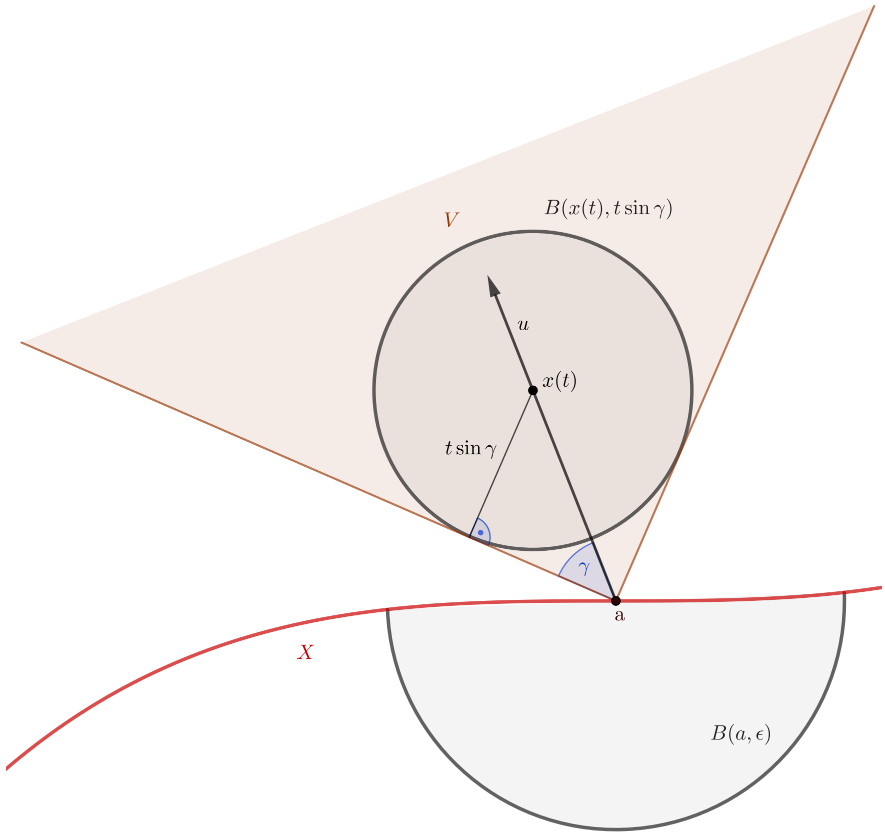

Figure 1.3: The set X, the vector u, the cone V and following points from Lemma 7.

We can see from Figure 1.3 that the open cone V contains the open ball $B(x(t), t\sin\gamma)$. We can see that $|x(t) - a(t)| \leq t$, and since $a(t) \in X$, it must lie outside V and we have thus

$$t\sin\gamma \leq |x(t) - a(t)| \leq t, \quad 0 < t < \epsilon.$$

Further, Equation 1.2 implies

$$(a - a(t)) \cdot n(t) \leq \frac{|a - a(t)|^2}{2\operatorname{reach} X}, \quad 0 < t < \epsilon.$$

Combining this two estimates, we obtain

$$\begin{aligned} u \cdot n(t) &= t^{-1}((x(t) - a(t)) - (a - a(t))) \cdot n(t) \\ &= t^{-1}(|x(t) - a(t)| - (a - a(t)) \cdot n(t) \\ &\geq \sin\gamma - \frac{|a(t) - a|}{\operatorname{reach} X}. \end{aligned}$$

In view of the compactness of the unit sphere, we can find a sequence $t_i \to 0_+$ such that $n(t_i) \to n \in S^{d-1}$. Since also $a(t_i) \to a$, we have $(a, n) \in \operatorname{nor} X$ by Lemma 6, hence, $n \in \operatorname{Nor}(X, x)$. As

$$u \cdot n \geq \liminf_{t \to 0} u \cdot n(t) > 0,$$

since $\frac{|a(t)-a|}{\operatorname{reach} X} \to 0$, we infer that $u \notin \operatorname{Nor}(X, x)^{\circ}$.

$\square$

In the proof of the next Theorem we will use the Separation theorem: let $A \subset \mathbb{R}^d$ nonempty closed convex and $K \subset \mathbb{R}^d$ nonempty convex and compact, $A \cap K = \emptyset$. Then there exists a hyperplane E strictly separating A from K, i.e., A and K are in the opposite open half-planes with a boundary E.

We now conclude equivalent statement for $\operatorname{reach} X \geq r$.

Theorem 8. *Given a closed set $X \subset \mathbb{R}^d$ and $r > 0$, the following statements are equivalent:*

(i) $\operatorname{reach} X \geq r$*,*

(ii) for any $a, b \in X$, $d_{\operatorname{Tan}(X,a)}(b-a) \leq \frac{|b-a|^2}{2r}$.

Remark. Note that condition (ii) is equivalent to the following statement:
If $a, b \in X$ with $0 < |b-a| < 2r$ then there exists a tangent vector $0 \neq u$ to X at a with $\beta := \angle(b-a, u) \leq \arcsin \frac{b-a}{2r} =: \alpha$.

We can see that equivalency as followed. We infer that $\frac{|b-a|^2}{2r} = \cos\gamma\,|b-a|$, for $\gamma = \frac{\pi}{2} - \alpha$, and $|(b-a)-u| = \sin\beta\,|b-a|$ (see Figure 1.4). Thus

$$\sin\beta\,|b-a| \leq \cos\gamma\,|b-a| \Leftrightarrow \sin\beta \leq \cos\gamma = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \Leftrightarrow \beta \leq \alpha,$$

hence we see the equivalency we wanted.

Proof. We prove that $(i) \Rightarrow (ii)$. Assume that $\operatorname{reach} X \geq r$ and $a, b \in X$. If $a = b$ or $|b-a| \geq 2r$ then condition (ii) is satisfied with $0 \in \operatorname{Tan}(X, a)$, since $|(b-a) - 0| \leq |b-a| \frac{|b-a|}{2r}$. Assume thus that $0 < |b-a| < 2r$ and denote $u_0 := \frac{b-a}{|b-a|}$, $\gamma := \arcsin \frac{|b-a|}{2r}$ and

$$C := \{u \mid u \cdot u_0 \geq |u| \cos\gamma\}.$$

We have to show that $\operatorname{Tan}(X, a)$ has nontrivial intersection with C. Assume, for contrary, that the intersection is trivial. Since both are closed convex cones, there must be a hyperplane strictly separating them. Thus there exists a unit vector w such that $v \cdot w < 0$ if $v \in \operatorname{Tan}(X, a)$ and $u \cdot w > 0$ if $u \in C$. Note that such w lies in $\operatorname{Nor}(X, a)$. Consider the vector $w_0 := u_0 - (\sin\gamma)w$. The angle formed by u_0 and w_0 is less or equal to γ (see Figure 1.5), hence, $w_0 \in C$.

Further, we have

$$w \cdot \frac{b-a}{|b-a|} = w \cdot u_0 = w \cdot (u_0 - w_0) + w \cdot w_0 > w \cdot (u_0 - w_0) = \sin\gamma = \frac{|b-a|}{2r},$$

which is in contradiction with Equation 1.2.

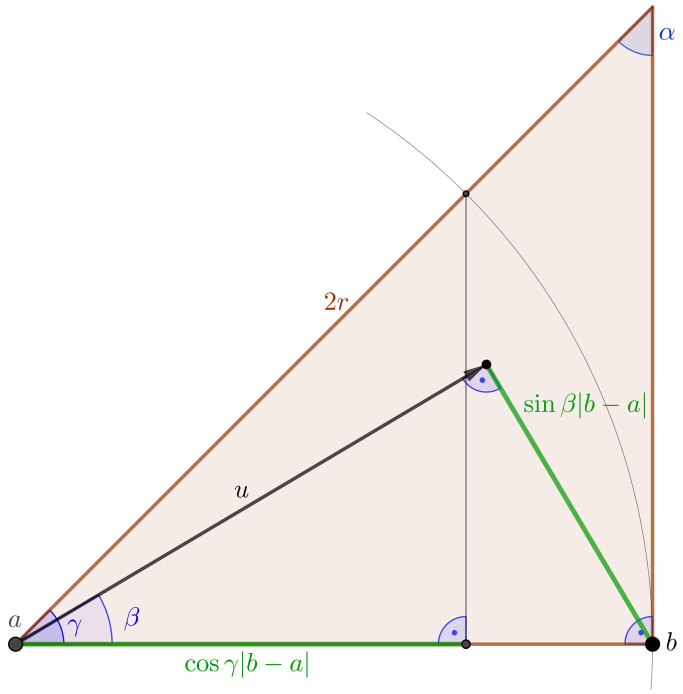

Figure 1.4: Remark after the Theorem 1.4.

Now we show $(ii) \Rightarrow (i)$. Assume that (ii) holds but reach $X < r$. Then there exists a point x with $s := d_X(x) < r$ which has at least two different points $a, b \in X \cap \partial\overline{B(x,s)}$. Since X does not intersect $B(x,s)$, any tangent vector to X at a must form an angle with $b - a$ of size at least $\arcsin \frac{|b-a|}{2s}$, which contradicts the property (ii).

□

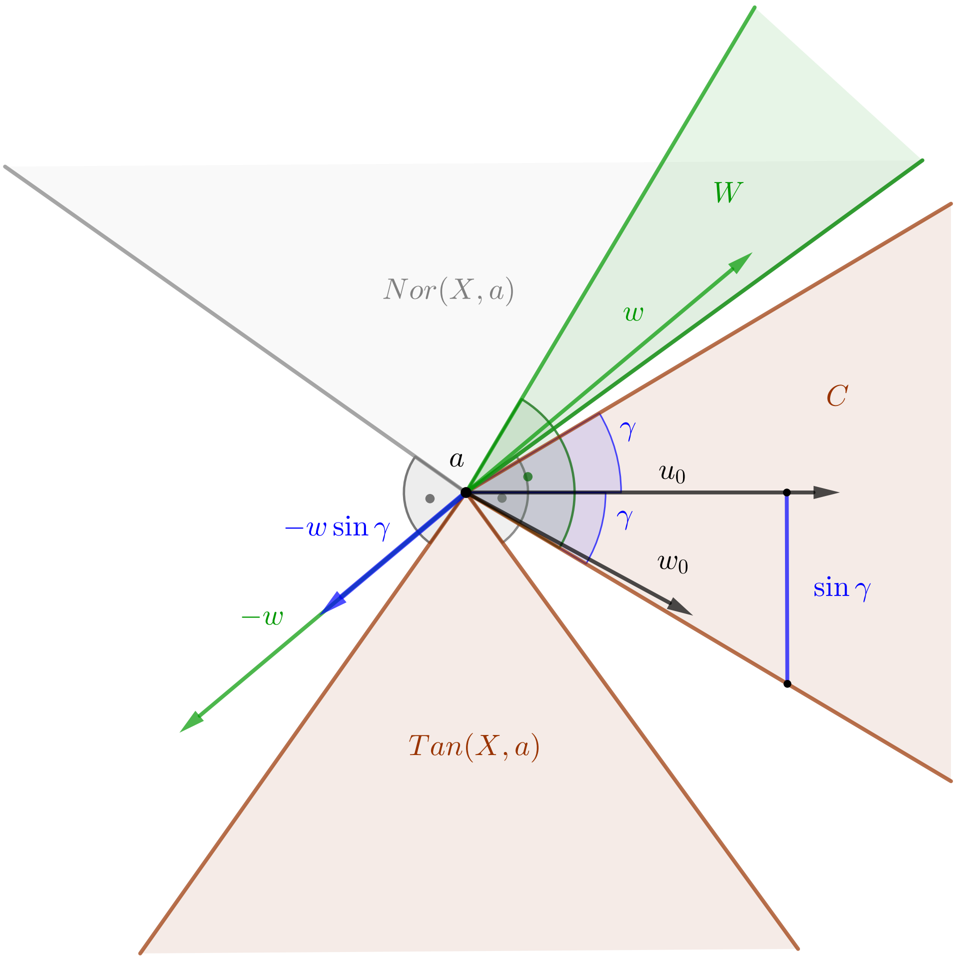

Figure 1.5: Visualization of C and other cones from Theorem 8. A cone W is a set of all w such that $v \cdot w < 0$ if $v \in \mathrm{Tan}(X, a)$ and $u \cdot w > 0$ if $u \in C$.

2. Reach of Intersection

In this chapter we will describe when an intersection of two sets with positive reach is a set with positive reach. As an example take the intersection of x-axis with the graph of the function $f(x) = x^4 \sin \frac{1}{x}$ for $x \in [-1, 0) \cap (0, 1]$ and $f(0) = 0$, then this intersection has reach 0.

Then, we will show some properties of mappings from or onto the Normal Bundle which gives us important properties of X_r and $\operatorname{nor} X$ about its differentiability, from which we infer existence of (generalized) principal curvatures of an set with positive reach, as it is in Rataj and Zähle [2019]. Lastly we describe the curvature of an intersection of a set with positive reach and a plane in $\mathbb{R}^3$.

Theorem 9. *Let* $\operatorname{reach} X, \operatorname{reach} Y \geq r > 0$. *Given* $a \in \partial(X \cap Y)$, *denote*

$$\eta(a) = \inf \left\{ \frac{|u+v|}{|u|+|v|} \;\middle|\; u \in \operatorname{Nor}(X, a), v \in \operatorname{Nor}(Y, a), |u| + |v| > 0 \right\}.$$

Set $\eta = \inf_{a \in \partial(X \cap Y)} \eta(a)$. *Assume that* $\eta > 0$. *Then for all* $a \in \partial(X \cap Y)$,

(i) $\operatorname{Tan}(X \cap Y, a) = \operatorname{Tan}(X, a) \cap \operatorname{Tan}(Y, a)$,

(ii) $\operatorname{Nor}(X \cap Y, a) = \operatorname{Nor}(X, a) + \operatorname{Nor}(Y, a)$,

(iii) $\operatorname{reach}(X \cap Y) \geq r\eta$.

Proof. First note, that (i) and (ii) are equivalent by Equation 1.3. We will show (ii).

Assume that $\operatorname{reach} X, \operatorname{reach} Y \geq r$ and let $a \in X \cap Y$ be given. If $a \notin \partial X \cap \partial Y$ then (ii) is obvious. If $a \in \partial X \cap \partial Y$ and $m \in \operatorname{Nor}(X, a), n \in \operatorname{Nor}(Y, a)$ are unit vectors, then $B(a + rn, r) \cap X = B(a + rn, r) \cap Y = \emptyset$ by Lemma 4 (iii). Hence,

$$(B(a + rn, r) \cup B(a + rn, r)) \cap (X \cap Y) = \emptyset.$$

This implies that m, n belong to $\operatorname{Nor}(X \cup Y, a)$, and if m, n are linearly independent, then for any point z from the line segment $[rm, rn]$, the open ball $B(z, |z - a|)$ is contained in $B(a+rm, r) \cup B(a+rn, r)$. Thus $B(z, |z - a|)$ is disjoint with $X \cap Y$, which implies that $z - a \in \operatorname{Nor}(X \cap Y, a)$, since $\operatorname{Tan}(X \cap Y, a) \subset \operatorname{Tan}(\mathbb{R}^d \backslash B(z, |z - a|), a) = \{u \mid u \cdot (z - a) \leq 0\}$. We have shown inclusion

$$\operatorname{Nor}(X, a) + \operatorname{Nor}(Y, a) \subset \operatorname{Nor}(X \cap Y, a).$$

For the opposite inclusion, take a vector $u \notin \operatorname{Nor}(X, a) + \operatorname{Nor}(Y, a)$, let H be a hyperplane separating u and the convex cone $\operatorname{Nor}(X, a) + \operatorname{Nor}(Y, a)$ and let v be a vector perpendicular to H and forming an acute angle with u. Then $u \in \operatorname{int} \operatorname{Tan}(X, a) \cap \operatorname{int} \operatorname{Tan}(Y, a)$, hence, there exists $\epsilon > 0$ such that $a + \epsilon v \in X \cap Y$ by the corollary of Lemma 6. Thus $v \in \operatorname{Tan}(X \cap Y, a)$ and $u \notin \operatorname{Nor}(X \cap Y, a)$. Thus (ii) is verified.

In order to show (iii), we will use Theorem 8. Let $a, b \in X \cap Y$ be given. We will show that

$$(b - a) \cdot w \leq \frac{|b - a|^2 \, |w|}{2\eta r}, \quad w \in \operatorname{Nor}(X \cap Y, a). \tag{2.1}$$

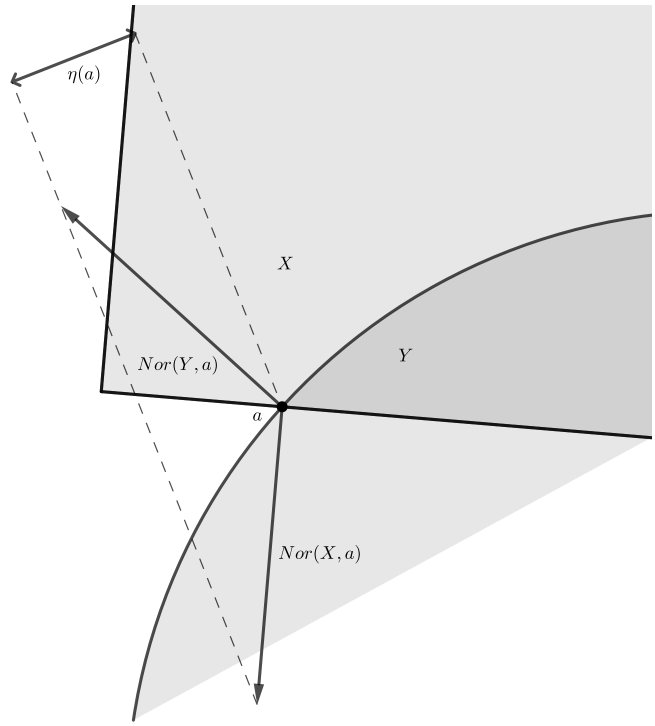

Figure 2.1: Two sets X, Y, their intersection and $\eta(a)$ from Theorem 9.

For $w = 0$ the inequality is clear. If $w \neq 0$ we can represent $w = u + v$ with some $u \in \mathrm{Nor}(X, a), v \in \mathrm{Nor}(Y, a)$ by (ii), where at least one of u, v is nonzero. From Equation 1.2 we have $(b-a)\cdot u \leq \frac{|b-a|^2|u|}{2r}$ and $(b-a)\cdot v \leq \frac{|b-a|^2|v|}{2r}$. Hence

$$(b-a)\cdot(u+v) \leq \frac{|b-a|^2\,(|u|+|v|)}{2r} \leq \frac{|b-a|^2\,|u+v|}{2\eta r},$$

proving 2.1.

We know already from (i) and (ii) that $\mathrm{Tan}(X \cap Y, a)$ is a convex cone and $\mathrm{Nor}(X \cap Y, a) = \mathrm{Tan}(X \cap Y, a)^\circ$. Denote $u = \Pi_{\mathrm{Tan}(X\cap Y,a)}(b-a)$ and $w = b-a-u$. We have $u \cdot w = 0$, since $\mathrm{Tan}(X \cap Y, a)$ is a cone, hence, $|w| = |b-a|\cos\alpha$, where α is the angle formed by w and $b-a$. Also $w \in \mathrm{Nor}(X \cap Y, a)$ and we infer from 2.1 that $\cos\alpha \leq \frac{|b-a|}{2\eta r}$, hence $|v| \leq \frac{|b-a|^2}{2\eta r}$. Thus it holds that

$$d_{\mathrm{Tan}(X\cap Y,a)}(b-a) \leq \frac{|b-a|^2}{2\eta r},$$

which implies (iii) by Theorem 8.

□

2.1 Unit Normal Bundle

Now we show some properties of the Unit Normal Bundle in order to show that X_r is a closed $C^{1,1}$-domain and nor X is a $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^{2d}$, if $0 < r <$ reach X.

Lemma 10. *Let $X \subset \mathbb{R}^d$. If*

$$\begin{aligned}
&N := \{(a, v) \mid a \in X, v \in \operatorname{Nor}(X, a)\},\\
&\sigma\colon N \to \mathbb{R}^d, \sigma(a, v) := a + v \text{ for } (a, v) \in N,\\
&\psi\colon \operatorname{Unp}(X) \to \mathbb{R}^d \times \mathbb{R}^d, \psi(x) := (\Pi_X(x), x - \Pi_X(x)) \text{ for } x \in \operatorname{Unp}(X),
\end{aligned}$$

then

$$\begin{aligned}
&\sigma \text{ is Lipschitzian},\\
&\psi(\operatorname{Unp}(X)) \subset N, \quad \psi \text{ is a homeomorphism}, \quad \psi^{-1} = \sigma|\psi(\operatorname{Unp}(X)).
\end{aligned}$$

If furthermore

$$\begin{aligned}
&K \subset X, 0 < r < q, \operatorname{reach}(X, a) \geq q \text{ for } a \in K,\\
&W := \operatorname{Unp}(X) \cap \{x \mid \Pi_X(x) \in K, x \in X_r\},
\end{aligned}$$

then

$$\begin{aligned}
&\psi(W) = N \cap \{(a, v) \mid a \in K, |v| \leq r\},\\
&\psi|W \text{ is Lipschitzian};
\end{aligned}$$

in case K is compact and $0 \leq t < \infty$, then

$$N \cap \{(a, v) \mid a \in K, |v| \leq t\} \text{ is compact.}$$

Proof. From Lemma 4 one can see that if $x \in \mathbb{R}^d, a \in X$ and $d_X(x) = |x - a|$, then

$$x - a \in \operatorname{Nor}(X, a), \quad (a, x - a) \in N, \quad \sigma(a, x - a) = x.$$

In case $x \in \operatorname{Unp}(X)$, then $a = \Pi_X(x), \psi(x) = (a, x - a), \sigma(\psi(x)) = x$. This implies the first part of this lemma. The second part follows from Lemmas 4, 5 and 7 with its proof. In case K is compact, then $W, \psi(W)$ are also compact and the image of $\psi(W)$ under the transformation mapping (a, v) onto $(a, tr^{-1}v)$. □

Corollary. If $r > 0$ and X_r is r-parallel body of X, then

$$\begin{aligned}
d_{X_r}(x) &= d_X(x) - r \text{ whenever } d_X(x) \geq r,\\
\Pi_X(\Pi_{X_r}(x)) &= \Pi_X(x) \text{ whenever } d_X(x) < \operatorname{reach}(X),\\
\operatorname{reach}(X_r) &\geq \operatorname{reach}(X) - r.
\end{aligned}$$

Furthermore, if $0 < r < \operatorname{reach}(X)$ and $X'_r := \{x \mid d_X(x) \geq r\}$, then

$$\begin{aligned}
d_{X'_r}(x) &= r - d_X(x) \text{ whenever } 0 < d_X(x) \leq r,\\
\Pi_X(\Pi_{X'_r}(x)) &= \Pi_X(x) \text{ whenever } 0 < d_X(x) \leq r,\\
\operatorname{reach}(X'_r) &\geq r.
\end{aligned}$$

Proof. The formula for d_{X_r} follows from the definitions, and the formula for $d_{X'_r}$ may be obtained with the aid of Lemma 4. Then the statement concerning reach and Π can be obtained from Lemma 3, applied to X, X_r, X'_r.

□

Recall that for a set X with C^2-boundary ∂X, the *Gauss map* of X is defined as the mapping

$$\nu_X \colon \partial X \to S^{d-1},$$

where $\nu_X(x)$ denotes the outer unit normal at $x \in \partial X$. The differential $D\nu_X(x)$ exists everywhere on ∂X and it is a self-adjoint linear map from $\mathrm{Tan}(X,x) \cong \nu_X(x)^\perp$ to $\mathrm{Tan}(S^{d-1}, \nu_X(x)) \cong \nu_X(x)^\perp$. The eigenvalues

$$\kappa_1(x), \ldots, \kappa_{d-1}(x) \geq 0$$

of $D\nu(x)$ are called *principal curvatures* of X at x and the associated eigenvectors $b_1(x), \ldots, b_{d-1}(x)$ we called *principal directions.* Note that

$$(b_1(x), \ldots, b_{d-1}(x), \nu(x))$$

is a positively oriented basis of $\mathbb{R}^d$.

Corollary. If $0 < r < \mathrm{reach}\, X$ then X_r is a closed C^1-domain (i.e., a d-dimensional C^1-submanifold with boundary) with the Lipschitzian Gauss map $x \mapsto \nu_{X_r}(x)$. Further, $\mathrm{nor}\, X$ is a $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^{2d}$.

Proof. From Lemma 3, since d_X is C^1 on $\mathrm{int}(\mathrm{Unp}\, X \setminus X)$, we see that X_r is a C^1-submanifold. From Lemmas 3, 5 we infer that the Gauss map $x \mapsto \frac{x - \Pi_X(x)}{|x - \Pi_X(x)|}$ on ∂X_r is Lipschitzian. From the second part of the previous lemma choose $K = X$, then $\psi|W$ is Lipschitzian on X_r, thus $\mathrm{nor}\, X$ is a Lipschitz submanifold of $\mathbb{R}^{2d}$.

□

2.2 Principal Curvatures and Directions

Let $X \subset \mathbb{R}^d, \mathrm{reach}\, X > r > 0$ and let $(x,n) \in \mathrm{nor}\, X$ be given. For such r it holds that $x + rn \in \partial X_r$ and the Gauss map ν_r on X_r satisfies $\nu_r(x+rn) = n$. Thus, if $0 < t < t+s < \mathrm{reach}\, X$, then the mapping $y \mapsto y + s\nu_t(y)$ is a bi-Lipschitzian homeomorphism between ∂X_t and ∂X_{t+s}, and its inverse is $z \mapsto z - s\nu_{t+s}(z)$, because of Lemma 10 and its two corollaries. As the consequence, ν_t is differentiable at y if and only if ν_{t+s} is differentiable at $y + sn$.

Since ν_r is Lipschitzian on the C^1-submanifold ∂X_r, it is differentiable almost everywhere on ∂X_r, according to the Rademacher's theorem. Thus for almost all points $(x,n) \in \mathrm{nor}\, X, \nu_r$ is differentiable at $x + rn$. We call points $(x,n) \in \mathrm{nor}\, X$ with this property *regular.*

Theorem 11. *If* $\mathrm{reach}\, X > 0$ *and* $(x,n) \in \mathrm{nor}\, X$ *is regular then* $\mathrm{Tan}(\mathrm{nor}\, X, (x,n))$ *is a* $(d-1)$*-dimensional subspace and there exist vectors* $b_1(x,n), \ldots, b_{d-1}(x,n)$ *in* $\mathbb{R}^d$ *and numbers*

$\kappa_1(x,n),\dots,\kappa_{d-1}(x,n) \in [-\operatorname{reach} X, \infty]$ *such that* $(b_1(x,n),\dots,b_{d-1}(x,n),n)$ *form a positively oriented orthonormal basis of* $\mathbb{R}^d$ *and the vectors*

$$\left(\frac{1}{\sqrt{1+\kappa_i^2(x,n)}}b_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i^2(x,n)}}b_i(x,n)\right), \quad i=1,\dots,d-1,$$

form an orthonormal basis of $\operatorname{Tan}(\operatorname{nor} X,(x,n))$. *(We set* $\frac{1}{\sqrt{1+\infty^2}}=0$ *and* $\frac{\infty}{\sqrt{1+\infty^2}}=1$.*)*

Proof. Let a regular point $(x,n) \in \operatorname{nor} X$ be fixed and $0<r<\operatorname{reach} X$. Recall that $\nu_r(y)=n$ at $x+rn$, the Weintgarten mapping $-D\nu_r(y)$ exists and, consequently, there exist principal values (curvatures) $\kappa_i^r(y)$ and principal directions $b_i^r(y) \in \operatorname{Tan}(\partial X_r, y)$. We could assume that the vectors $b_1^r(y),\dots,b_{d-1}^r(y),n$ form a positively oriented orthonormal basis of $\mathbb{R}^d$. We will show that

$$-\frac{1}{\operatorname{reach} X - r} \le \kappa_i^r(y) \le \frac{1}{r}, \quad i=1,\dots,d-1. \tag{2.2}$$

This is a consequence of the fact that all directional derivatives of ν_r at y lie within the given bounds which follow from

$$-\frac{1}{\operatorname{reach} X - r} \le \frac{\nu_r(z)-\nu_r(y)}{|z-y|}\cdot\frac{z-y}{|z-y|} \le \frac{1}{r}, \quad z,y \in \partial X_r.$$

Left-hand side of the inequality follows from 1.2 as: since $\operatorname{reach} X_r \ge \operatorname{reach} X - r$ (corollary of Lemma 10) we get $(z-y)\cdot\nu_r(z) \le \frac{|y-z|^2}{2(\operatorname{reach} X - r)}$,
$(y-z)\cdot\nu_r(y) \le \frac{|y-z|^2}{2(\operatorname{reach} X - r)}$, and by subtracting them we are done. For the right-hand inequality, note that

$$\begin{aligned}\frac{\nu_r(z)-\nu_r(y)}{|z-y|}\cdot\frac{z-y}{|z-y|} &= \frac{1}{r}\frac{((z-\Pi_X z)-(y-\Pi_X y))\cdot(z-y)}{|z-y|^2}\\ &= \frac{1}{r}\left(1-\frac{(\Pi_X z-\Pi_X y)\cdot(z-y)}{|z-y|^2}\right) \le \frac{1}{r},\end{aligned}$$

since we have shown in the proof of Lemma 5 that $(\Pi_X z - \Pi_X y)\cdot(z-y) \ge 0$. Thus, 2.2 is proved.

The vectors $b_i^r(y), i=1,\dots,d-1$, form a basis of $\operatorname{Tan}(\partial X_r, y)$. Note that for σ from Lemma 10 we can have slightly modified map $\sigma^r\colon \operatorname{nor} X \to \mathbb{R}^d$, $\sigma^r(a,n)=a+rn$ which is also Lipschitzian, and its inverse map can be written as

$$(\sigma^r)^{-1}(y) = (y - r\nu_r(y), \nu_r(y)),$$

hence, it is differentiable as well, with differential

$$D(\sigma^r)^{-1}(y) = (I - rD\nu_r(y), D\nu_r(y)),$$

mapping $\operatorname{Tan}(\partial X_r, y)$ onto $\operatorname{Tan}(\operatorname{nor} X,(x,n))$. Thus, the vectors

$$D(\sigma^r)^{-1}(y)b_i^r(y) = ((1-\kappa_i^r)b_i^r, \kappa_i^r b_i^r), \quad i=1,\dots,d-1,$$

form a basis of $\mathrm{Tan}(\mathrm{nor}\, X, (x,n))$. Setting $b_i(x,n) := b_i^r(y)$ and

$$\kappa_i(x,n) := \begin{cases} \frac{\kappa_i^r(y)}{1-\kappa_i^r(y)r} & \text{if } \kappa_i^r r < 1, \\ \infty & \text{if } \kappa_i^r r = 1, \end{cases} \qquad i = 1, \dots, d-1,$$

we get the assertion.

□

Lemma 12. *The values $\kappa_i(x,n)$ from Theorem 11 are uniquely determined at any regular point $(x,n) \in \mathrm{nor}\, X$, up to the order. Furthermore, for any $1 \le i \le d-1$, the subspace*

$$\mathrm{Lin}\{b_j(x,n) \mid \kappa_j(x,n) = \kappa_i(x,n)\}$$

is uniquely determined.

Proof. Throughout the proof, we shall omit the argument (x,n) at κ_i and b_i. Assume that

$$\left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right), \quad i = 1, \dots, d-1,$$

and

$$\left(\frac{1}{\sqrt{1+(\kappa_i')^2}} b_i', \frac{\kappa_i'}{\sqrt{1+(\kappa_i')^2}} b_i' \right), \quad i = 1, \dots, d-1,$$

are two orthonormal bases of $\mathrm{Tan}(\mathrm{nor}\, X, (x,n))$, where $\{b_i\}, \{b_i'\}$ are two orthonormal bases of $n^\perp$. Then there exist coefficients c_{ij} such that

$$\frac{1}{\sqrt{1+(\kappa_i')^2}} b_i' = \sum_j c_{ij} \frac{1}{\sqrt{1+\kappa_j^2}} b_j, \tag{2.3}$$

$$\frac{\kappa_i'}{\sqrt{1+(\kappa_i')^2}} b_i' = \sum_j c_{ij} \frac{\kappa_j}{\sqrt{1+\kappa_j^2}} b_j. \tag{2.4}$$

We fix some $1 \le i \le d-1$ and assume that $\kappa_i < \infty$. Multiplying 2.3 with κ_i' and comparing it with 2.4, we get

$$c_{ij} \left(\frac{\kappa_j}{\sqrt{1+\kappa_j^2}} - \frac{\kappa_i'}{\sqrt{1+\kappa_j^2}} \right) = 0$$

for all j. Consequently, we have $\kappa_j < \infty$ and $c_{ij}\kappa_i' = c_{ij}\kappa_j$ for all j. Thus

$$c_{ij} = 0 \text{ or } \kappa_i' = \kappa_j \tag{2.5}$$

for any j.

Now assume that $\kappa_i' = \infty$. Then we have zero on the left-hand side of 2.3 which implies that $c_{ij}/\sqrt{1+\kappa_j^2} = 0$ or $\kappa_j = \infty$ for all j. Thus 2.5 holds for all j. It follows from 2.5 that the sets of numbers $\{\kappa_i \mid 1 \le i \le d-1\}$ and $\{\kappa_i' \mid 1 \le i \le d-1\}$ coincide and that any b_i' is a linear combination of those b_j belonging to the same κ_i.

□

Corollary. Let $(x,n) \in \operatorname{nor} X$ be a regular point and $0 < r < \operatorname{reach} X$. Then the principal curvatures κ_i^r of X_r at $x+rn$ and the (generalized) principal curvatures $\kappa_i(x,n)$ of X at (x,n) are related by

$$\kappa_i(x,n) = \begin{cases} \frac{\kappa_i^r(x+rn)}{1-r\kappa_i^r(x+rn)} & \text{if } \kappa_i^r(x+rn) < \frac{1}{r}, \\ \infty & \text{if } \kappa_i^r(x+rn) = \frac{1}{r}, \end{cases} \qquad i = 1,\dots,d-1,$$

after an eventual change of order. Consequently,

$$\lim_{r\to 0_+} \kappa_i^r(x+rn) = \kappa_i(x,n), \quad i = 1,\dots,d-1.$$

The (generalized) principal directions $b_i(x,n), i = 1,\dots,d-1$ are the principal directions $b_i^r(x+rn)$ of ∂X_r at $x+rn$ for all $r < \operatorname{reach} X$.

2.2.1 Generalized Curvature of a Planar Section

Now we describe generalized curvatures of the intersection of a set with positive reach and a plane.

Lemma 13. *Let $X \subset \mathbb{R}^3, \operatorname{reach} X > 0$ and $F \subset \mathbb{R}^3$ be a plane. For $x \in X \cap F$ and for a* regular point $(x,n) \in \operatorname{nor} X$ *let* $\operatorname{reach}(X \cap F, x) > 0$. *Let us denote L a linear subspace such that $F = x + L$. We define the perpendicular projection $P_L\colon \operatorname{Nor}(X,x) \to L$ and $\Pi_L(n) := \frac{P_L(n)}{|P_L(n)|}$.*

Then for $(u,v) \in \operatorname{Tan}(\operatorname{nor} X,(x,n))$ holds that

$$\left(u, \frac{\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta}{\sin\beta} u\right) \in \operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F),(x,\Pi_L(n))),$$

where κ_1, κ_2 are generalized principal curvatures at (x,n), θ is an angle between u and the principal direction b_1, β is an angle between n and a normal to the plane F and $\operatorname{nor}^{(F)}(X \cap F) := \operatorname{nor}(X \cap F) \cap (F \times S^1)$.

Proof. From Theorem 9 we infer

$$(x,n) \in \operatorname{nor} X \Leftrightarrow (x,\Pi_L(n)) \in \operatorname{nor}^{(F)}(X \cap F). \tag{2.6}$$

If $(u,v) \in \operatorname{Tan}(\operatorname{nor} X,(x,n))$ then from the definition of the tangent cone there exist $(x_i,n_i) \in \operatorname{nor} X$, $x_i \in X \cap F$ and $\alpha_i > 0$, $x_i \to x$, $n_i \to n$ such that $\alpha_i((x_i,n_i) - (x,n)) \to (u,v)$. Note that such u lies in $P_1(\operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F),(x,n)))$, where P_1 is a projection on the first coordinate.

From Theorem 11 we conclude

$$\begin{aligned}(u,v) &= \delta\left(\frac{1}{\sqrt{1+\kappa_1^2}}b_1, \frac{\kappa_1}{\sqrt{1+\kappa_1^2}}b_1\right) + \epsilon\left(\frac{1}{\sqrt{1+\kappa_2^2}}b_2, \frac{\kappa_2}{\sqrt{1+\kappa_2^2}}b_2\right) \\ &= \left(\frac{\delta}{\sqrt{1+\kappa_1^2}}b_1 + \frac{\epsilon}{\sqrt{1+\kappa_2^2}}b_2, \frac{\delta\kappa_1}{\sqrt{1+\kappa_1^2}}b_1 + \frac{\epsilon\kappa_2}{\sqrt{1+\kappa_2^2}}b_2\right),\end{aligned}$$

for some $\delta,\epsilon \in \mathbb{R}$. Thus $\alpha_i(x_i - x) \to u = \frac{\delta b_1}{\sqrt{1+\kappa_1^2}} + \frac{\epsilon b_2}{\sqrt{1+\kappa_2^2}} \in L$ and $\alpha_i(n_i - n) \to v = \frac{\delta\kappa_1 b_1}{\sqrt{1+\kappa_1^2}} + \frac{\epsilon\kappa_2 b_2}{\sqrt{1+\kappa_2^2}}$. Since $|b_1| = |b_2| = |u| = 1$, we see that $\frac{\delta}{\sqrt{1+\kappa_1^2}} = \cos\theta, \frac{\epsilon}{\sqrt{1+\kappa_2^2}} = \sin\theta$, where θ is an angle between b_1 and u.

Now we want to describe v relatively in $\operatorname{nor}^{(F)}(X \cap F)$. It holds for $(x_i, \Pi_L(n_i)), (x, \Pi_L(n)) \in \operatorname{nor}^{(F)}(X \cap F)$ that

$$\begin{aligned}
\alpha_i(\Pi_L(n_i) - \Pi_L(n))) &= \alpha_i \left(\frac{P_L(n_i)}{|P_L(n_i)|} - \frac{P_L(n)}{|P_L(n)|} \right) \\
&= \alpha_i \left(\frac{P_L(n_i - n)}{|P_L(n)|} + \frac{P_L(n_i)(|P_L(n)| - |P_L(n_i)|)}{|P_L(n_i)|\,|P_L(n)|} \right) \\
&\to \frac{P_L(v)}{|P_L(n)|} + 0 = \frac{P_L\left(\cos\theta\kappa_1 b_1 + \sin\theta\kappa_2 b_2\right)}{|P_L(n)|},
\end{aligned}$$

since $P_L(n_i) \to P_L(n)$ and thus the second fraction in the second equality go to the zero. Thus by 2.6 and the definition of tangent vectors

$$\left(u, \frac{P_L(v)}{|P_L(n)|} \right) \in \operatorname{Tan}(\operatorname{nor}^{(F)}(X \cap F), (x, \Pi_L(n))).$$

Finally we describe $P_L(v)$. Since a unit normal vector is perpendicular to its derivation and because we operate in the plane, then the unit normal vector is a multiple of a tangent vector. Thus $P_L(v) = \langle u, v \rangle u$, where $\langle \cdot, \cdot \rangle$ denotes the dot product and since $u \in L$. Thus

$$\begin{aligned}
P_L(v) &= P_L\left(\cos\theta\kappa_1 b_1 + \sin\theta\kappa_2 b_2\right) = \left(\cos\theta\kappa_1\langle u, b_1\rangle + \sin\theta\kappa_2\langle u, b_2\rangle\right) u \\
&= \left(\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta\right) u.
\end{aligned}$$

Lastly let β be an angle as it is in the lemma statement, then $|P_L(n)| = \sin\beta$.

□

We can now conclude, from Theorem 11 and the previous lemma, a generalization of the normal curvature of a point in some direction.

Corollary. If $(x, \Pi_L(n))$ is a *regular point* of $\operatorname{nor}^{(F)}(X \cap F)$, then we describe a (generalized) normal curvature in direction u of x as $\kappa_u := \frac{\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta}{\sin\beta}$.

Note that κ_1, κ_2 could be ∞.

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