FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Tereza Tížková

H-compactifications of Topological Spaces

Department of Mathematical Analysis

Supervisor of the master thesis: doc. Mgr. Benjamin Vejnar, Ph.D.
Study programme: Mathematics
Study branch: Mathematical Structures

Prague 2022

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to thank my thesis supervisor, doc. Mgr. Benjamin Vejnar, Ph.D., for his guidance and willingness to help and discuss anything, from technical constructions to set theory and illustrations of "mathematical intuition". I am grateful that he brought my attention to the topic of H-compactifications in the first place, but also navigated my further learning and motivation for mathematics in general. Likewise, I want to thank very much to all the other teachers from our faculty who answered my questions and provided interesting sources to read. Last but not least, I would like to thank my classmates and friends for proofreading my thesis and for the critical view of the work.

Title: H-compactifications of Topological Spaces

Author: Tereza Tížková

Department: Department of Mathematical Analysis

Supervisor: doc. Mgr. Benjamin Vejnar, Ph.D., Department of Mathematical Analysis

Abstract: H-compactifications form an important type of compactifications, carrying the extra property that all automorphisms of a given topological space can be continuously extended over such compactifications. Van Douwen proved there are only three H-compactifications of the real line and only one of the rationals. Vejnar proved that there are precisely two H-compactifications of higher dimensional Euclidean spaces. The concept of H-compactifications is introduced at the beginning, extra emphasis being put on the Alexandroff and Stone-Čech compactification. We summarize findings that exist about H-compactifications of some well-known spaces. The result we come with in the Chapter 3 is that there is only one H-compactification of the set of all rational sequences, which is precisely the Stone-Čech compactification. The third chapter describes various properties of the set of all rational sequences and its clopen subsets. Some of them - mainly strong zero-dimensionality and strong homogeneity - are then used to reach the said result. In the final Chapter 4, we ask a question about the set of all H-compactifications of the Hilbert space of all square summable real sequences and propose three ways to tackle this problem. We show that under certain conditions, any H-compactification of a space is homeomorphic to its Stone-Čech compactification. We look at H-compactifications of Euclidean spaces that have some properties in common with l^2. Finally, we construct a compactification of a space which is homeomorphic to l^2 and hence possesses the same set of H-compactifications.

Keywords: Compactification, Tychonoff space, H-compactification, Homeomorphism, Category theory

Contents

List of Notation

β	Stone-Čech compactification as a functor
$\mathcal{C}$	1) Cantor set 2) Arbitrary category
$C^{\star}(X)$	Set of bounded real valued continuous functions from X to $\mathbb{R}$
$cHaus$	Category of compact Hausdorff spaces
F_{σ}	F-sigma set
$F_{\sigma\delta}$	F-sigma-delta set
$\mathfrak{F}$	Z-filter
G_{δ}	G-delta set
$G_{\delta\sigma}$	G-delta-sigma set
grf	Graph of f
$\mathbb{H}$	Halfline
$\mathcal{H}$	Set of all automorphisms on X
$\mathcal{H}^{+}([0,1])$	Space of increasing homeomorphisms of the interval $[0,1]$
$Hom(X,Y)$	Morphisms from X to Y
$K(X)$	Set of all compactifications of X
$\mathcal{K}(X)$	Hyperspace of X (equivalently denoted 2^{X})
l^{2}	Space of all square summable real sequences
$\mathbb{N}$	Natural numbers (equivalently denoted ω)
$Ob(X)$	Objects of X
P	1) Irrational numbers 2) Partially ordered set
P_{X}	Lattice of all H-compactifications of X
$\mathbb{Q}$	Rational numbers
$\mathbb{Q}^{\omega}$	Set of all rational sequences
Q_{X}	Lattice of all compactifications of X
$\mathbb{R}$	Real numbers
$\mathbb{R}^{X}$	Set of all functions from X in to $\mathbb{R}$
S^{n}	n-dimensional sphere
Top	Category of topological spaces
T_{π}	Category of Tychonoff topological spaces

X_1	Class of all zero-dimensional absolute $F_{\sigma\delta}$-spaces which are nowhere σ-complete and of first category
$\mathbb{Z}$	Integers
αX	Alexandroff one-point compactification of X
βX	Stone-Čech compactification of X
γX	Arbitrary compactification of X
Δ^0_α	Class of sets that is both in Σ^0_α and Π^0_α
Π^0_α	Class of sets whose complement is in Σ^0_α
ρ_H	Hausdorff metric
Σ^0_α	Class of sets closed under countable unions and finite intersections
ω_1	Set of all countable ordinals
$[0,1]^{(0,1)}$	Tychonoff cube
$\{0,1\}^X$ or 2^X	Cantor cube
$\{0,1\}^{\mathbb{N}}$	Set of infinite sequences consisting of 0s and 1s
∞	Point at infinity
2^ω	Cantor space

Introduction and Preliminaries

Let us start the thesis by recalling some basic facts from general topology and mathematical analysis that will make it easier to provide some intuition behind the notion of H-compactifications and its importance.

Definition 1. *A topological space X is Hausdorff if and only if for every two distinct points x and y, there exist disjoint open sets U and V with $x \in U$ and $y \in V$.*

Hausdorff spaces represent a nice world for proving theorems. They formalize the idea of separating points from each other, they allow limits of sequences to be unique, they have many more of the properties one would intuitively associate with a space.

Metric spaces (such as real numbers), manifolds and many other objects of interest are Hausdorff spaces and in many mathematical texts, all spaces are automatically assumed Hausdorff.

Hausdorff spaces that are compact are even more favorable to work with.

Definition 2. *A compact space is a space in which every open covering of X has a finite subcovering, in other words, if for any collection $\{U_\alpha\}_{\alpha \in A}$ of open sets with $X \subset \cup_{\alpha \in A} U_\alpha$ there exists a finite set of indices $\{\alpha_1, ...\alpha_n\}$ such that $X \subset U_{\alpha_1} \cup ... \cup U_{\alpha_n}$.*

In layman terms,

"If a city is compact, it can be guarded by a finite number of arbitrarily near-sighted policemen."

This paraphrase of the finite subcover definition of compactness is attributed to Hermann Weyl. However, compact spaces can be also described using other properties, for example the following statements define compactness for metric spaces:

1. *All continuous functions are bounded*,
2. *All continuous functions attain a maximum*,
3. *Every sequence has a convergent subsequence.*

Note that all these characteristics are deducible from each other. However, without more specific assumptions, only the first and most general definition via open covers can be used in all cases.

Proposition 3. *Compactness is a topological property.*

Proof. Let X be a compact space. The claim follows from the fact that if a function $f : X \to Y$ is continuous, then $f(X)$ is compact, which is proven in the part (iii) of the following theorem. Note that this is implied by the definition of homeomorphism (open sets are preserved). □

Theorem 4. *(Chandler, 1976, Theorem 1.11)*

(i) Closed subsets of compact spaces are compact.

(ii) Compact subsets of Hausdorff spaces are closed.

(iii) If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

(iv) If $f : X \to Y$ is one-to-one and continuous, X is compact, and Y is Hausdorff then f is a homeomorphism onto $f(X)$.

Proof. (i) Choose a compact space X and its closed subset F. Let $\{U_\alpha \mid \alpha \in A\}$ be an open covering of F. We take the open covering $\{U_\alpha \mid \alpha \in A\}$ and add it to the set $X \setminus F$ which is open. $\{U_\alpha \mid \alpha \in A\} \cup \{X \setminus F\}$ is still an open covering of F, in fact, it is also an open covering of X, since $(X \setminus F) \cup \bigcup_{\alpha \in A} U_\alpha \supset (X \setminus F) \cup F = X$. In conclusion, there exists a finite subcovering of X and it already contains the finite subcovering of the set F as well.

(ii) For a fixed point $x \in X \setminus F$ and each point y of F select disjoint open sets U_y, V_y containing x, y. $\{V_y \mid y \in F\}$ is an open covering of F. Take a finite subcovering (which we can do by assupmtion) and so the intersection of the corresponding $U's$ will be a neighborhood of x in $X \setminus F$.

(iii) Let X be a compact space and let $f : X \to Y$ be a continuous function. If $\{U_\alpha \mid \alpha \in A\}$ is an open cover of $f(X)$, then $\{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open cover of X, since the inverse image of an open set is open. By the assumption, X is compact, so it has a finite subcover $\{f^{-1}(U_{\alpha_i}) \mid i = 1, 2, 3, ...n, n \in \mathbb{N}\}$. Then $\{U_{\alpha_i} \mid i = 1, 2, 3, ...n, n \in \mathbb{N}\}$ makes a finite subcover of $f(X)$, which proves that $f(X)$ is compact.

(iv) For an open set $U \subset X$ we have $X \setminus U$ is closed. Then, of course, it has to be compact, from which $f(X \setminus U)$ is also compact. Hence $f(X \setminus U)$ is closed; hence $f(U)$ is open. Thus, $f : X \to f(X) \subset Y$ is an open mapping . ☐

Theorem 5 (Tychonoff)**.** *The product topological space (endowed with product topology) of an arbitrary set of compact topological spaces is also compact.*

This theorem, originally from Čech (1937), has several ways of proving this very famous theorem available across the literature, three of them given by a relatively new Matheron´s paper (see Matheron (2020)). Tychonoff's proof from 1930 used the concept of a complete accumulation point, the Cartan's proof using ultrafilters or Chernoff's proof using nets. The theorem also comes as a corollary of the Alexander subbase theorem saying that if (X, τ) is a topological space and X has a subbasis S such that every cover of X by elements from S has a finite subcover, then X is compact.

Example 5.1. $[0, 1]^{\mathbb{N}}$ *is compact, being a product of closed intervals* $[0, 1]$*.*

Compact spaces indeed carry many advantageous properties; however, compactness is not hereditary (unlike other properties like metrizability, Hausdorffness, regularity etc.) This is actually the reason why "putting a non-compact space into something compact" can broaden our knowledge about that space- which brings us to the concept of compactifications.

Remark. *We say that a topological space X embeds into a compact Hausdorff topological space Y if X is homeomorphic to a subspace of Y. If we take a closure of the embedding, the name for this process, or equivalently for the resulting compact Hausdorff space, is compactification.*

Definition 6. *A (Hausdorff) compactification γX of a topological space X is a compact (Hausdorff) space γX together with an embedding*

$$\gamma : X \to \gamma X$$

so that $\gamma(X)$ is dense in γX.

Some authors instead use a pair (X, γ) as a notation for a compactification, where X is a topological space γ is the embedding described in the definition. The term *Hausdorff compactification* is often abbreviated to *compactification* and further on, we always mean Hausdorff compactifications, if not stated otherwise.

Definition 7. *A topological space X is called completely regular if given any point $x \in X$ and any closed $S \subset X$ such that $x \notin S$, there is a continuous map $f : X \to [0, 1]$ such $f(x) = 0$ and $f(S) = 1$.*

Loosely speaking, points of a complete regular space can be separated from closed sets via (bounded) continuous real-valued functions.

Definition 8. *A topological space X is called a Tychonoff space (alternatively a $T_{3\frac{1}{2}}$ space) if it is a completely regular Hausdorff space.*

Proposition 9. *A topological space has a Hausdorff compactification if and only if it is Tychonoff.*

Proof. It is a known fact (shown by Tychonoff) that every compact Hausdorff space is automatically a Tychonoff space. Since every subspace of a Tychonoff space is Tychonoff, we conclude that any space possessing a Hausdorff compactification must be a Tychonoff space. The converse, i.e. that every Tychonoff space has a Hausdorff compactification was proved in the famous Tychonoff´s 1930 article (see Tychonoff (1930)). □

Remark. *Every Tychonoff space embeds into a product of type $[0, 1]^I$, whence it always admit a compactification (take the closure of the embedded copy of X).*

Regarding the facts above, all of the topological spaces that we will work with across the text will be considered Hausdorff spaces, if not stated otherwise.

Introducing H-compactifications

It is useful to introduce a special type of compactifications - such that they do not depend on a specific "representation" of a given space, but exclusively on its topological properties.

Definition 10. *Let $h : X \to Y$ be a continuous mapping and γX a compactification on X. We say that h has an extension to γX if there exists a continuous* *mapping $\gamma h : \gamma X \to Y$ such that $\gamma h \circ \gamma = h$. In other words, the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr, "h"'] & \gamma X \arrow[d, "\gamma h"] \\ & Y \end{tikzcd}$$

Alternatively, $\gamma h \restriction_X = h$.

Definition 11. *Given a topological space X, a compactification γX of X is said to be an H-compactification if each automorphism of X can be continuously extended to an automorphism (a mapping of γX into γX).*

Remark. *For a space X, if an automorphism*

$$f : X \to X$$

extends continuously to

$$\gamma f : \gamma X \to \gamma X,$$

then also γf^{-1} exists. Therefore, γf is an automorphism of γX too.

Because the inverses of automorphisms are automorphisms and we always count the identity morphism as an automorphism, this is a hint that all automorphisms on a given space naturally form a group. Moreover, if the group of automorphisms admits a topology, we get a topological group.

Many common topological spaces like $\mathbb{R}, \mathbb{Q}$, the set of irrational numbers or the Cantor set, carry "rich" automorphism groups, which manifests that a lot of their points behave topologically the same. Spaces with large number of automorphisms have their special name - *homogeneous spaces.*

Definition 12. *The above-mentioned notion of a homogeneous space is defined as follows. Let $\mathcal{H}(X)$ be the group of all automorphisms of a space X. We say that a continuous mapping $f : X \to Y$ is homogeneous if for every $h \in \mathcal{H}(X)$ there exists $g \in \mathcal{H}(X)$ such that*

$$f \circ h = g \circ f.$$

In some literature, a homogeneous space X is also defined as a space X such that for every $x, y \in X$ there is an automorphism of X that maps x to y. It is intuitive that more automorphisms a space admits, less H-compactifications of that space exist.

Example 12.1. *Rigid space is a space on which the only homeomorphisms are the trivial ones (i.e. the identity homeomorphisms). It is not difficult to observe that all compactifications of rigid spaces are automatically H-compactifications.*

Ordering of H-compactifications

It has been shown that H-compactifications certainly form a mathematical structure, for instance they have been analyzed from lattice-theoretic perspective by Lubben (1941), Chandler (1976) or Vejnar (2011).

First, we introduce a reasonable ordering, which, for H-compactifications is inherited from the natural ordering of compactifications.

Definition 13. *We say that, given a topological space X, a H-compactification γ_1 is finer than a H-compactification γ_2 if for the embeddings*

$$\gamma_1 : X \to \gamma_1 X \text{ and } \gamma_2 : X \to \gamma_2 X$$

there exists a continuous map $f : \gamma_1 X \to \gamma_2 X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\gamma_1"] \arrow[dr, "\gamma_2"'] & \gamma_1 X \arrow[d, "f"] \\ & \gamma_2 X \end{tikzcd}$$

commutes.

Further on, we will use the notation $\gamma_2 X \leq \gamma_1 X$ for the fact that $\gamma_1 X$ is finer than $\gamma_2 X$. The complementary notion is a coarser H-compactification.

An equivalent definition states that $\gamma_2 X \leq \gamma_1 X$ if there exists a continuous $f : \gamma_1 X \to \gamma_2 X$ that *fixes all points of* X, which means the points stay unaffected by applying f.

The f is a morphism and if it exists, it is automatically surjective and unique. Recall that in the definition of Hausdorff compactifications, $\gamma_i(X)$ is dense in $\gamma_i X$.

Definition 14. *Recall that a lattice L is a partially ordered set where any pair of its elements has both a greatest lower bound (called meet and denoted $\wedge$) and a least upper bound (called join and denoted $\vee$) in L (which, by induction, implies this for all finite subsets).*

Definition 15. *A complete lattice L is a lattice where all subsets have a greatest lower bound and a least upper bound in L.*

Definition 16. *A complete sub-lattice L of a (complete) lattice M is a lattice where the meet and join operations in L agree with the meet and join operations in M.*

Definition 17. *An upper semi-lattice L is a partially ordered set that has a join operation $\vee$ for any nonempty finite subset. Analogously, a lower semi-lattice has $\wedge$ operation defined for each such subset.*

The ordering of H-compactifications brings for each topological space X a partially ordered set (shortly called poset) P_X endowed with a binary relation which is precisely the above defined ordering $\leq$. We will preserve the notation P_X for such poset (or, with more properties, even a lattice or semi-lattice).

Definition 18. *Given a Hausdorff topological space X, a partially ordered set of all H-compactifications of X is defined as*

$$P_X = \{\gamma X, \leq \mid \gamma X \text{ is a H-compactification of } X\},$$

where $\leq$ is the natural ordering of H-compactifications.

Indeed, the structure of H-compactifications depends on the nature of the space X. Therefore, we will look separately at properties of P_X in the case when X is locally compact and then in the general case.

Theorem 19. *(Chandler, 1976, Theorem 2.19), based on Lubben (1941) The set of all compactifications of X forms a complete lattice if and only if X is a locally compact topological space.*

Proof. We provide an idea and main steps for proving the theorem, detailed proof is conducted for example in (Chandler, 1976, Chapter 2). Denote the set of all compactifications of X by $K(X)$.
(i) First, we show by contradiction that if $K(X)$ (as a lattice) is complete, then X is locally compact. Suppose X is chosen such that it is not locally compact. Then for any $\gamma X \in K(X)$, $\gamma X \setminus X$ consists of more than one point. If X is open in γX, then $\forall x \in X$, there is a continuous map $h : \gamma X \to [0, 1]$ such that

$$h(x) = 0$$
$$h(\gamma X \setminus X) = \{1\}.$$

Now, we shrink the $[0, 1]$ to $[0, \frac{1}{2}]$. Then, $h^{-1}([0, \frac{1}{2}])$ is a closed neighborhood of x which is still contained in X and the closedness implies compactness of $h^{-1}([0, \frac{1}{2}])$.

We then proceed by determining for each compactification $\gamma X \in K(X)$ another element $\gamma' X$ of $K(X)$ such that $\gamma X > \gamma' X$ and hence conclude that $K(X)$ has no greatest lower bound, hence it cannot be complete. This way the contradiction is reached.
(ii) Conversely, let X be locally compact and define

$$C = \{f \in C^{\star}(X) \mid \forall \epsilon > 0 \, \exists \text{ a compact } K_\epsilon \subset X \text{ and } |f(x)| < \epsilon \, \forall x \in X \setminus K_\epsilon\}.$$

Chandler proceeds in the proof with observing that C separates points from closed sets and $C \subset C_\gamma$ for any $\gamma X \in K(X)$. From that, we conclude that for any $\gamma X \in K(X)$, $C \subset \cap_{i \in I} C_{\gamma_i}$ for an arbitrary subset $\{\gamma_i X\}_{i \in I} \subset K(X)$. Finally, we will use the fact from (Chandler, 1976, Theorem 2.18) stating that the greatest lower bound of any set of compactifications $\{a_i X\}_{i \in I} \subset K(X)$ exists if and only if $\cap_{i \in I} C_{\alpha_i}$ separates points and separates points from closed sets. From this, $K(X)$ is complete. □

The following proposition describes H-compactifications of a locally compact space in terms of lattices and sublattices.

Proposition 20. *The set of all H-compactifications of a locally compact space X is a complete sub-lattice of the (complete) lattice of all compactifications of X.*

Proof. Let X be locally compact and denote by $\mathcal{H}(X)$ the group of all automorphisms of X.
For a given family of H-compactifications $\mathcal{K}_i = \{\gamma_i X \mid i \in I\}$ denote by C_i the set of all continuous functions from $\mathcal{C}^{\star}(X)$ that admit an extension over $\mathcal{K}_i$. Let $C = \bigcap_{i \in I} \mathcal{K}_i$. The greatest lower bound of the elements of $\mathcal{K}_i$ is given by $\overline{\gamma(X)}$ where γ is a map

$$\gamma : X \to \mathbb{R}^C$$
$$f(x)_e = e(x).$$

We now choose any $f \in \mathcal{C}^\star(X)$ such that it is continuously extendable over $\overline{\gamma X}$ and an arbitrary $h \in \mathcal{H}(X)$. We will verify that $f \circ h$ can be continuously extended over $\overline{\gamma X}$, which is equivalent with $\overline{\gamma X}$ being H-compactification, following (Vejnar, 2011, Proposition 5) which is used in the same paper for proving this proposition. Let $h \in \mathcal{H}(X)$ and let $f \in \mathcal{C}^\star(X)$ be continuously extendable over $\overline{\gamma(X)}$. Then $f \in C$. Consequently, $f \circ h \in \mathcal{K}_i$ for every $i \in I$ since $\gamma_i X$ is an H-compactification. Since $f \circ h$ is in each $\mathcal{K}_i$, it follows that $f \circ h \in \bigcap \mathcal{K}_i = C$ can be continuously extended over $\overline{\gamma X}$. We have shown that $\mathcal{K}_i$ is closed under the join operation, which is enough to observe that the set of all H-compactifications as a lattice is complete. $\square$

In the general case when X is not required to be locally compact, it still possesses a semi-lattice structure. Assume for the following two proposition that X is not necessarilly locally compact.

Proposition 21. *Lubben (1941) The set of all compactifications of a space X with natural order is a complete upper semi-lattice.*

Proof. Let X be a topological space and let $K(X)$ be the set of all compactifications of X. For a subset $\{\gamma_i X \mid i \in I\}$ of $K(X)$, denote for each $i \in I$ by γ_i the map $X \to \gamma_i X$. Define a map e such that

$$\begin{gathered} e : X \to \textstyle\prod_{i \in I} \gamma_i X \\ e(x)(i) = \gamma_i(x). \end{gathered}$$

and denote by π_i the projections for each $i \in I$. The e can be understood as a function evaluating the functions γ_i and is determined by $\{\gamma_i X \mid i \in I\}$, hence from (Chandler, 1976, Theorem 1.24), since each γ_i is a homeomorphism, e is also a homeomorphism. This means that $eX = \overline{e(X)}$ is a compactification of X. For each $i \in I$, $f_i : eX \to \gamma_i X$ is the projection map restricted to eX. Then $(f_i \circ e)(x) = e(x)(i) = \gamma_i X$ so that $f_1 \circ e = \gamma_i$ which implies $eX \geq \gamma_i X\ \forall i \in I$. Now suppose for all $i \in I$ $e_1 X \geq \gamma_i X$ and define g_i such that

$$\begin{gathered} g_i : e_1 X \to \gamma_i X \\ g_i \circ e_1 = \gamma_i. \end{gathered}$$

Define

$$\begin{gathered} f : e_1 X \to \textstyle\prod_{i \in I} \gamma_i X \\ f(y)(i) = g_i(y). \end{gathered}$$

Then $\pi_i \circ f = g_i$, hence f is continuous and moreover $f(e_1(x)(i) = g_i(e_1(x)) = \gamma_i(x) = e(x)(i)$. In conclusion, $f \circ e_1 = e$ so that $f(e_1 X) = eX$ and $e_1 X \geq eX$ which implies that eX is the desired least upper bound of $\{\gamma_i X \mid i \in I\}$ with respect to the binary relation $\geq$. $\square$

Once we have established the properties for a locally compact space, we can use similar approach in the general case.

Proposition 22. *The set of all H-compactifications of a space X is a complete (upper) sub-semi-lattice of the complete (upper) semi-lattice of all compactifications of X.*

Proof. Let $\mathcal{K}_i = \{\gamma_i X \mid i \in I\}$ be a set of H-compactifications of X. For each $i \in I$ denote the corresponding inclusion mapping by $\gamma_i : X \to \gamma_i X$. Define a diagonal mapping $\Delta\gamma_i$ by $\Delta\gamma_i(x) = \prod_{i\in I} \gamma_i X$. Observe that the least upper bound of $\mathcal{K}_i$ is given by $\overline{\gamma(X)}$ where $\gamma = \Delta\gamma_i$.
The final step is to show that $\overline{\gamma(X)}$ is also a H-compactification, whence a part of $\mathcal{K}_i$. This is proven as a part of (Vejnar, 2011, Proposition 4) by showing that for any autohomeomorphism h of X there exists an autohomeomorphism g of $\overline{\gamma(X)}$ such that $\gamma \circ h = g \circ \gamma$. Analogously we can find such h and g for γ. Two lemmas are used, saying that if this property holds for a mapping $X \to Y_i, i \in I$, then it can be extended to mappings $X \to \prod_{i\in I} Y_i$ and $X \to \overline{f(X)}$. $\square$

1. Stone-Čech and Alexandroff Compactification

The most common H-compactifications one comes across are the Stone-Čech compactification and the Alexandroff one-point compactification. The former is the "most general, the latter is the "smallest". In the language of the ordering defined earlier, we say that the Stone-Čech compactification is the finest compactification of a given space and the Alexandroff compactification (provided it exists) is the coarsest.

1.1 The Stone-Čech Compactification

Embedding a (general) topological space into a compactification that continuously extends all its automorphisms is not always that straightforward. However, for a Tychonoff space X, the most famous H-compactification of any space βX called the Stone-Čech compactification always exists.

Definition 23. *The Stone–Čech compactification of the topological space X is a compact Hausdorff space βX together with a continuous map β with the universal property, which means that any continuous map $k : X \to K$, where K is a compact Hausdorff space, extends uniquely to a continuous map $\beta f : \beta X \to K$, i.e. $\beta f \circ \beta = k$.*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "k"'] & \beta X \arrow[d, "\beta f"] \\ & K \end{tikzcd}$$

The universal property, expressed by the commutative diagram, will get more attention in the last chapter of this thesis, where functorial properties of βX are studied.

βX is the most "general" compactification in the sense that it is characterized by the universal property. Any continuous function from X to a compact Hausdorff space K can be extended to a continuous function from βX to K in a unique way.

Definition 24. *For any topological space X and the cartesian product $X \times X$ of X with itself, a map $\Delta : X \to X \times X$, defined by $\Delta(x) = (x, x) \forall x \in X$ is called the diagonal mapping.*

We can extend the definition of diagonal mapping to $\Delta : X \to \prod_{s \in S} X_s$, defined by $\Delta(x) = (x, ...x) \forall x \in X$ where each X_s denotes one copy of X.

Definition 25. *Given a topological space X and a family $\{Y_s\}_{s \in S}$ of topological spaces and a family $\mathcal{F} = \{f_s\}_{s \in S}$ of continuous mappings $f_s : X \to Y_s$, we say that the family $\mathcal{F}$ separates points if for any two distinct points $x, y \in X$, we can find a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \neq f_s(y)$.*

Moreover, we say that the family $\mathcal{F}$ separates points and closed sets if for every closed set $V \in X$ and every point $x \in X, x \notin V$ there exists a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \notin \overline{f_s(V)}$.

Theorem 26. *(Engelking, 1989, 2.3.20 - the Diagonal Theorem) If* $\mathcal{F} = \{f_s\}_{s \in S}$ *is a family of continuous mappings where for each* $s \in S$ *the* $f_s : X \to Y_s$ *separates points, then the diagonal mapping* $\Delta_{s \in S} f_s : X \to \prod_{s \in S} Y_s$ *is a one-to-one mapping. Moreover, if* $\mathcal{F}$ *separates points and closed sets, then* $\Delta_{s \in S} f_s$ *is a homeomorphic embedding.*

Proof. Let $\mathcal{F} = \{f_s\}_{s \in S}$, $f_s : X \to Y_s$ be a family of continuous mappings that separates points. We have to show that the mapping $\Delta_{s \in S} f_s$ never assigns the same value in $\prod_{s \in S} Y_s$ to two distinct elements from X. Choose a pair $(x, y) \in X$ such that $x \neq y$. There exists a $s \in S$ such that $f_s(x) \neq f_s(y)$. Therefore, we have for the diagonal mapping $\Delta_{s \in S} f_s(x) \neq \Delta_{s \in S} f_s(y)$, hence $\Delta_{s \in S} f_s$ is injective.

Finally, we apply a lemma from (Engelking, 1989, Lemma 2.3.19) saying that for a continuous, one-to-one mapping $f : X \to Y$ and the one-element family $\{f\}$ that separates points and closed sets, the mapping f is a homeomorphic embedding.

Suppose $\mathcal{F}$ separates points and closed sets and let $V = \overline{V}$ be a subset of X and let $p_s : \prod_{s \in S} \gamma_s X \to \gamma_s X$ be a projection for each $s \in S$.

Then if $\Delta_{s \in S} f_s(x) \in \overline{\Delta_{s \in S} f_s(V)}$, then for every $s \in S$, $f_s(x) = p_s f(x) \in p_s(\overline{\Delta_{s \in S} f_s(V)}) \subset \overline{p_s(\Delta_{s \in S} f_s(V))} \subset \overline{f_s(V)}$. This yields that the family $\{\Delta_{s \in S} f_s\}$ also separates points and closed sets. But this is exactly the one-element family $\{f\}$ from the lemma, which implies that $\Delta_{s \in S} f_s$ is a homeomorphic embedding, as we wanted to show. $\square$

The proof of the following theorem was originally conducted by Čech.

Theorem 27. *Čech (1937) Let* X *be a Tychonoff space. Then its Stone-Čech compactification exists and it is unique (up to homeomorphism, being an identity on* X*).*

Proof. For uniqueness, we first prove that if a compactification of a Tychonoff space X satisfying the given universal property exists, then it is unique up to equivalence. Suppose we have two compactifications of X satisfying the property, namely $\gamma_1 : X \to \gamma_1 X$ and $\gamma_2 : X \to \gamma_2 X$. Then γ_1 and γ_2 have to be continuous functions from X into compact Hausdorff spaces, and so by the universal property we can find other continuous functions $f : \gamma_1 X \to \gamma_2 X$ and $g : \gamma_2 X \to \gamma_1 X$ such that $\gamma_2 = f \circ \gamma_1$ and $\gamma_1 = g \circ \gamma_2$. From this it immediately follows that $g \circ f$ is the identity map on $\gamma_1 X$, and that $f \circ g$ is the identity map on $\gamma_2 X$. Therefore f is a continuous function with a continuous inverse, making it the homeomorphism we require.

Now we show that for any continuous mapping $f : X \to K$ where K is a compact space, we can find an extension to a continuous mapping $\beta f : \beta X \to K$, that is, we show the universal property of the Stone-Čech compactification. Denote by γ the embedding of X in its arbitrary compactification γX and identify X with the subspace $\gamma(X)$ of γX. We can use the Diagonal theorem: For a family $\{f_s\}_{s \in S}$ of continuous mappings, where $f_s : X \to Y_s$ separates points for each $s \in S$, then the diagonal $\Delta_{s \in S} f_s : X \to \prod_{s \in S} Y_s$ is a one-to-one mapping. Moreover, if the said family separates points and closed sets, then $\Delta_{s \in S} f_s$ is a homeomorphic embedding. This theorem implies that $\beta \Delta f : X \to \beta X \times K$ is a homeomorphic embedding. Hence, $\overline{\gamma(X)} \subset \beta X \times K$ is a compactification of X

and by the maximality of βX, we can find a continuous mapping $g : \beta X \to \gamma X$ such that $g\beta = \gamma$.

Now, let $p : \gamma X \to K$ be the restriction of the projection of $\beta X \times K$ onto K to γX. Then the mapping $p \circ g : \beta X \to K$ is the desired extension of f, since $p \circ g\beta = p \circ \gamma = f$.

For existence, we will show that for every Tychonoff space has a largest element in the family of all compactifications of X with respect to the ordering $\leq$ as defined in the introductory part. This is corollary of the fact that every non-empty subfamily of the family of all compactifications of X has a least upper bound with respect to the order $\leq$ in the family of all compactifications. (see Engelking (1989)[Theorem 3.5.9]).

To show that, let K be a non-empty subfamily of the family of all compactifications of X and let $\Delta_{s\in S} : X \to \prod X$ be a diagonal mapping. Denote by $\gamma_i X$ an arbitrary compactification of X and let $\Delta_{s\in S}\gamma_s : X \to \prod_{s\in S} \gamma_s X$, where S is a set of indices and $\gamma_s : X \to \gamma_s X$ is a homeomorphic embedding of X in $\gamma_s X$ such that $\overline{\gamma_s(X)} = \gamma_s X$.

We need to make sure that the $\Delta_{s\in S}\gamma_s$ is still a homeomorphic embedding. That is true from the Diagonal theorem, since each $\gamma_s : X \to \gamma_s X$ separates points and closed sets.

Now we have to prove that $\Delta_{s\in S}\gamma_s X = \overline{\Delta_{s\in S}\gamma_s(X)} \subset \prod_{s\in S} \gamma_s X$ is the least upper bound of the subfamily K.

Denote the $\Delta_{s\in S}\gamma_s$ by f and observe that for each $s \in S$, the projection p_s: $\prod_{s\in S} \gamma_s X \to \gamma_s X$ commutes with γ_s:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\gamma_s"'] & \prod_{s\in S} \gamma_s X \arrow[d, "p_s"] \\ & \gamma_s X \end{tikzcd}$$

This, by the definition of ordering of compactifications, is just another way of saying that $\gamma_s X \leq \Delta_{s\in S}\gamma_s X$ for every $s \in S$.

Finally, assume that a compactification λX of X satisfies for each $s \in S$ $\gamma_s X \leq \lambda X$. This means that there are mappings $f_s : \gamma X \to \gamma_s X$ for which $f_s \circ \lambda = \Delta_{s\in S}\gamma_s$ holds for each $s \in S$. For every $s \in S$ $\Delta_{s\in S} f_s$ satisfies the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[dr, "\Delta_{s\in S}\gamma_s"'] & \lambda X \arrow[d, "\Delta_{s\in S} f_s"] \\ & \prod_{s\in S} \gamma_s X \end{tikzcd}$$

Then we obtain that $\Delta_{s\in S}\gamma_s X \leq \lambda X$, which shows that the non-empty subfamily K of the family of all compactifications of X has a least upper bound, as required.

□

There are several ways to construct the Stone-Čech compactification - some of them are presented in the Chapter 3 and for at least three different detailed construction, see (Chandler, 1976, Chapters 2, 3).

Remark. *The (Hausdorff) Stone-Čech compactification can be seen as a functor*

$$\beta : T_\pi \to cHaus$$

where cHaus refers to the category of compact Hausdorff spaces and T_π refers to the category of Tychonoff topological spaces (T_π is sometimes used to denote a Tychonoff space).

1.2 The Alexandroff Compactification

The other very typical H-compactification is the Alexandroff one-point compactification αX.

Definition 28. *For a locally compact X, denote the topology of X by τ_X and define the topology $\tau_{\alpha X}$ on a compact space αX the following way:*

$$\tau_{\alpha X} := \tau_X \cup \{U \subseteq \alpha X \mid \infty \in U \text{ and } X \setminus U \text{ is a compact subset of } X\}.$$

A Hausdorff compactification αX of a locally compact (i.e. such that each point of it has a compact neighborhood), non-compact, Hausdorff space X, obtained by adding a single point ∞ to X and endowed with the topology $\tau_{\alpha X}$ is called the Alexandroff one-point compactification.

The extra point ∞ is usually called "a point at infinity".

Remark. *In the situation where X is not locally compact, we can still construct the one-point compactification, however, such compactification is not Hausdorff (and hence not Tychonoff). Moreover, the neighborhood of the point attached to X consists of complements of the closed compact sets of X. This general case of one-point compactification is usually called Alexandroff extension.*

2. Known sets of all H-compactifications

This chapter synthesizes all that is known about sets of H-compactifications of more or less common topological spaces. What interests us is description of the set of *all* H-compactifications of a given space.

Different sources use a lot of different names for H-compactification, e.g. "G-compactification", where G is a subgroup of the group of all automorphisms of a space X, in Groot de, McDowell (1959/60) or "equivariant extension" in Smirnov (1994). We find the most suitable for this notion the term topological compactification" from Douwen van (1979) and "H-compactification" used in Vejnar (2011), which we will keep throughout this text.

For a space X, the βX usually does not look that nice - on the other hand, constructing αX is often more convenient. Hence, for some spaces we will look closely at their one-point compactifications.

2.1 Halfline

Let $\mathbb{H}$ be the halfline $[0, \infty)$. Van Douwen's observation about $\mathbb{H}$ leads to the description of H-compactifications of the real line.

Proposition 29. *(Douwen van, 1979, Proposition 4) The set of all H-compactifications of $\mathbb{H}$ consists of just two elements: $\alpha\mathbb{H}$ and $\beta\mathbb{H}$.*

Proof. Let $\lambda\mathbb{H}$ be any H-compactification of $\mathbb{H}$ with $|\lambda\mathbb{H} \setminus \mathbb{H}| > 1$ (so it is distinct from $\alpha\mathbb{H}$). We show that $\lambda\mathbb{H} = \beta\mathbb{H}$ by showing that disjoint closed subsets of $\mathbb{H}$ have disjoint closures in $\lambda\mathbb{H}$. (A part of the Chapter 3 is devoted to explaining how this property characterizes the Stone-Čech compactification). So let F and G be disjoint closed subsets of $\mathbb{H}$. Without loss of generality, assume that $0 \in F$. We can define two discrete families. Generally, a discrete family $\mathcal{F}$ is a family of subsets of a topological space such that every point of the space has a neighbourhood intersecting at most one element of $\mathcal{F}$. In our case, we can define a discrete family $\mathcal{A}$ of closed intervals in $\mathbb{H}$ such that every point of $\mathbb{H}$ has a neighborhood intersecting at most one interval from $\mathcal{A}$. Analogously, define another discrete family $\mathcal{B}$, such that it consists of closed intervals in $\mathbb{H}$ such that every point of $\mathbb{H}$ has a neighborhood intersecting at most one interval from $\mathcal{B}$. One can construct $\mathcal{A}$ and $\mathcal{B}$ such that

$$F \subseteq \cup \mathcal{A},$$
$$G \subseteq \cup \mathcal{B} \text{ and}$$
$$(\cup \mathcal{A}) \cap (\cup \mathcal{B}) = 0.$$

Let p and q be any two distinct points of $\lambda\mathbb{H} \setminus \mathbb{H}$. Let U and V be neighborhoods of p and q in $\lambda\mathbb{H}$ with $0 \in U$ and $\overline{U} \cap \overline{V} = \emptyset$. Without difficulty one can construct an autohomeomorphism h of $\mathbb{H}$ such that $h(A) \subseteq U$ for $A \in \mathcal{A}$ and $h(B) \subseteq V$ for $B \in \mathcal{B}$. If λh is the extension of h then

$$(\lambda h)(\overline{F}) \cap (\lambda h)(\overline{G}) \subseteq \overline{U} \cap \overline{V} = \emptyset,$$

hence $\overline{F} \cap \overline{G} = \emptyset$. $\square$

2.2 Real Line

Proposition 30. *(Douwen van, 1979, Proposition 2) $\alpha\mathbb{R}$, the two-point compactification of $\mathbb{R}$ and $\beta\mathbb{R}$ are the only three H-compactifications of $\mathbb{R}$.*

Proof. Denote by $\gamma\mathbb{R}$ an arbitrary H-compactification of $\mathbb{R}$. We will refer to the halflines $[0,\infty), (-\infty,0]$ as $\mathbb{R}_+, \mathbb{R}_-$ respectively. Denote for any space X the set of remainders of a compactification γX by $\gamma_R X = \gamma X \setminus X$ and the intersection of closures of $\mathbb{R}_+$ and $\mathbb{R}_-$ in $\gamma\mathbb{R}$ with $\gamma_R\mathbb{R}$ by $\gamma_R\mathbb{R}_\pm$. (Note that $\gamma_R\mathbb{R} = \gamma_R\mathbb{R}_+ \cup \gamma_R\mathbb{R}_-$.

We will show that any $\gamma\mathbb{R}$ is (homeomorphic to) either the one-point compactification $\alpha\mathbb{R}$, the two-point compactification of $\mathbb{R}$, or the Stone-Čech compactification $\beta\mathbb{R}$.

Choose a H-compactification $\gamma\mathbb{R}$. Observe that for any $x \in \mathbb{R}$, there is a map

$$f : x \to -x.$$

This map is a homeomorphism and by definition of a H-compactification, it can be continuously extended over $\gamma\mathbb{R}$. Hence, for the corresponding compactifications $\gamma\mathbb{R}_+, \gamma\mathbb{R}_-$ the remainder $\gamma_R\mathbb{R}_+$ is homeomorphic to $\gamma_R\mathbb{R}_-$, which implies that $\gamma\mathbb{R}_+, \gamma\mathbb{R}_-$ must be of the same "type" - either both are one-point compactifications, or both are the Stone-Čech compactifications of $\mathbb{R}_+, \mathbb{R}_+$ respectively.

In the former case, $\gamma\mathbb{R}$ is either the one-point or the two-point compactification of $\mathbb{R}$. Assume that $\gamma\mathbb{R}_+, \gamma\mathbb{R}_-$ are both the Stone-Čech compactifications. Because $\gamma\mathbb{R}$ is a H-compactification by assumption and each automorphism $f_+ : \mathbb{R}_+ \to \mathbb{R}_+$ or $f_- : \mathbb{R}_- \to \mathbb{R}_-$ continuously extends to an automorphism $f : \mathbb{R} \to \mathbb{R}$ (say, by the identity of the complement), the $\mathbb{R}_+ \cup \gamma_R\mathbb{R}_\pm$ and $\mathbb{R}_- \cup \gamma_R\mathbb{R}_\pm$ are H-compactifications. This implies that both $\gamma_R\mathbb{R}_\pm$ and $\beta_R\mathbb{R}_\pm$ are homeomorphic to the set of remainders of the Stone-Čech compactifications of $\mathbb{R}_+$ and $\mathbb{R}_-$. Denote these homeomorphisms by $h_\pm : \beta_R\mathbb{R}_\pm \to \gamma_R\mathbb{R}_\pm$. These homeomorphisms extend the identity maps $\mathbb{R}_\pm \to \mathbb{R}_\pm$.

Denote by $X \sqcup Y$ the disjoint union of X and Y. Since we assumed that $\gamma\mathbb{R}_+$ and $\gamma\mathbb{R}_-$ are not one-point compactifications, $\beta_R\mathbb{R} = \beta_R\mathbb{R}_+ \sqcup \beta_R\mathbb{R}_-$ and $\gamma_R\mathbb{R} = \gamma_R\mathbb{R}_+ \sqcup \gamma_R\mathbb{R}_-$. For $\gamma\mathbb{R}$ being two-point, this is easily seen. If $\gamma\mathbb{R}$ is (homeomorphic to) the $\beta\mathbb{R}$, then the remainders $\gamma_R\mathbb{R}_+$, $\gamma_R\mathbb{R}_-$ cannot be singletons and there exists a map $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for two distinct points $x_1, x_2 \in \mathbb{R}_+$, $f(x_1) = x_2$. Extending this f by the identity to $\mathbb{R}_-$, we obtain a homeomorphism $\mathbb{R} \to \mathbb{R}$ which, by assumption, extends continuously over $\gamma\mathbb{R}$. Denote this extension by h. If we choose a point $y \in \gamma_R\mathbb{R}_- \subset \gamma\mathbb{R}$, this point is fixed by the automorphism $h : \gamma\mathbb{R} \to \gamma\mathbb{R}$, hence $x_1 \neq y$, so the desired decomposition to disjoint union holds.

The above implies that the maps $h_\pm$ combine to a bijective continuous map $h : \beta\mathbb{R} \to \gamma\mathbb{R}$ extending the identity map $\mathbb{R} \to \mathbb{R}$. Reversing the roles of $\beta\mathbb{R}, \gamma\mathbb{R}$, we conclude that h is a homeomorphism. (Alternatively, we could use the fact that both compactifications are Hausdorff.) □

Proposition 31. *The one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.*

Proof. The construction can be given explicitly as an inverse stereographic projection. Consider the map

$$s : \mathbb{R} \to S^1 \text{ given by}$$
$$x \to \left(\tfrac{1-x^2}{1+x^2}, \tfrac{2x}{1+x^2}\right)$$

Then s is a homeomorphism between $\mathbb{R}$ and $S^1 \setminus \{(-1,0)\}$. It is because for any real x, taking $f(x) = 2arctan(x)$ is in the interval $(-\pi,\pi)$ and $s(x) = (cos(f(x)), sin(f(x)))$ is a continuous bijection. The fact that s is an open map follows from the observation that s maps any bounded open real interval (x, y) to $\{(cos(u), sin(u)) : f(x) < u < f(y)\}$, which is an open set in $S^1 \setminus \{(-1,0)\}$. So, since $S^1 \setminus \{(-1,0)\}$ is dense in S^1 and $S^1 \setminus (S^1 \setminus \{(-1,0)\})$ consists of a single point, S^1 is the one-point compactification of $\mathbb{R}$. □

2.3 Higher dimensional Euclidean spaces $\mathbb{R}^n$

Obviously, any space $\mathbb{R}^n, n \geq 2$ is still locally compact, however, the previous argument using the half-line H-compactifications fails and a more sophisticated proof is needed.

Proposition 32. *(Vejnar, 2011, Corollary 28) For $\mathbb{R}^n$ for $n \geq 2$, there are exactly two H-compactifications, namely $\alpha\mathbb{R}$ and $\beta\mathbb{R}$.*

We will rephrase the proof of this proposition in the second part of the chapter which explores compactifications of l^2.

Example 32.1. *We have already constructed $\alpha\mathbb{R}$, so it is not surprising that for $n \geq 2$, $\alpha\mathbb{R}^n$ is (homeomorphic to) the n-sphere S^n. What is impressive is that this allows proving things about S^n by passing to $\mathbb{R}^n$ and vice versa. The embedding of $\mathbb{R}^n$ in S^n is the inverse of the stereographic projection.*

One example is the fact that S^n is simply connected (it is path-connected and any loop can be contracted to a point). We can see this if we show that a path in S^n can be deformed so that it misses one point. From here, removing the missed point gives a path in $\mathbb{R}^n$, which can be deformed into a constant path using linear functions.

2.4 $\mathbb{N}$

It is easy to embed $\mathbb{N}$ (with the usual discrete topology) into a compact space. It only admits two H-compactifications - that is the one-point H-compactification and the Stone-Čech H-compactification. The one-point compactification $\alpha\mathbb{N}$ is homeomorphic to the generic convergent sequence. $\alpha\mathbb{N}$ is described for example in (Escardó, 2013, Parts 5-9). It is obtained from the discrete space $\mathbb{N}$ as

$$\alpha\mathbb{N} = \{x \in \{0,1\}^{\mathbb{N}} \mid \forall i \in \mathbb{N}(x_i \geq x_{i+1})\},$$

Regarding the construction of $\beta\mathbb{N}$, one example can be found in a paper by Tychonoff (1935) where it is built as a closure of a countable set

$$A = \{a_n(x) : n \in \mathbb{N}\}$$

of points in the Tychonoff cube $[0,1]^{(0,1)}$. The $a_n(x)$ refers to a dyadic expansion of every $x \in (0,1)$ - that is, a representation of each such x with just zeros and ones. One can also identify $\beta\mathbb{N}$ with the set of ultrafilters on $\mathbb{N}$, with the topology generated by sets of the form $\{F : U \in F\}$ where U is a subset of $\mathbb{N}$. We define filters and ultrafilters in the chapter about the space l^2.

Overall, $\beta\mathbb{N}$ is notoriously elusive and highly sensitive to various set-theoretic axioms - a lot of the research on $\beta\mathbb{N}$ instead concentrates on its remainder - $\beta\mathbb{N} \setminus \mathbb{N}$, see for example the chapter Dow, Hart (2003) in the Encyclopedia of General Topology.

Remark. *The set of all H-compactifications of* $\mathbb{Z}$ *is the same as for* $\mathbb{N}$ *(since these are homeomorphic).*

2.5 $\mathbb{Q}$

Proposition 33. *(Douwen van, 1979, Proposition 1) The Stone-Čech compactification* $\beta\mathbb{Q}$ *is the only H-compactification of* $\mathbb{Q}$*.*

Proof. This is an immediate consequence of the following proposition which we will also find very useful in the following chapter about $\mathbb{Q}^\omega$.

All the assumptions except non-compactness are separately analyzed in the next chapter about $\mathbb{Q}^\omega$.

Obviously, $\mathbb{Q}$ - with a topology inherited from $\mathbb{R}$ is non-compact. It is a known fact that a subset of real numbers is compact if and only if it is closed and bounded. The set of rational numbers has neither of these properties in $\mathbb{R}$.

The strong zero-dimensionality is proved as a separate proposition in the next chapter. The proposition in the section 3.1.1 shows that $\mathbb{Q}$ is zero-dimensional and since this is equivalent for all separable, metrizable spaces with strong zero-dimensionality, this yields the desired result.

The fact that every nonempty clopen subspace $U \subset \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$ is true because of the Sierpinski's theorem. $\square$

Proposition 34. *(Douwen van, 1979, Proposition 3) If* X *is a non-compact strongly zero-dimensional space in which every nonempty clopen subspace is homeomorphic to* X*, then the only H-compactification of* X *is the Stone-Čech compactification.*

Proof. Let γX be an arbitrary H-compactification of X. In order to show that γX is the same as βX we prove that every clopen subset of X has an open closure in γX, which will, according to one of the equivalent characteristics of the Stone-Čech compactification, lead to the desired conclusion. Denote again by $\bar{\ }$ the closure operator in γX.

We will use the strong zero-dimensionality of X which provides that disjoint zero-sets are separated by clopen sets. Together with the fact that clopen sets in X have clopen closures in γX, we obtain that disjoint zero-sets (i.e. sets of the form $\{x : f(x) = 0\}$ for a continuous function $f : X \to \mathbb{R}$) in X have disjoint closures in γX.

If U is a clopen subset of X, we can assume $\emptyset \neq U \neq X$. Then we can find a nonempty clopen subset V of X such that $\overline{U} \cap \overline{V} = \emptyset$. U and V are indeed homeomorphic. Then there exists an automorphism h of X such that $h(U) = V$ and h sends any $x \notin U \cup V$ to itself. By the assumption, γX is a H-compactification, so we can define an extension of h over γX. Denote such extension by γh. Since the intersection of closures of U and V is assumed to be empty and h maps every $x \in U$ to $y \in V$, we see that γh satisfies

$$\overline{U} \cap (\gamma h)(\overline{U}) = \overline{U} \cap \overline{V} = \emptyset.$$

Observe that from our definition of h, $\gamma h(x) = x$ for each $x \in \overline{(X \setminus (U \cup V))}$. Therefore, we have

$$\overline{U} \cap \overline{(X \setminus U)} = (\overline{U} \cap (\gamma h)(\overline{U})) \cup (\overline{U} \cap \overline{(X \setminus (U \cup V))}) = \emptyset.$$

Now, since $\overline{U}$ is the complement of $\overline{X \setminus U}$ in γX (which is a closed set), the closure $\overline{U}$ is open in γX. $\square$

2.6 Other spaces

Van Douwen noted that even for many other zero-dimensional spaces, e.g. irrationals P or Sorgenfrey line, the conclusion about H-compactifications mentioned above still works and hence the set of all H-compactifications of each such space consists only of the Stone-Čech compactification. Some modifications of these spaces, for instance the product of $\mathbb{Q}$ and P, also only admit the Stone-Čech compactification.

There are other spaces studied by several authors, often as "variations" of already analyzed spaces. For example Vejnar described 26 H-compactifications of the space $\omega \times \mathbb{R}$ in (Vejnar, 2011, Theorem 33) and proved that the spaces $\omega \times S_n, n > 1$ have only three H-compactifications in (Vejnar, 2011, Corollary 32) and from the same paper, the space $\omega \times S$ has exactly four H-compactifications. Recall that spaces that are topologically the same (homeomorphic) have the same set of H-compactifications. Therefore, we can use topological characterization of the spaces introduced here to find similar conclusion on other spaces.

3. H-compactifications of $\mathbb{Q}^\omega$

In this section, we find the set of all H-compactifications of the space $\mathbb{Q}^\omega$ - the set of all rational sequences. Our result will be shown using characteristics from (Douwen van, 1979, Proposition 3). To achieve that, we have to first analyze properties of $\mathbb{Q}^\omega$ and its nonempty clopen subsets to conclude that the only H-compactification on $\mathbb{Q}^\omega$ is $\beta\mathbb{Q}^\omega$.

Definition 35. *Let I be a non-empty index set and for each $i \in I$, let X_i be a topological space. If $\prod_{i\in I} X_i$ is a Cartesian product and $\pi_i : \prod_{i\in I} X_i \to X_i$ are canonical projections, then the product topology on $\prod_{i\in I} X_i$ is defined as the coarsest topology (i.e. the topology with the fewest open sets) for which every π_i is continuous. $\mathbb{Q}^\omega$ is defined as a set of all rational sequences endowed with the standard product topology.*

Throughout some literature, $\mathbb{Q}^\omega$ is denoted as $\mathbb{Q}^\infty$ or $\mathbb{Q}^\mathbb{N}$ which emphasizes the fact that we can practically interpret $\mathbb{Q}^\omega$ as the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$. It is important to have in mind that $\mathbb{Q}^\omega$ is built as a countable product of $\mathbb{Q} \subset \mathbb{R}$, where we assume that rational numbers have their standard topology inherited from real numbers (with the standard metric).

$\mathbb{Q}^\omega$ is non-compact, metrizable and separable (note that it is a countable product of $\mathbb{Q}$). Sine the H-compactifications of the space $\mathbb{Q}$ have been described in Douwen van (1979), we now want to study the same problem in the more complicated case of $\mathbb{Q}^\omega$.

Question 1. *What is the set of all H-compactifications of $\mathbb{Q}^\omega$?*

3.1 Characterization and properties of $\mathbb{Q}^\omega$

The following sections 3.1.1 - 3.1.4 characterize the space $\mathbb{Q}^\omega$. Among other properties, strong zero-dimensionality is shown, which is later used for answering the question about H-compactifications, using the Van Douwen's Proposition 34 from the Chapter 2. For this proposition, we verify in the section 3.1.5 the crucial assumption that every nonempty clopen subspace of $\mathbb{Q}^\omega$ is homeomorphic to $\mathbb{Q}^\omega$.

3.1.1 (Strong) zero-dimensionality

Further on, we will refer to spaces that are open and closed at the same time as *clopen*.

Definition 36. *A Hausdorff topological space X is zero dimensional if for every point x of X and every neighborhood U of x in X, there exists a nonempty clopen subset V of X such that $x \in V \subset U$. The clopen basis of any zero-dimensional space is a collection of clopen sets that is closed under complements and finite intersections.*

Thus, a space is zero dimensional if and only if it has a basis of clopen sets.

Proposition 37. *The space $\mathbb{Q}$ endowed with the topology inherited from $\mathbb{R}$ is a zero-dimensional space.*

Proof. To show zero dimensionality of $\mathbb{Q}$ (regarded as a subspace of $\mathbb{R}$, which itself is not zero-dimensional), pick $q \in \mathbb{Q}$ and let V be a neighborhood of q in $\mathbb{Q}$. Then there exists an open subset U of $\mathbb{R}$ such that $U \cap \mathbb{Q} = V$. We can find irrational numbers x, y with $x < y$ such that $q \in (x, y)$ and $(x, y) \subset U$. Hence, applying set intersection yields that also $q \in (x, y) \cap \mathbb{Q} \subset U \cap \mathbb{Q} = V$. Moreover, $(x, y) \cap \mathbb{Q} = [x, y] \cap \mathbb{Q}$. But that means exactly that $(x, y) \cap \mathbb{Q}$ is a clopen subset of $\mathbb{Q}$. □

It is a well-known fact that any product of zero-dimensional spaces is zero-dimensional, whence $\mathbb{Q}^\omega$ has zero-dimensionality guaranteed, being a product of zero-dimensional spaces $\mathbb{Q}$.

Definition 38. *A Tychonoff space X is said to be strongly zero-dimensional if its Stone-Čech compactification βX is totally disconnected (that is if the only connected subspaces of βX are singletons).*

It is also true that strong zero-dimensionality of X is equivalent to zero-dimensionality of βX. Trivially, every strongly zero-dimensional space is zero-dimensional, but conversely, it is not necessarily true.

Proposition 39. *Zero-dimensionality and strong zero-dimensionality are equivalent for all separable metrizable spaces.*

Proof. A topological space is said to be a Lindelöf space if every open cover of the space has a countable sub-cover. According to the result in (Engelking, 1989, Theorem 6.2.7), every zero-dimensional Lindelöf space is strongly zero-dimensional. It follows from the theorem in (Willard, 2004, Theorem 16.11) that for a (pseudo)metric space, being a Lindelöf space is equivalent to separability of the space. In conclusion, every separable and metrizable space is Lindelöf and therefore it is strongly zero-dimensional if and only if it is zero-dimensional. □

The proposition guarantees strong zero-dimensionality of $\mathbb{Q}^\omega$ which we will utilize at the end of this chapter. we need to introduce some more concepts.

3.1.2 Borel hierarchy

Let us recall some basic definitions and facts concerning the Borel hierarchy - a classical and widely studied topic.

Definition 40. *Given a set S, a σ-algebra over S is a family of subsets of S closed under countable union, countable intersection and complement. The Borel algebra over $\mathbb{R}$ is the smallest σ-algebra containing the open sets of $\mathbb{R}$. A Borel set of real numbers is an element of the Borel algebra over $\mathbb{R}$.*

Informally, a Borel set is any set in X that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and complement.

Definition 41. *Choose a separable metric space X and $\alpha \in [1, \omega_1]$ where ω_1 means the set of all countable ordinals. We will define the Borel classes Σ^0_α and Π^0_β. First let Σ^0_1 be the class of open sets. For each $\alpha > 1$ let Σ^0_α be the class of countable unions of elements of*

$$\cup\{\Pi^0_\beta \mid \beta < \alpha\}$$

where

$$\Pi^0_\beta = \{X - A \mid A \in \Sigma^0_\beta\}.$$

We will not deep dive into the Borel sets and hierarchy, so for more information, see Miller (1979) or, within the framework of separable metrizable spaces, Kuratowski (1966) and Hausdorff (1962)).

Definition 42. *Each Borel set is assigned a unique countable ordinal number called the rank of the Borel set. A Borel set is said to have a finite rank if it belongs in Σ^0_α for a finite ordinal α - else it is said to have an infinite rank.*

The following theorem summarizes the properties of Borel classes in a more intuitive manner.

Theorem 43. *(Miller, 1995, Theorem 2.1) Σ^0_α is closed under countable unions and finite intersections, Π^0_α is closed under countable intersections and finite unions.*

Proof. From the definition of Σ^0_α, it is clearly closed under countable unions. Now denote by $A_n, B_m, n, m \in \mathbb{N}$ arbitrary families of sets. Since

$$(\bigcup_{n\in\mathbb{N}} A_n) \cap (\bigcup_{n\in\mathbb{N}} B_n) = \bigcup_{m,n\in\mathbb{N}} (A_n \cap B_m),$$

Σ^0_α is closed under finite intersections. It also follows from the De Morgan's laws

$$(A \cup B)' = A' \cap B' \text{ and}$$
$$(A \cap B)' = A' \cup B' \text{ and}$$

that Π^0_α is closed under finite unions and countable intersections (for the latter, take complements). $\square$

Definition 44. *Some authors add also the ambiguous classes Δ^0_α to the definition of Borel hierarchy. A set is in Δ^0_α if and only if it is in both Σ^0_α and Π^0_α, whence it is closed under finite intersections, finite unions and complements.*

Example 44.1. *On $\mathbb{R}$, the first level of Borel hierarchy consists of all open and closed subsets of $\mathbb{R}$, and upon having defined levels $2, 3, 4, ..., n-1$, level n is obtained by taking countable unions and intersections of the previous level.*

When working on higher levels of Borel hierarchy, the classical notation introduced above comes handy. We will, however, work with just the first two levels, hence, further on, we will stick to an alternative notation of Borel hierarchy which uses letters F and G.

Remark. *$\mathbb{Q}^\omega$ is not a $G_{\delta\sigma}$-subset in the completely metrizable space $\mathbb{R}^\omega$. (See Engelen van (1984)). It is, however, $F_{\sigma\delta}$, which can be rewritten as $\cap_{n\in\mathbb{N}}\cup_{m\in N_n} F_m$ where each N_n is a countable set and all $F_m, m \in N_n, n \in \mathbb{N}$ are closed.*

Definition 45. *A metrizable space X is an absolute $F_{\sigma\delta}$-set (or an absolute $F_{\sigma\delta}$-space) provided that X is an $F_{\sigma\delta}$-subset in every metrizable space in which X is embedded.*

It follows from Engelen's characterization of $\mathbb{Q}^\omega$ in (Engelen van, 1985, Part 2) that it is absolute $F_{\sigma\delta}$.

3.1.3 Sets of first and second category

For making a conclusion about the H-compactifications of $\mathbb{Q}^\omega$, we have to demand $\mathbb{Q}^\omega$ is of first category (note that this terminology should not be confused with category theory concepts).

Informally, a set of first category is a set whose elements are not tightly clustered together anywhere in the space.

Definition 46. *A subset $Y \subseteq X$ of a topological space X is called nowhere dense (or rare) in X if its closure has empty interior. Equivalently, Y is nowhere dense in X if we cannot find any nonempty open subset of X which would be contained in Y*

Definition 47. *A subset of a topological space X is said to be of first category in X if it is a countable union of nowhere dense subsets of X.*

The sets of first category are also called meagre sets or meager sets. Informally, one thinks of a first category subset as a "small" subset of the host space. A subset that is not of first category in X is said to be of the second category in X

Proposition 48. *$\mathbb{Q}$ with the usual topology (as the countable union of one-point subsets of $\mathbb{R}$) is of first category in itself.*

Proof. We need to write $\mathbb{Q}$ as a countable union of nowhere dense subsets of $\mathbb{Q}$. Choose an enumeration $\{q_1\}, \{q_2\}, ...$ of $\mathbb{Q}$. Then $\mathbb{Q} = \bigcup_{i=1,...}\{q_1\}$ and $\{q_i\}$ is nowhere dense for each i. Equivalently, each $\mathbb{Q} - \{q_i\}$ is an open subset of $\mathbb{Q}$ and observe that $\bigcap_{i=1,...}(\mathbb{Q} - \{q_i\}) = \emptyset$, which means that the empty set can be expressed as a countable intersection of open dense subsets of $\mathbb{Q}$. $\square$

The case of $\mathbb{Q}$ is not that difficult to visualize. The question is, however, whether the same holds for the infinite product of $\mathbb{Q}$.

Throughout this sub-section, we will, for the sake of convenience, assume all spaces embedded in the Cantor space 2^ω.

Definition 49. *Define and denote the following:*

1. $\Gamma =$ *an arbitrary class of spaces*
2. $\Gamma' = \{X \mid 2^\omega - X \in \Gamma\}$ *(the dual class of Γ).*
3. $Q_0 = \{x \in 2^\omega \mid \exists m : \forall n \geq m : x_n = 0\}$
4. $Q_1 = \{x \in 2^\omega \mid \exists m : \forall n \geq m : x_n = 1\}$

5. $P = 2^\omega - (Q_0 \cup Q_1)$

6. *Mapping* $\phi : P \to 2^\omega$ *by* $\phi(x)_n = 0$ *if and only if the* $n^t h$ *block of zeros in* x *has even length.*

Definition 50. *Let* Γ *be a class of topological spaces and everything as defined above. We say that* Γ *has a property* $(\star)$ *if* Γ *is continuously closed (meaning closed under continuous preimage), and for each* $X \in \Gamma$, $\phi^{-1}(X) \cup Q_0 \in \Gamma$.

Definition 51. *Let* Γ *be a class of sets. We say that a space* X *has a property* $(\star\star)$ *with respect to* Γ *if for each non-empty clopen subset* U *of* 2^ω, $U \cap X \in \Gamma \setminus \Gamma'$.

Definition 52. *A topological space is called a Baire space if the countable intersection of open dense subsets is also dense. Equivalently, it is a space such that a countable union of closed sets each with empty interior also has empty interior.*

Theorem 53. *If a class of Borel sets* Γ *has the property* $(\star)$*, and* A *and* B *both have the property* $(\star\star)$ *with respect to* Γ*, and are either both of first category or both Baire, then* $A \simeq B$.

For the proof of this theorem, which is rather technical and three pages long, see (Steel, 1980, Theorem 2). What is of higher interest here is the implication for $\mathbb{Q}^\omega$.

Corollary 53.1. $\mathbb{Q}^\omega$ *densely embedded in* 2^ω *is of first category.*

Proof. If we set $\Gamma = \Pi^0_3 = F_{\sigma\delta}$ or $\Gamma = \Sigma^0_3 = G_{\delta\sigma}$, then either of them have the property $(\star)$. Assume $\mathbb{Q}^\omega$ is densely embedded in 2^ω. For $\Gamma = \Pi^0_3$, $\mathbb{Q}^\omega$ has the property $(\star\star)$ and for $\Gamma = \Sigma^0_3$, $2^\omega - \mathbb{Q}^\omega$ has the property $(\star\star)$. Then it is concluded in (Engelen van, 1996, Part 2) that the theorem above implies exactly that $\mathbb{Q}^\omega$ is of first category. $\square$

We will further comment the fact that $\mathbb{Q}^\omega$ is of first category (in itself) in the proof of the van Engelen's theorem from 1985.

3.1.4 Characterization of $\mathbb{Q}^\omega$

The notions from the previous sections will be beneficial for describing $\mathbb{Q}^\omega$ in terms of a special class of spaces denoted $\mathcal{X}$, which is inspired by (Engelen van, 1985, Part 3).

Definition 54. *A separable metrizable space* X *is said to be* σ*-complete if* $X = \cup_{i=1}^\infty X_i$ *where each* X_i *is complete (i.e. an absolute* G_δ *space), Equivalently, this definition says that* X *is an absolute* $G_{\delta\sigma}$.

Definition 55. *A Polish space is a separable completely metrizable topological space. We say that a subset of a Polish space* X *is an analytic set if it is a continuous image of a Polish space.*

Notation. *Denote by* $\mathcal{X}$ *the class of all zero-dimensional absolute* $F_{\sigma\delta}$*-spaces which are nowhere* σ*-complete and of first category.*

Prior to the main theorem of this subsection, we provide three lemmas that lead to the result that $\mathbb{Q}^\omega \in \mathcal{X}$. Proofs of the lemmas are omitted here and can be found in Engelen van (1985). Recall that Y is nowhere dense in X if its closure in X has empty interior.

Lemma 56. *If X is an analytic space which is not σ-complete then X contains a closed nowhere σ-complete subspace Y which is nowhere dense in X.*

Definition 57. *A topological space is said to be σ-compact if it is the union of countably many compact sets.*

Example 57.1. *The set of rational numbers $\mathbb{Q}$ with its usual topology is countable (rational numbers are countable infinite), which implies that $\mathbb{Q}$ is σ-compact. In fact, any countable space is σ-compact.*

Lemma 58. *Let A be a Borel set in $\mathcal{C}$ which is not σ-complete, and let F be a σ-compact space such that $A \subset F \subset \mathcal{C}$. Then A contains a closed nowhere dense subset Y which is nowhere σ-complete and first category, such that the closure $\overline{Y}$ in $\mathcal{C}$ is a subset of F.*

This lemma helps prove the following lemma which is crucial for the characterization of $\mathbb{Q}^\omega$ provided in the subsequent theorem. We need more definition to proceed.

Definition 59. *Let X_i be a subset of a metrizable space X. Define the diameter of X_i as*

$$diam(X_i) = \sup\{\rho(x_1, x_2) \mid x_1, x_2 \in X_i\}$$

where ρ is a metric on X.

Lemma 60. *Let $X \in \mathcal{X}$, let F be a σ-compact space such that $X \subset F \subset \mathcal{C}$ and let $\epsilon > 0$. Then there exist closed nowhere dense subsets X_i of X such that*

(i) $X = \cup_{i=1}^{\infty} X_i$

(ii) $X_i \in X$ *for each* $i \in \mathbb{N}$

(iii) $\overline{X_i} \subset F$ *(the closure is in* $\mathcal{C}$*)*

(iv) $diam(X_i) < \epsilon$.

Theorem 61. *(Engelen van, 1985, Theorem 3.4) Up to homeomorphism, $\mathbb{Q}^\omega$ is the only element of $\mathcal{X}$.*

Proof. We have already seen that $\mathbb{Q}^\omega$ is zero-dimensional. Here we will show the remaining properties, plus the fact that there is no other element in $\mathcal{X}$.
(i) $\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$:
We can write $\mathbb{Q}$ as the union of singletons $\bigcup_{q\in\mathbb{Q}}\{q\}$ where each singleton is compact. It is easy to see that $\mathbb{Q}$ is σ-compact and $\mathbb{Q}^\omega$ is a product of σ-compacta. Therefore, $\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$.

(ii) $\mathbb{Q}^\omega$ is of first category:
Consider the finite sequences of rational numbers $(q_0, \ldots, q_n)$, $n \in \mathbb{N}$ consisting

of elements of $\mathbb{Q}^\omega$, i.e. rational sequences where each element is itself a rational sequence. Formally,

$$[(q_0, \dots, q_n)] = \{x \in \mathbb{Q}^\omega : x_i = q_i, 0 \leq i \leq n, n \in \mathbb{N}\}.$$

Clearly $\mathbb{Q}^\omega$ is the countable union of all the sets of the form $[(q_0, \dots, q_n)]$ and such sets are closed in $\mathbb{Q}^\omega$ and have empty interiors.

(iii) $\mathbb{Q}^\omega$ is nowhere σ-complete:
To show that $\mathbb{Q}^\omega$ is nowhere σ-complete, we take inspiration in (Engelen van, 1984, Lemma 2.1 (b)). Suppose $\{A_i \mid i \in \mathbb{N}\}$ is a countable family of subsets of $\mathbb{Q}^\omega$ that are complete. $\mathbb{Q}^\omega$ is not Baire space, hence $\overline{A_1} \neq \mathbb{Q}^\omega$, which implies there is a basic non-empty open subset U of $\mathbb{Q}^\omega$ such that $U \cap A_1 = \emptyset$. Recall the notation of $\{X_i : i \in \mathbb{N}\}$ introduced earlier in this section and let $n_1 \in \mathbb{N}$ and $(q_1, ..., q_{n_1}) \in \mathbb{Q}^{n_1}$ be such that

$$X_1 = (q_1, ..., q_n) \times \mathbb{Q} \times \mathbb{Q} \times ... \subset U.$$

Observe that trivially, $X_1 \simeq \mathbb{Q}^\omega$. Since $A_2 \cap X_1$ is closed in A_2, it is complete, and since X_1 is not Baire, there exist $n_1, n_2 \in \mathbb{N}$ such that $n_1 < n_2$ and $(q_{n_1+1}, ..., q_{n_2}) \in \mathbb{Q}^{n_2 - n_1}$ such that

$$X_2 = (q_1, ..., q_{n_2}) \times \mathbb{Q} \times \mathbb{Q} \times ... \subset \mathbb{Q}^\omega \setminus (A_1 \cup A_2).$$

Proceeding in this way, we discover a point $(q_i)_{i \in \mathbb{N}} \in \mathbb{Q}^\omega \setminus (\cup_{i=1}^\infty A_i)$. Hence $\mathbb{Q}^\omega$ is not σ-complete, and because it is strongly homogeneous (see the next part of this chapter), it is nowhere σ-complete.

(iv) $\mathbb{Q}^\omega$ is the *only* element of $\mathcal{X}$:
Finally, we need to see that there is no other element in $\mathcal{X}$. Further on, denote by M the set of all finite sequences of natural numbers, including the empty sequence $\emptyset$. Take an arbitrary X such that $X \in \mathcal{X}$ which we embed in the Cantor set $\mathcal{C}$. Let $\{F_n \mid n \in \mathbb{N}\}$ be a family of σ-compact subsets of $\mathcal{C}$ such that $X = \cap_{n=1}^\infty F_n$ and put $F_0 = \mathcal{C}$. We will construct closed subspaces X_s of X, for each $s \in M$, satisfying the following conditions:

1. $X = X_\emptyset$ and $X_s = \cup_{i=1}^\infty$ for each $s \in M$.
2. For each $i \in \mathbb{N}$ and each $s \in M, X_{s,i}$ is nowhere dense in X_s.
3. For each $s \in M$, $X_s \in \mathcal{X}$.
4. For each $s \in M$, $diam(X_s) < (|s| + 1)^{-1}$.
5. For each $s \in M$, $\overline{X}_s \subset F_{|s|}$ (the closure is in $\mathcal{C}$).

To construct such X_s, put $X_\emptyset = X$, and if X_s has been defined for all $s \in M$ with $|s| \leq n$, then we obtain the sets $X_{s,i}$ by applying the Lemma 60 from this subsection to $X_s \subset F_{|s|+1} \subset \mathcal{C}$ where $\epsilon = (|s| + 2)^{-1}$. Now, we claim that the sets X_s satisfy the following condition: If $\sigma \in \mathbb{N}^\omega$ and $p_n \in X_{\sigma|n}$ for each $n \in \mathbb{N}$, then the sequence $(p_n)_{n \in \mathbb{N}}$ converges. For that, let $\sigma \in \mathbb{N}^\omega$ and since $\overline{X}_{\sigma|1} \supset \overline{X}_{\sigma|2} \supset ...$ is a decreasing sequence of compacta, $\cap_{n=1}^\infty \overline{X}_{\sigma|n} = \emptyset$, let $x \in \cap_{n=1}^\infty \overline{X}_{\sigma|n}$ By the condition 5, $x \in \cap_{n=1}^\infty F_n = X$. Thus, $x \in \cap_{n=1}^\infty \overline{X}_{\sigma|n}$ and if U is any open neighborhood of x in X, then by the condition 4, $X_{\sigma|k} \subset U$ for some $k \in \mathbb{N}$. Hence, if $p_n \in X_{\sigma|n}$ for each $n \in \mathbb{N}$, then $p_n \in U$ for $n \geq k$, which implies that $(p_n)_n$. converges to X as we wanted to prove. □

3.1.5 Strong homogeneity

Once we have described all relevant properties of $\mathbb{Q}^\omega$, we will proceed to the conclusion about the set of all H-compactifications, using the notion of strong homogeneity.

Definition 62. *A space X is homogeneous if for each $x, y \in X$, there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.*

Example. $\mathbb{Q}^\omega$ is homogeneous. This is overt, considering the homogeneity of $\mathbb{Q}$, since any product of homogeneous spaces is homogeneous. This implies $\mathbb{Q}^\omega$, similarly as $\mathbb{Q}$ has a rich group of automorphisms and hence we can come up with a hypothesis that the set of all H-compactifications would be quite narrow.

Definition 63. *A separable metrizable topological space X is strongly homogeneous if for each non-empty clopen subset U of X, we have $U \simeq X$.*

We now introduce three statements that will be used to show that $\mathbb{Q}^\omega$ is strongly homogeneous.

Lemma 64. *For each sequence $s = (S_n)_{n\in\mathbb{N}}$ of clopen subsets of $\mathbb{Q}$, let A_s be the subset $\prod_{n\in\mathbb{N}} S_n$ of $\mathbb{Q}^\omega$. The complement of any basic clopen A_s is a finite union of basic clopens, where by basic clopen we mean a (closed and open) set A_s such as $S_n = \mathbb{Q}$ for all n except finitely many.*

Proof. For each sequence ξ of elements of $\{-1, 1\}$, consider the clopen A_{s_ξ}, where s_ξ is the sequence given by S_n if $\xi_n = 1$ or $\mathbb{Q} \setminus S_n$ if $\xi_n = -1$. Note that $A_{s,\xi}$ is empty for all ξ except finitely many, and if for some ξ we have that $A_{s,\xi}$ is nonempty, then it is clopen. Moreover, the sets $\{A_{s_\xi}\}_{\xi\in\{-1,1\}^\omega}$ are pairwise disjoint and their union is all $\mathbb{Q}^\omega$. Hence, the complement of $A_s = A_{s_{(1,1,1,\dots)}}$ is just the union of the rest of clopens $A_{s,\xi}$, with $\xi \neq (1, 1, 1, \dots)$, which are nonempty. ☐

Corollary 64.1. *If $A_1, \dots, A_n$ are basic clopens, then $A_n \setminus \bigcup_{i=1}^{n-1} A_i$ is a finite union of basic clopens.*

Proof. $A_n \setminus \bigcup_{i=1}^{n-1} A_i = \bigcap_{i=1}^{n-1} (A_n \setminus A_i)$. Each set $A_n - A_i$ is a finite union of basic clopens, because it is $A_n \cap (\mathbb{Q}^\omega \setminus A_i)$. So their intersection is also a finite union of basic clopens. ☐

Theorem 65. *Sierpinski (1920) Any countable metric space (X, d) without isolated points is homeomorphic to $\mathbb{Q}$ (considered with the standard topology.)*

Lemma 66. *(Engelen van, 1984, Lemma 2.1) $\mathbb{Q}^\omega$ is strongly homogeneous.*

Proof. First, for each sequence $s = (S_n)_{n\in\mathbb{N}}$ of clopen sets of $\mathbb{Q}$, let A_s be the subset $\prod_{n\in\mathbb{N}} S_n$ of $\mathbb{Q}^\omega$. Note that if $S_n = \mathbb{Q}$ for all n except finitely many, then A_s is clopen and as in the previous lemma, refer to such A_s as "clopens". As $\mathbb{Q}$ has a basis formed by clopen sets (e.g. intervals with irrational ends), basic clopens form a basis for the topology of $\mathbb{Q}^\omega$.

Also note that nonempty basic clopens are homeomorphic to $\mathbb{Q}^\omega$. This is a consequence of the fact that any nonempty clopen of $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}$ by the Sierpinski's theorem.

For any two basic clopens A_s and A_t, the clopen $A_s \cap A_t$ is given by A_r, where $r = (R_n)_n$ is given by $R_n = S_n \cap T_n$. Hence, it is also a basic clopen. From the lemma presented earlier in this section, any complement of any basic clopen is a finite union of basic clopens. As a corollary, if $A_1, \ldots, A_n$ are basic clopens, then $A_n \setminus \bigcup_{i=1}^{n-1} A_i$ is a finite union of basic clopens.

Now consider any open subset U of $\mathbb{Q}^\omega$. As $\mathbb{Q}^\omega$ is second countable and basic clopens form a basis for its topology, we can express $U = \cup_{n\in\mathbb{N}} U_n$, where the U_n are all basic clopens. Letting $V_n = U_n \setminus \bigcup_{i=1}^{n-1} U_i$, we also have $U = \cup_n V_n$. The V_n are pairwise disjoint, and each one is a finite union of disjoint clopens. So we have obtained a form to express U as a union of disjoint clopens.

We also want infinitely many of the clopens to be non-empty: for that, it is enough to have infinitely many nonempty V_n: this can be achieved by choosing the cover U_n of U in such a way that no finite union of the U_n cover U.

To obtain a cover of U by clopens such that no finitely many of them cover U, we can do the following: let $\pi_m : \mathbb{Q}^\omega \to \mathbb{Q}$ be the m$^{\text{th}}$ projection of the product $\mathbb{Q}^\omega$ onto its factors. As U is open, we have $\pi_m(U) = \mathbb{Q}$ for some $m \in \mathbb{N}$. Now take an arbitrary open cover U_n of U and let $U_{n,k} = U_n \cap \pi_m^{-1}((-k\pi, k\pi))$. The cover $(U_{n,k})_{n,k}$ of U satisfies what we want because for any union X of finitely many of the $U_{n,k}$, $\pi_m(X)$ is bounded, so $X \neq U$.

So, for any open set U of $\mathbb{Q}^\omega$, we have obtained U as an infinite union of disjoint basic clopens, which in turn are homeomorphic to $\mathbb{Q}^\omega$. So, U is homeomorphic to $\mathbb{N} \times \mathbb{Q}^\omega$. Note that this also aplies to the open set $U' = \mathbb{Q}^\omega$. So $\mathbb{Q}^\omega$ is also homeomorphic to $\mathbb{N} \times \mathbb{Q}^\omega$, thus U is homeomorphic to $\mathbb{Q}^\omega$. $\square$

Remark. *A similar, easier proof works to show that any open subset of the Cantor set $\mathcal{C} \subseteq [0,1]$ which is not compact is homeomorphic to $\mathbb{N} \times \mathcal{C}$: we can take a base of clopen sets of the Cantor set (which are homeomorphic to $\mathcal{C}$ by Brouwer's theorem). Then we can express any open subset U of $\mathcal{C}$ as a union of a sequence A_n of these clopen sets. Letting $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$, B_n are clopen so they are either empty or homeomorphic to $\mathcal{C}$. Moreover, as $U = \bigcup_n B_n$ is not compact, infinitely many of the B_n have to be nonempty. So $U \simeq \mathbb{N} \times \mathcal{C}$.*

The strong homogeneity of $\mathbb{Q}^\omega$ implies the following fact.

3.2 Conclusion

The previous analysis of $\mathbb{Q}^\omega$ and its nonempty clopen subsets above allows us to answer the Question 1. $\mathbb{Q}^\omega$ satisfies all the assumptions from the Proposition 34 - it is non-compact, strongly zero-dimensional space whose every nonempty clopen subspace is homeomorphic to $\mathbb{Q}^\omega$. We have seen that $\mathbb{Q}^\omega$ complies with all assumptions of this theorem and that every non-empty clopen subset of $\mathbb{Q}^\omega$ is homeomorphic to $\mathbb{Q}^\omega$. This answers our question.

Corollary 66.1. *The only H-compactification of $\mathbb{Q}^\omega$ is precisely the Stone-Čech compactification $\beta\mathbb{Q}^\omega$.*

Remark. *Homeomorphic spaces generally have identical sets of compactifications (and H-compactifications). Many authors studied spaces homeomorphic to $\mathbb{Q}^\omega$. For instance,*

$$Y = \{(y_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega : lim_{i\to\infty} y_i = \infty\} \simeq \mathbb{Q}^\omega$$

(see Engelen van (1985)), or

$$X \times \mathbb{Q}^\omega \simeq \mathbb{Q}^\omega \textit{ for every zero-dimensional } F_{\sigma\delta}\textit{-space } X$$

(see Engelen van (1984)).

4. H-compactifications of l^2

The purpose of this chapter is to speculate about H-compactifications of the space l^2. Our, maybe too ambitious, hypothesis that this set only has one element - βl^2 has been neither successfully proved nor disproved, so the question stays unanswered.
However, we analyze several topics that could help understanding behaviour of l^2 and its compactifications and motivate future aims in this problem.
Throughout this chapter, l^2 will be considered as the Hilbert space of all square summable real sequences.

Definition 67. *ℓ^2 is the space of sequences $x = (x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$, such that $\sum_{n\in\mathbb{N}} |x_n|^2 < \infty$. The norm on l^2 is given by $\|x\| = \sqrt{\langle x, x\rangle}$ and makes it a Hilbert space.*

The space l^2 can be regarded as a topological vector space and a topological group at the same time. It is endowed with the classical norm topology.

Question 2. *What H-compactifications does the space l^2 admit?*

Since l^2 possesses rich group of automorphisms, one would expect that it only admits the most general H-compactification, that is βl^2. In the subsequent section, we present three ways to look at l^2 that could help prove this hypothesis - the first is the characterization of the Stone-Čech compactification and showing that under certain conditions, any other H-compactification is homeomorphic to it. The second is a technique for describing the set of all H-compactifications of Euclidean spaces that have some properties in common with l^2. In the final section, we construct a compactification of a space homeomorphic to l^2.

4.1 Characterization of βX

As one of the possible tools to verify the hypothesis above, we will present one of the interesting characterizations of the Stone-Čech compactification. We will prove that if an arbitrary H-compactification meets certain criteria, it has to be homeomorphic to the Stone-Čech compactification. Consequently, if one succeeds to show that $\gamma l^2 \simeq \beta l^2$ for every H-compactification γl^2, that would be enough to prove that βl^2 is the only H-compactification of l^2.

Definition 68. *Take a completely regular Hausdorff space X, a compact space K and a mapping h such that $h : X \to K$. Define the Stone extension as the extension βh of h into K such that $\beta h : \beta X \to K$.*

Definition 69. *For a topological space X, we introduce the following notation and definitions:*

1. *Denote by $C(X)$ a ring of all real-valued continuous functions on a topological space X.*

2. *Denote by $C^\star(X)$ = subring of $C(X)$ of continuous bounded functions from X to $\mathbb{R}$.*

3. *For a topological space X, $C^\star(X)$ is the ring of bounded continuous functions on X. A subspace $A \subseteq X$ is said to be $C^\star$-embedded in X if every $f \in C^\star(A)$ can be extended to some $g \in C^\star(X)$.*

4. *Let $f \in C(X)$. A subset Z of X which is of the form $Z = \{x \in X \mid f(x) = 0\}$ is called the zero-set of f. Clearly, such sets are closed.*

5. *Denote by $Z(X)$ the family of all zero-sets in X.*

The notion of zero sets is useful for introducing z-filters - special cases of filters that play an important role giving a connection between βX and $C(X)$.

Definition 70. *A nonempty subfamily $\mathfrak{F}$ is called a z-filter on X provided that $\mathfrak{F}$ is a part of $Z(X)$*

1. $\emptyset \notin \mathfrak{F}$

2. *if $Z_1, Z_2 \in \mathfrak{F}$, then $Z_1 \cap Z_2 \in \mathfrak{F}$ and*

3. *if $Z_1 \in \mathfrak{F}$, $Z_2 \in Z(X)$, and $Z_1 \subset Z_2$ then $Z_2 \in \mathfrak{F}$.*

Definition 71. *We call a z-filter on X a z-ultrafilter on X if it is a maximal z-filter, i.e., one not contained in any other z-filter. We call a z-filter free if the intersection of all its members is empty.*

Definition 72. *The z-filter $\mathfrak{F}$ is said to converge to the limit p if every neighborhood of p contains a member of $\mathfrak{F}$.*

The Stone-Čech compactification βX of X can be constructed by adjoining to X one new point for each free z-ultrafilter and it is essentially unique. For βX, distinct free z-ultrafilters on X converge to distinct points of βX. These properties are captured by the following two theorems which demonstrate equivalent characterizations of βX and methods of construction.

Theorem 73. *(Gillman, Jerison, 1960, Theorem 6.4, (III)) Let X be dense in a compact space γX. Within this theorem, we mean by the symbol $\bar{\ }$ the closure of a space in γX. The following five statements are equivalent.*

1. *Every continuous mapping γ from X into any compact space K has an extension to a continuous mapping from γX into K.*

2. *X is $C^\star$-embedded in γX.*

3. *Any two disjoint zero-sets in X have disjoint closures in γX.*

4. *If we take any two zero-sets Z_1 and Z_2 in X, their closures in γX satisfy $\overline{Z_1 \cap Z_2} = \overline{Z_1} \cap \overline{Z_2}$*

5. *Every point of γX is the limit of a unique z-ultrafilter on X.*

Proof. We will prove the most relevant implications for this chapter, which are 2. $\implies$ 3. and 3. $\implies$ 4.. The complete proof can be found in (Gillman, Jerison, 1960, Theorem 6.4, (III))
The former is due to the Urysohn's theorem saying that any subspace of X is $C^\star$-embedded in X if and only if any two completely separated sets in that subspace

are completely separated in X. For the latter implication, we will show $\overline{Z_1} \cap \overline{Z_2}$ is contained in $\overline{Z_1 \cap Z_2}$ (the reverse inclusion is done trivially). Take a point $x \in \overline{Z_1} \cap \overline{Z_2}$. Then if V is an arbitrary zero-set neighborhood (in γX) of x, we have $x \in \overline{V \cap Z_1}$ and $x \in \overline{V \cap Z_2}$. Hence, by 3., $V \cap Z_1$ intersects $V \cap Z_2$ i.e., V intersects $Z_1 \cap Z_2$ Therefore $p \in \overline{Z_1 \cap Z_2}$ and hence, $\overline{Z_1} \cap \overline{Z_2}$ is contained in $\overline{Z_1 \cap Z_2}$. □

Theorem 74. *(Gillman, Jerison, 1960, Theorem 6.5, (III)) Every (completely regular and Hausdorff) space X has a unique compactification βX such that any two disjoint zero-sets in X have disjoint closures in βX. If another compactification γX of X satisfies the said condition, then there exists a homeomorphism of βX onto γX that leaves X point-wise fixed.*

The proof follows from the equivalences in the previous theorem. Recall that in the introductory section, we have defined the Stone-Čech compactification via its universal property, that is, the first statement in the Gillman and Jerison's theorem.

4.2 H-compactifications of $\mathbb{R}^2$

The goal of this chapter is to describe the set of all H-compactifications of $\mathbb{R}^2$. We will rephrase findings from (Vejnar, 2011, Part 3.3) who proved the same for the general case $\mathbb{R}^n$, $n = 2, 3, 4, ...$
With a bit of imagination, we can envision l^2 like a generalized version of Euclidean spaces, where each "vector" has infinite number of coordinates. (Here, the "vectors" are represented by the infinite sequences). Higher dimensional Euclidean spaces are therefore something we can begin with when exploring the H-compactification of l^2.
Recall that $\mathbb{R}^n, n \geq 2$ only admit $\beta\mathbb{R}^n$ and $\alpha\mathbb{R}^n$ as H-compactifications, the latter due to local compactness of $\mathbb{R}^n$.

Definition 75. *Let $\mathcal{U}$ be a collection of subsets of a metric space X. Recall that $diam(X) = \sup\{\rho(x_1, x_2) \mid x_1, x_2 \in X\}$ where ρ is a metric on X and define the mesh of $\mathcal{U}$ such that:*

$$mesh\, \mathcal{U} = sup\{diam(U) : U \in \mathcal{U}\} \in [0, +\infty].$$

Definition 76. *We say that an open subset U of a space X has property $(\star)$ if for every $E \subseteq U$ closed in X and for every $F \subseteq U$ which is open and non-empty, we can find an automorphism h of X such that $h(E) \subseteq F$ and $h(x) = x$ on $X \setminus U$.*

Definition 77. *We say that a space X has property $(\star\star)$ if there exists a number $N \in \mathbb{N}$ such that for arbitrary small $\epsilon > 0$ there is an open covering $\mathcal{U}$ which can be expressed as a union of N discrete sub-collections $U \in \mathcal{U}$ with mesh less than ϵ, where each U has property $(\star)$, i.e. as in the previous definition, we can find an automorphism h that sends closed subsets of U to open subsets of U and is an identity on $X \setminus U$.*

Lemma 78. *(Vejnar, 2011, Lemma 22) Let X be a separable locally compact metric space with property $(\star\star)$. Let $M = 2N$. Then:*

(i) *Let F be a closed set contained in an open set U. Then, for F there exist a closed discrete set $C \subseteq F$ and closed sets $F_0, ..., F_{M-1}$ such that $F = \cup F_i$ and for every $i < M$ and each open neighborhood G of C, there exists an automorphism h of X such that $h(F_i) \subseteq G$ and h is the identity on the complement of U.*

(ii) *For every pair of closed sets F, $F' \subseteq X$ such that $F \cap F' = \emptyset$, we can find closed discrete sets $C, C' \subseteq X$, $C \cap C' = \emptyset$ and closed sets $F_0, ..., F_{M-1}$, $F'_0, ...F'_{M-1}$ such that*

$$F = \cup F_i,$$
$$F' = \cup F'_j$$

and for any $i, j < M$ and neighborhoods G and G' of C and C' respectively there is an automorphism h of X such that $h(F_i) \subseteq G$ and $h(F'_j) \subseteq G'$.

For will skip the proof of this lemma, which is quite lengthy and technical and refer to (Vejnar, 2011, Lemma 22). More important for this chapter will be the Theorem 82 and its corollary about H-compactifications of $\mathbb{R}^2$.

Definition 79. *A space X is called strongly locally homogeneous if for every $x \in X$ and each neighborhood U of x there exists a neighborhood V of x in U such that for every $y \in V$ we can find a homeomorphism of X which sends x to y and is the identity on the complement of V.*

Proposition 80. *(Vejnar, 2011, Proposition 26) Every bijection of closed discrete subsets of $\mathbb{R}^2$ can be extended to a homeomorphism of the whole space.*

We will also omit conducting a proof this proposition, which, as was remarked by Vejnar, is more convenient to prove for Euclidean spaces of higher dimension than 2, else it is rather complicated.

Definition 81. *Let X be a topological space and $\mathcal{P}$ a collection of subspaces of X. Then X is called 2-homogeneous with respect to $\mathcal{P}$ if for any sets $C_1, C_2, D_1, D_2 \in \mathcal{P}$ such that $C_1 \simeq D_1$, $C_2 \simeq D_2$ and $C_1 \cap C_2 = \emptyset = D_1 \cap D_2$ there exists an automorphism h of X such that $h(C_1) = D_1$ and $h(C_2) = D_2$.*

Theorem 82. *(Vejnar, 2011, Theorem 24) Let X be a separable locally compact but non-compact metric space with property $(\star\star)$ and 2-homogeneous with respect to closed discrete sets. Then αX and βX are the only H-compactifications of X.*

Proof. First, let γX be an H-compactification of X such that γX is distinct from αX. Take $F, G \subset X$ such that both F and G are closed and $F \cap G = \emptyset$. In metric spaces, zero sets and closed sets are the same. Our goal is therefore to show that the closures of F and G in γX, $\overline{F}$ and $\overline{G}$, do not intersect. Then, from findings in the previous section, we can tell that γX is equivalent to βX.

Clearly, X obeys the assumptions of the Lemma 78, hence, there are two closed discrete sets C_1, C_2 and families of closed sets $F_0, ...F_M$ and $G_0, ...G_M$ having the properties mentioned in the lemma. Since $F = \cup F_i$ and $G = \cup G_i$, if every pair (F_i, G_j), $i, j < M$ has empty intersection, then also $F \cap G = \emptyset$. Hence, to conclude that $\overline{F} \cap \overline{G} = \emptyset$. we need to prove that $\forall i, j < M$, $\overline{F_i} \cap \overline{G_j} = \emptyset$ (considering the closures being closures in γX).

We can find two countable infinite closed discrete sets D_1 and D_2 of X such that $\overline{D_1}$ and $\overline{D_2}$ in γX are disjoint, since γX is not a one-point compactification by assumption, whence $\gamma X \setminus X$ contains at least two points. By the Proposition 80, X is 2-homogeneous with respect to closed discrete sets and since γX is an H-compactification, we infer that $\overline{C_1} \cap \overline{C_2} = \emptyset$ in γX. Hence, we are able to separate them by open sets U_1 and U_2 in X with disjoint closures in γX.

Since the F and G are closed sets with an empty intersection, by the Lemma 78, we can find an automorphism h of X with $h(F_i) \subseteq U_1$ and $h(G_j) \subseteq U_2$. Consequently, the closures of $h(F_i)$ and $h(G_j)$ in γX are disjoint and since γX is an H-compactification, indeed the closures of F_i and G_j are also disjoint, hence, from $F = \cup F_i$ and $G = \cup G_i$, $\overline{F} \cap \overline{G} = \emptyset$, as we wanted.

Thus we have proved that $\gamma X \simeq \beta X$. □

The following corollary has already been introduced in the Chapter 2 for general $n \in N, n \geq 2$.

Corollary 82.1. *(Vejnar, 2011, Corollary 28) The set of all H-compactifications of $\mathbb{R}^2$ has exactly two elements - $\alpha\mathbb{R}^2$ and $\beta\mathbb{R}^2$.*

Proof. The corollary follows from the previous theorem - to see that, we are going to play with the premises and outcomes of the lemma preceding this theorem, together with the proposition about bijections of discrete subsets of $\mathbb{R}^2$.

(i) Lemma 78: Consider the maximum metric ρ on $\mathbb{R}^2$, defined, for each $x, y \in \mathbb{R}^2$, $x = (x_1, x_2), y = (y_1, y_2)$ by

$$\rho(x, y) = max(|x_1 - y_1|\,; |x_2 - y_2|)$$

and denote by $B_\rho(x, r)$ the open ball with centre x and diameter r. To verify the property $(\star\star)$, choose $N \in \mathbb{N}$ equal to 4 so it corresponds to the number of sequences of length 2 formed by zeros and ones. Now we have to find for arbitrary $\epsilon > 0$ an open covering $\mathcal{U}$ for which there exist discrete sub-collections U_i such that $\mathcal{U} = \cup_i U_i$ and for any pair of closed set $E_i \subset U_i$ and non-empty open $F_i \subset U_i$, there is an automorphism h of U_i such that $h(E_i) \subset F_i$ and $h(u) = u$ on $U_i \subset \mathbb{R}^2 \setminus U_i$. In other words, we have to formulate $\mathcal{U}$ as in definition of the property $(\star\star)$. Pick an $\epsilon > 0$ and put

$$\mathcal{U}_j = \{B_\rho(\tfrac{\epsilon i}{3}, \tfrac{\epsilon}{4}) : i \in \mathbb{Z}^2, i_k \equiv j_k (\text{mod } 2)\}.$$

.

Then, define the required open cover $\mathcal{U}$ as

$$\mathcal{U} = \cup\{\mathcal{U}_j \mid j \in 4\} \text{ of } \mathbb{R}^2.$$

Observe that the sets $B_\rho(\frac{\epsilon i}{3}, \frac{\epsilon}{4})$ from the definition of $\mathcal{U}_j$ obey the property $(\star)$ because every ball $B_\rho(x, r)$ in $\mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and for an arbitrary closed subset of $B_\rho(x, r)$ and any non-empty open subset of $B_\rho(x, r)$, we can find an automorphism mapping the closed subset to the open subset and the complement of the in $\mathbb{R}^2$ to itself. That is, every ball $B_\rho(x, r)$ in $\mathbb{R}^2$ has property $(\star)$.

In conclusion, since all collections U_j are discrete and made of sets with property $(\star)$, we obtain that $\mathbb{R}^2$ has property $(\star\star)$.

(ii) Proposition 80: Let $\mathcal{P}$ be a collection of closed discrete subsets of $\mathbb{R}^2$. From the Proposition 80, we know that every bijection of any $P \in \mathcal{P}$ can be extended to a homeomorphism of the whole space $\mathbb{R}^2$. This implies that $\mathbb{R}^2$ is 2-homogeneous with respect to closed discrete sets, which is the last assumption left to verify for the Vejnar's theorem. $\square$

Remark. *The two H-compactifications of* $\mathbb{R}^2$ *are distinct from one another. This is an immediate consequence of the fact that there exists a continuous bounded function with no limit at infinity.*

4.3 Compactifying the space $\mathcal{H}^+([0,1])$

This section provides a different way of investigating the H-compactifications of l^2. We will study the space $\mathcal{H}^+([0,1])$ which is topologically the same as l^2 and hence can provide a new intriguing view on the compactifications of l^2. Inspired by Kennedy (1988), we will construct an example of a compactification of $\mathcal{H}^+([0,1])$, using a hyperspace.

Definition 83. $\mathcal{H}^+([0,1])$ *is the space of increasing homeomorphisms of the closed interval* $[0,1]$*, endowed with the supremum metric. Analogously we could define the space* $\mathcal{H}^-([0,1])$ *of decreasing homeomorphisms of* $[0,1]$*.*

The topology on $\mathcal{H}^+([0,1])$ is the topology of uniform convergence, induced by the supremum metric. The $\mathcal{H}^+([0,1])$ also possesses a group structure, where the group multiplication operation is the composition of homeomorphisms.

Perhaps not so intuitively, l^2 is homeomorphic to $\mathcal{H}^+([0,1])$.

Theorem 84. *(Keesling, 1971, Theorem III.1)* $\mathcal{H}^+([0,1]) \simeq l^2$*.*

Proof. In this proof, we consider $\mathcal{H}^+([0,1]) \simeq l^2$ as a set of all functions on a unit interval $[0,1]$ that are monotone, increasing and onto. We endow $\mathcal{H}^+([0,1])$ with the compact open topology.

Here, we will just sketch the proof - verifying the details is a routine procedure. What we will show is that $\mathcal{H}^+([0,1]) \simeq \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$. Then, the homeomorphism $\mathcal{H}^+([0,1]) \simeq l^2$, is guaranteed by Anderson, Bing (1968) where the the authors prove the result that l^2 is homeomorphic to the space of all sequences $\{x_i\}_{i>1}$ of real numbers (endowed with the product topology.)

Let $\{x_{n,i}\} \in \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$. Define an orientation preserving homeomorphism h of $[0,1]$ associated with $\{x_{n,i}\}$. Suppose that we chose an arbitrary $n \in \mathbb{N}$ and that we have defined points

$$A_n = \{0 = \alpha_0^n < \alpha_1^n < ...\alpha_{2^n}^n = 1 \text{ and}$$
$$B_n = \{0 = \beta_0^n < \beta_1^n < ...\beta_{2^n}^n = 1$$

that we defined h such that $h(\alpha_i^n) = \beta_i^n$ for $i = 0, 1, ..., 2^n$. Now we have to extend the definition of h to a set of points $A_{n+1} \supset A_n$ onto $B_{n+1} \supset B_n$ where each A_{n+1} and B_{n+1} contain exactly $2^{n+1} + 1$ points. We investigate the two cases of n:

(i) n is odd: Let z_i be the midpoint of the interval $[\alpha_{i-1}^n, \alpha_i^n]$ for $i = 0, 1, ..., 2^n$ and let $y_i = h(z_i) = x_{n,i}(\beta_i^n - \beta_{i-1}^n) + \beta_{i-1}^n$.

(ii) n is even: Let y_i be the midpoint of the interval $[\beta_{i-1}^n, \beta_i^n]$ and let

$$z_i = h^{-1}(y_i) = x_{n,i}(\alpha_i^n - \alpha_{i-1}^n) + \alpha_{i-1}^n.$$

Then let $A_{n+1} = A_n \cup \{z_i\}^2 N_{i=1}$ and $\beta_{n+1} = \beta_n \cup \{y_i\}^2 N_{i=1}$. Then we will obtain $A_{n+1} = \{0 = \alpha_0^{n+1} < ... < \alpha_{2^n+1}^{n+1} = 1\}$ and $\{0 = \beta_0^{n+1} < ... < \beta_{2^n+1}^{n+1} = 1\}$ with $h(\alpha_i^{n+1}) = \beta_i^{n+1}$. Proceeding in this way and put $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$. Then A and B will be dense in $[0,1]$ and $h[A] = B$ will be order preserving. Hence h admits a continuous extension to an orientation preserving homeomorphism of $[0,1]$ onto itself which will be also denoted by h.

Finally, define

$$F : \prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i} \to \mathcal{H}^+([0,1]) \text{ by}$$
$$F(\{x_{n,i}\}) = h.$$

Then F is the desired homeomorphism of $\prod_{n=0}^{\infty} \prod_{i=1}^{2n} (0,1)_{n,i}$, onto $\mathcal{H}^+([0,1])$. $\square$

Definition 85. *Let X be a metric topological space equipped with a metric ρ. A hyperspace $\mathcal{K}(X)$ of X is defined as*

$$\mathcal{K}(X) := \{K \subseteq X, K \text{ compact, } K \neq \emptyset\}.$$

The metric on $\mathcal{K}(X)$ is the Hausdorff metric

$$\rho_H(K,L) := max\{sup_{x\in K}\rho(x,L), sup_{y\in L}\rho(y,K)\}.$$

The hyperspace is sometimes denoted 2^X, which can refer to hyperspace defined with closed, not compact sets. The Hausdorff metric induces the so-called Vietoris topology. It is a well-known fact that if the space X is compact, then the hyperspace $\mathcal{K}(X)$ is compact as well.

Note that the subsets of the X in the definition of hyperspace have to be compact, while for the whole X, this condition is not necessary. However, when X happens to be compact too, one interesting fact about the hyperspace arises.

Notation. *Denote by d a metric on X, compatible with its topology and by grf a graph of f.*

If $X = [0,1]$, we can think of any $f \in \mathcal{H}^+([0,1])$ as a closed collection of ordered pairs in $[0,1] \times [0,1] = [0,1]^2$. Hence, we can associate any such f with its graph, grf. Therefore, if we take $[0,1]^2$, which is compact, then we can use a hyperspace to construct a nice compactification of $\mathcal{H}^+([0,1]) \simeq l^2$.

Proposition 86. *(Kennedy, 1988, Observation 1) If $X = ([0,1], d)$ is a compact metric space, then the space $\mathcal{H}([0,1])$ can be embedded in its hyperspace $\mathcal{K}([0,1])$.*

Proof. For any $f \in \mathcal{H}([0,1])$, define its graph as $grf = \{(x, f(x) \mid x \in [0,1]\}$. Denote $G = \{grf \mid f \in \mathcal{H}([0,1])\}$ (note that $G \subseteq \mathcal{K}([0,1]^2)$).

Then, define the function

$$\phi : \mathcal{H}([0,1]) \to G \text{ such that}$$
$$\phi(f) = grf.$$

Observe that ϕ is a one-one function from $\mathcal{H}([0,1])$ onto G. Then, we need to show that ϕ is a homeomorphism, which is proven in (Kennedy, 1988, Observation 1). Using relationship between metrics on $[0,1]$ and $[0,1]^2$, it is shown that ϕ is continuous and then, by contradiction, that ϕ^{-1} is continuous as well). $\square$

The statements from Keesling and Kennedy allow us to construct a specific compactification of the space l^2. The implication is that we are able to embed l^2 into a compact hyperspace of closed sets on $[0, 1]$.

4.4 Conclusion

There are more ways to go about the analysis of l^2. If we stick to the hypothesis that there is no other H-compactification except βl^2, one can find useful the characterization via disjoint zero-sets or any of the equivalent conditions in the Theorem 73.
Another possibility is recreating the process that Vejnar used on the case of $\mathbb{R}^n$, $n \geq 2$ and compensate the local compactness used in the Lemma 78 and the Theorem 82 with different properties.
Finally, there exist spaces homeomorphic to l^2, which admit the same set of H-compactifications and can serve as an intermediary in this problem.

Remark. *By the well-known Anderson–Kadec theorem, all separable Banach spaces of infinite dimension are homeomorphic to* $\mathbb{R}^\omega$ *- the Cartesian product of countably many copies of the real line* $\mathbb{R}$*. The problem therefore mainly concerns the countable power of the space* $\mathbb{R}$*.*

Conclusion

The overall purpose of this thesis is to synthesise all the scattered findings about H-compactifications into a complex and structured overview. We have summarized the main findings about H-compactifications of several well-known spaces and analysed properties of two outstanding types of compactifications - Alexandroff and Stone-Čech.

We devoted the Chapter 3 to taking a closer look at the space $\mathbb{Q}^\omega$ for which the set of H-compactifications has not been shown before. We have proven that the only H-compactification of $\mathbb{Q}^\omega$ is $\beta\mathbb{Q}^\omega$, which is automatically true for all spaces homeomorphic to $\mathbb{Q}^\omega$ as well.

In the Chapter 4, we have studied the problem of describing the set of all H-compactifications of l^2. The question stays unanswered, but we provide several ways that can motivate the future aims in solving this problem.

Bibliography

Anderson R. D., Bing R. H. A complete Elementary Proof that Hilbert Space is Homeomorphic to the Countable Infinite Product of Lines // Bull. Amer. Math. Soc. 1968. 74.

Chandler Richard E. Hausdorff Compactifications. 1976. vii+146. (Lecture Notes in Pure and Applied Mathematics, Vol. 23).

Douwen Eric K. van. Characterizations of $\beta\mathbf{Q}$ and $\beta\mathbf{R}$ // Arch. Math. (Basel). 1979. 32, 4. 391–393.

Dow Alan, Hart Klaas Pieter. d-18. The Čech-Stone Compactifications of $\mathbb{N}$ and $\mathbb{R}$. 12 2003.

Engelen Fons van. Countable Products of Zero-dimensional Absolute $F_{\sigma\delta}$ Spaces // Nederl. Akad. Wetensch. Indag. Math. 1984. 46, 4. 391–399.

Engelen Fons van. Characterizations of the Countable Infinite Product of Rationals and Some Related Problems // Rend. Circ. Mat. Palermo (2) Suppl. 1985. 11. 37–54 (1987).

Engelen Fons van. A Non-homogeneous Zero-dimensional X Such That $X \times X$ is a Group // Proc. Amer. Math. Soc. 1996. 124, 8. 2589–2598.

Engelking Ryszard. General topology. 6. 1989. Second. viii+529. (Sigma Series in Pure Mathematics). Translated from the Polish by the author.

Escardó Martín H. Infinite Sets That Satisfy the Principle of Omniscience in any Variety of Constructive Mathematics // J. Symbolic Logic. 2013. 78, 3. 764–784.

Gillman Leonard, Jerison Meyer. Rings of Continuous Functions. 1960. 82–89. (The University Series in Higher Mathematics).

Groot J. de, McDowell R. H. Extension of Mappings on Metric Spaces // Fund. Math. 1959/60. 48. 251–263.

Hausdorff Felix. Set Theory. 1962. Second. 352. Translated from the German by John R. Aumann et al.

Keesling James. Using Flows to Construct Hilbert Space Factors of Function Spaces // Trans. Amer. Math. Soc. 1971. 161. 1–24.

Kennedy Judy. Compactifying the Space of Homeomorphisms // Colloq. Math. 1988. 56, 1. 41–58.

Kuratowski K. Topology. Vol. I. 1966. xx+560. New edition, revised and augmented, Translated from the French by J. Jaworowski.

Lubben R. G. Concerning the Decomposition and Amalgamation of Points, Upper Semi-continuous Collections, and Topological extensions // Trans. Amer. Math. Soc. 1941. 49. 410–466.

Matheron Étienne. Three Proofs of Tychonoff's Theorem // Amer. Math. Monthly. 2020. 127, 5. 437–443.

Miller Arnold W. On the Length of Borel Hierarchies // Ann. Math. Logic. 1979. 16, 3. 233–267.

Miller Arnold W. Descriptive Set Theory and Forcing. 4. 1995. ii+130. (Lecture Notes in Logic). How to prove theorems about Borel sets the hard way.

Sierpinski Waclaw. Sur une propriété topologique des ensembles dénombrables denses en soi // Fundamenta Mathematicae. 1920. 1. 11–16.

Smirnov Yu. M. Can Simple Geometric Objects be Maximal Compact Extensions for $\mathbf{R}^n$? // Uspekhi Mat. Nauk. 1994. 49, 6(300). 213–214.

Steel John R. Analytic Sets and Borel Isomorphisms // Fund. Math. 1980. 108, 2. 83–88.

Tychonoff A. Über die Topologische Erweiterung von Räumen // Math. Ann. 1930. 102, 1. 544–561.

Tychonoff A. Über einen Funktionenraum // Math. Ann. 1935. 111, 1. 762–766.

Vejnar Benjamin. Topological Compactifications // Fund. Math. 2011. 213, 3. 233–253.

Willard Stephen. General topology. 2004. xii+369. Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581].

Čech Eduard. On bicompact spaces // Ann. of Math. (2). 1937. 38, 4. 823–844.