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Dynamical properties of continua

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Abstract: This thesis investigates long–term topological behaviour of continuous self–maps or sets of continuous self–maps of metric spaces, mostly Peano continua. The first chapter is preparatory for the following two and summarize some properties of compact spaces with emphasis on Peano continua. In the second chapter, we give an overview of chaotic features and then we prove that for every Peano continuum X there exists a LEO self–map of X with a dense set of periodic points. In particular, such f is chaotic with respect to widely accepted Devaney' definition of chaos. The third chapter deals with topological fractals, we prove there a new sufficient condition under which a Peano space is a topological fractal, namely that any Peano continuum with uncountably many local cutpoints is a topological fractal. We use this result to partially answer problems concerning regenerating fractals.

Keywords: dynamical system continuum continuous map transitive map

Contents

Introduction

A topological dynamical system is a topological space X and a set $\mathscr{F}$ of continuous self–maps of X. When studying such a space, we are interested in a long–term topological behaviour of the space when we are repeatedly applying maps in $\mathscr{F}$. These properties vary depending on both the space X and the set $\mathscr{F}$. In this thesis we will investigate only dynamical systems where $\mathscr{F}$ is finite, and in a part of this thesis even one–element.

This introduction is based on a book on topological dynamics Brin and Stuck [2002].

Notation. We denote by I the unit interval $[0, 1]$.

Let us develop the idea of topological dynamics using several examples. Consider the following self–maps of I: $id : x \mapsto x$, a constant function $c : x \mapsto c$, where $c \in I$ is arbitrary, $f_1 : x \mapsto 1 - x$, $f_2 : x \mapsto x^2$, a contraction $f_3 : x \mapsto s \cdot x$, where $s \in (0, 1)$, and the tent map

$$f_4(x) := \begin{cases} 2x, & x \leq 1/2, \\ 2(1-x), & x \geq 1/2. \end{cases}$$

Intuitively, very different things happen when we are repaeatedly applying different maps of the above list: when applying id literally nothing happens, while any constant function degenerates the whole space to one point in the first iteration and other iterations do nothing.

The map f_1 keeps tossing the points of I from one side to the other periodically but does not change the diameter of any set and also maps open sets to open sets. In contrast, the map f_2 does change diameter of almost all subsets of I and does not evolve periodically although it is a homeomorphism too and thus maps open sets to open sets, proper subsets of I to proper subsets of I and does not change the cardinality of sets.

The map f_3 gradually pulls everything down to 0, but in a nondegenerative way: the image of I with respect to any iteration of f_3 is homeomorphic to I. Finally, the tent map f_4 is open and enlarge subsets of I in the sense that every subset of I with nonempty interior is mapped onto I after only finitely many iterations. We will discuss this in more detail in the second chapter.

On the other hand, there is no essential difference between the behavior of two constant maps $c_1 : x \mapsto c_1$, $c_2 : x \mapsto c_2$, $c_1, c_2 \in I$. Therefore we introduce a notion of equivalence between dynamical systems to compare and classify them.

Definition 1. *A semiconjugacy from $(Y, g) := (Y, \{g\})$ to (X, f) (or, briefly, from g to f) is a surjective map $\pi : Y \to X$ satisfying $f \circ \pi = \pi \circ g$. If there is a semiconjugacy π from g to f, then we say that (X, f) is a factor of (Y, g). A conjugacy is an invertible semiconjugacy.*

Note that conjugacy is an equivalence relation. Also note that for arbitrary map $\pi : I \to I$ and every $x \in I$ we have $c_1 = c_1(\pi(x))$ and $\pi(c_2(x)) = \pi(c_2)$. Thus for every $c_1, c_2 \in I$ the dynamical system (I, c_1) is a factor of (I, c_2) since there always exists an onto map $\pi : I \to I$ satisfying $\pi(c_2) = c_1$. If $c_1, c_2 \in \{0, 1\}$

or $c_1, c_2 \in (0, 1)$, then we can choose such π to be actually a bijection and thus in these cases the dynamical systems (I, c_1), (I, c_2) are conjugate.

Let us discuss one more example illustrating what conjugacy and semiconjugacy is. Let S^1 denote the plane circle $\{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$ and consider the dynamical system (S^1, R_α), where R_α is the rotation of S^1 by angle α. Thus repeated applying R_α to S^1 just keeps S^1 rotating, always by angle α. Again, intuitively we feel that the behaviour of the system $(S^1, k\alpha)$ is kind of included in the behaviour of the system (S^1, α) for every $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Thus there should be a semiconjugacy from (S^1, α) to $(S^1, k\alpha)$ and this is indeed true. It is justified by the continuous surjective map $\pi : S^1 \to S^1$ that wraps S^1 around itself k times and change the orientation if $k < 0$. Note that π is a bijection if and only if $k = \pm 1$ and thus in these cases we obtain a conjugacy. This also fits our intuition since rotation by α and rotation by $-\alpha$ behave the same way, they are mirror images of each other.

Let us just note without more details that under suitable parametrization it is possible to represented S^1 by $[0, 1)$, the rotation R_α by $x \mapsto (x + \alpha) \mod 1$ and π by $x \mapsto (k \cdot x) \mod 1$. This easily entails the required equality since for every $x \in [0, 1)$ it holds that $\pi(R_\alpha(x)) = k(x + \alpha) \mod 1 = kx + k\alpha \mod 1 = R_{k\alpha}(\pi(x))$.

Conjugacy and semiconjugacy are important tools that allow us to use known properties of well–behaved dynamical systems when analyzing other dynamical systems. Of course, analyzing well–behaved, canonical dynamical systems is useful in itself since in this manner we often obtain interesting properties worth further studying. For this reason we will next introduce probably the most fundamental dynamical system, but to do that, we need to recall some notation for the product of sets, topological spaces and metric spaces, respectively.

Notation. In this thesis we use the set–theoretic notion of natural numbers ω. That is, for us $0 = \emptyset \in \omega$ and $n + 1 = n \cup \{n\} \in \omega$ for every $n \in \omega$. Thus for every $n \in \omega$ also $n \subseteq \omega$. Whenever we do not need the inner structure of natural numbers, we use the symbol $\mathbb{N}$ to denote the set of positive natural numbers and $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

If S is any set, then by S^ω we understand the set of all functions from ω to S. If we write S^n for $n \in \omega$ we have in mind the set of functions from n to S, while if we write r^n for $r \in \mathbb{R}$ and $n \in \mathbb{N}$, we mean by it just the result of the standart arithmetic operation. We denote by $|\cdot|$ the cardinality of a set as well as an absolute value of a real number, but it should always be clear what we mean from the context. We denote by $S^{<\omega} := \bigcup\{S^n, n \in \omega\}$.

Now we are prepared to introduce the promised example. Let Λ be a finite set (with discrete topology) and consider the product space Λ^ω. This is a compact metric space, usually it is considered either together with the metric

$$d(x, y) := \sum_{n \in \omega} \frac{e(x(n), y(n))}{2^{-|n|}},$$

where e is a discrete metric on Λ, or with the product topology given by the basis containing the sets of the form $\{x \in \Lambda^\omega, x|_{dom(s)} = s\}$, for some $s \in S^{<\omega}$.

This space plays an important role in dynamics, since it is naturally endowed with a special kind of map called "the shift", more exactly "the shift to the left" and “the shift to the right”. The first one is defined by $L(s)(n) := s(n+1)$ and for the second one fix some $\lambda \in \Lambda$, then $R_\lambda(s)(n) := s(n-1)$ if $n \geq 1$ and $R_\lambda(s)(0) := \lambda$. It is easy to observe that these maps are continuous.

Although these maps are indeed canonical, from the topological point of view the “left” and “right” version of shift are opposites in the following way: while applying repeatedly the map L on any set of the basis we described above gives us the whole space Λ^ω after only finitely many steps, applying on the whole space any sequence of right shifts gives us a set as small as the applied sequence was long.

Let us discuss this in more detail. In the case of the left shift, every set with nonempty interior maps eventually onto the whole space. We can thus interpret L as a kind of a magnifier. But, we can also interpret this by saying that f is very chaotic, because arbitrarily close points can end after some time as far from each other as possible since arbitrary open set eventually maps onto the whole space. We will study maps with properties similar to these properties of the map L in more detail in the second chapter.

In the case of the right shifts, note first that they are contractions, and what is more, not just that since we do not have only inequality but equality:

$$d(R_\lambda(x),\, R_\lambda(y)) = 1/2 \cdot d(x,\, y);\ \lambda \in \Lambda$$

Thus every right shift shrinks the whole space but in a every regular way, i.e. there is no degeneration. Another sign of the fact that no essential information is lost can be express by

$$\Lambda^\omega = \bigcup_{\lambda \in \Lambda} R_\lambda(\Lambda^\omega);$$

that is, we are able to recover the original space from applying the right shifts. We will study sets of maps with properties similar to the above properties in the third chapter.

However to discuss the suggested problems we need some advenced theory of continua. Therefore before we start with the topics above, we first introduce some preliminary notions, definitions and known results.

1. Preliminaries

In this chapter we give a summary of some preliminary notions and results, most of them will be needed in the following chapters. Despite some of the presented notions and results for metric spaces can be generalised to topological spaces, we will only work with metric spaces.

In fact, we will restrict us even further and work mostly with continua.

Definition 2. *A continuum is a nonempty connected compact metric space.*

A singleton is a continuum and any other continuum is of size continuum. Another examples of continua are the closed interval I, the circle, the square or the Hilbert cube I^ω. There is a great book on the fundamentals of continuum theory containing also many other examples by S. Nadler, namely Nadler [1992].

Next, let us introduce some notation and state without proof one theorem from Nadler [1992] that will be used in many proofs throughout the thesis.

Notation. For a metric space X we denote by 2^X the set of all nonempty closed subsets of X.

Definition 3. *Let X, Y be metric spaces. We say that a function $F : X \to 2^Y$ is usc (upper semi–continuous) provided that for every $x \in X$ and every open set $V \subseteq Y$ such that $F(x) \subseteq V$ there exists an open set $U \subseteq X$, $x \in U$ such that $\bigcup\{F(z),\, z \in U\} \subseteq V$.*

Theorem 4 (General Mapping Theorem)**.** *[Nadler, 1992, Theorem 7.4] Let X, Y be nonempty compact metric spaces and $F_n : X \to 2^Y$, $n \in \mathbb{N}$, be usc functions. Assume that:*

1. *for all $n \in \mathbb{N}$ and $x \in X$ it holds $F_{n+1}(x) \subseteq F_n(x)$,*
2. *for every $x \in X$ it holds that $\lim_{n\to\infty} diam(F_n(x)) = 0$,*
3. *for every $n \in \mathbb{N}$ it holds that $\bigcup\{F_n(x),\, x \in X\} = Y$.*

Then we can define a map $f : X \to Y$ by $\{f(x)\} = \bigcap\{F_n(x),\, n \in \mathbb{N}\}$, for every $x \in X$, and this map f is continuous and onto.

1.1 Connectedness and the Cantor set

Next we will give just a short summary firstly of types of connectedness and secondly of some properties of the well–known Cantor set.

Let us recall that if X is a metric space, then a path in X is a continuous map $f : I \to X$, respectively its image, and an arc is a continuous one–to–one map $f : I \to X$, respectively its image. If $f(0) = x$ and $f(1) = y$, then we say that f is path, respectively an arc, from x to y. We say that X (or a subset of X) is path connected, if for every $x, y \in X$ there exists a path in X from x to y; and that X (or a subset of X) is arcwise connected, if for every $x, y \in X$ there exists an arc in X from x to y.

Connected sets

Next we will recall some well known facts about maximal connected, path connected and arcwise connected subsets of metric spaces.

Connected components

Let X be a metric space. For every $x \in X$ there exists the largest connected set containing x, namely the union of all connected subsets of X that contains x. This set is called the connected component of X. Since a closure of a connected set is connected, connected components are always closed. Further if $y \in X$, then the connected component of x and the connected component of y are either equal or disjoint since the union of intersecting connected sets is connected. Thus connected components form a partition of X. Note that X is connected if and only if it consists of a single connected component.

Path components

Let X be a metric space. Define on X a relation $\sim$ by letting $x \sim y$, x, $y \in X$, if and only if there exists a path in X from x to y. It is easy to see that $\sim$ is an equivalence. The classes of this equivalence are maximal path connected subsets of X, we call them path components. Note that if $x \sim y$, then the corresponding path $f : I \to X$ from x to y gives us the connected set $f(I) \ni x, y$ and thus x and y lie in the same connected component. Therefore every connected component is a union of path components. X is path connected if and only if it consists of a single path component.

Arcwise connected components

Let X be a metric space and analogously to the previous case, define on X a relation $\sim'$ by letting $x \sim' y$, x, $y \in X$, if and only if $x = y$ or there exists an arc in X from x to y. Then $\sim'$ is an equivalence, even though the proof of the transitivity of $\sim'$ is more challenging than it was in the case of $\sim$. Moreover since X is metric, there holds $\sim'=\sim$. Thus the classes of the equivalence $\sim'$ will be called path components too,they are exactly the maximal arcwise connected sets.

Note that even if X was not metrizable, $x \sim' y$ immediately entails that $x \sim y$ since every arc is a path. Therefore every path connected component is a union of arcwise connected components in any topological space. Note that again a metric space X is arcwise connected if and only if it consists of a single path connected component.

The Cantor set

In contrast to previous part where we focused on ”very connected” sets, the Cantor set $(2^\omega, |\cdot|)$ or for brevity just 2^ω, where

$$|s - t| = \sum_{n=0}^{\infty} \frac{||s(n)| - |t(n)||}{2^n},$$

is an example of a ”very disconnected” set. Recall that we understand the elements of 2^ω to be functions from ω to $2 = \{0, 1\}$ and thus it makes sense to write

$s(n)$ for every $s \in 2^\omega$ and $n \in \omega$. Moreover, it makes even sense to write $s|_n$ since every $n \in \omega$ also satisfies $n \subseteq \omega$.

Recall that the set $\{\{t \in 2^\omega,\ t|_n = s\};\ s \in 2^n,\ n \in \omega\}$ is a basis of the topology of $(2^\omega, |\cdot|)$, as in every product of discrete spaces. Note that $2^\omega \setminus \{t \in 2^\omega,\ t|_n = s\} = \bigcup\{\{t \in 2^\omega,\ t|_n = r\};\ r \in 2^n \setminus \{s\}\}$ and thus the basis above consists of clopen sets. Thus, in particular, 2^ω is not connected and hence not a continuum.

Let $x \neq y \in 2^\omega$, then there exists $n \in \omega$ such that $x|_n \neq y|_n$ and thus $x \in \{t \in 2^\omega,\ t|_n = x|_n\}$, $y \in \{t \in 2^\omega,\ t|_n = y|_n\}$ are open disjoint sets. This shows that the only connected subsets of the Cantor set are the singletons and thus the singletons are exactly the connected components of 2^ω. Note that singletons are not open in 2^ω and thus the connected components do not have to be open in general. Let us also note that the two conditions, i.e. having a basis consisting of clopen sets and all connected components being singletons, are for compact spaces equivalent to having the topological dimension equal to 0.

Define a map $\varphi : 2^\omega \to I$ by

$$\varphi(s) := \sum_{n=0}^{\infty} \frac{2 \cdot |s(n)|}{3^{n+1}}.$$

It is a well known fact that φ is a homeomorphism onto its image. Let us just note that the proof of this fact is just unfolding the definitions, so we will not present it here.

1.2 Peano continua

In this section we introduce Peano continua, an important class of continua. The entire chapter is based on the eight chapter of Nadler [1992]. Although we cover here most of that chapter, we have reformulated some statements and in those cases we will not refer to concrete theorems proven in Nadler [1992]. We also directly restrict ourself to the case of continua although it makes sense to consider the case of general topological and metric spaces too.

Definition 5. *We say that a continuum X is locally connected, if the set of all connected open subsets of X forms a basis of topology of X. Since X is compact, this is the case if and only if for every $\varepsilon > 0$ there exists a finite cover of X by open connected sets with diameter less than ε. A locally connected continuum is called Peano continuum.*

Definition 6. *We say that a metric space X has the property S if for every $\varepsilon > 0$ there exists a finite cover of X by connected sets (or equivalently, by closed connected sets, since the closure of a connected is connected) with diameter less than ε.*

Definition 7. *Let X be a set, a sequence $A_1, \dots, A_n$ of subsets of X is called a chain if for every $1 \leq i \leq n-1$ it holds that $A_i \cap A_{i+1} \neq \emptyset$.*

Remark. In fact, what we call here "a chain" is usually called "a weak chain", and by "chain" is understood a sequence $A_1, \dots, A_n$ of subsets of X satisfying that for every $1 \leq i, j \leq n$ it holds that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Nevertheless, we will not follow this usage since we will never need the notion of a chain in the whole thesis.

Theorem 8. *Let (X, d) be a metric space. Then the following conditions are equivalent:*

1. *X is a Peano continuum,*
2. *X is a continuum with property S,*
3. *X is a continuum and for every $\varepsilon > 0$ there exists a finite cover of X by Peano subcontinua of X with diameter less than ε,*
4. *for every $\varepsilon > 0$ and $x, y \in X$ there exists a chain $A_1, \ldots, A_n$ consisting of Peano subcontinua with diameter less than ε satisfying $x \in A_1$, $y \in A_n$ and $A_1 \cup A_2 \cup \cdots \cup A_n = X$,*
5. *there exists a continuous onto map $f : I \to X$.*

Proof. "1 ⇒ 2" is trivial,

"2 ⇒ 1": let $x \in X$ and $\varepsilon > 0$, then by 2 there exist connected $A_1, A_2, \ldots A_n$, $n \in \mathbb{N}$, covering X, each of diameter less then $\varepsilon/2$. Notice that their closures $\overline{A_1}, \ldots, \overline{A_n}$ are closed connected sets covering X, each of diameter less then $\varepsilon/2$. The set $\bigcup\{\overline{A_i};\, x \in \overline{A_i},\, 1 \leq i \leq n\}$ is connected (not necessarily open) neighbourhood of x since it contains the complement of the closed set $\bigcup\{\overline{A_i};\, x \notin \overline{A_i},\, 1 \leq i \leq n\} \not\ni x$ and moreover its diameter is less than ε. This proves that for every $x \in X$ and any neighbourhood of x there exists a smaller connected neighborhood of x.

To prove the claim, let again $x \in X$ and $\varepsilon > 0$ be arbitrary and consider C the connected component of x in $B(x, \varepsilon)$. We want to prove that C is open. Let $y \in C$, then since $y \in B(x, \varepsilon)$, there exists $A \subseteq B(x, \varepsilon)$ a connected neighborhood of y by the previous part. Then $C \cup A \subseteq B(x, \varepsilon)$ is a connected set containing x and thus by maximality of C we obtain $A \subseteq C$, in particular there exists U open satisfying $y \in U \subseteq A \subseteq C$ and thus C is open. Since $x \in X$ and $\varepsilon > 0$ were arbitrary, this shows that connected open sets form a basis of topology of X and therefore X is a Peano continuum.

"2 ⇒ 3": this is probably the most technical part of the proof. Let us introduce some notation. Whenever $\varepsilon > 0$, we say that a chain $L_1, \ldots, L_n$, $n \in \mathbb{N}$, is a $S(\varepsilon)$–chain if it is formed by connected sets and it holds that $diam(L_i) < \varepsilon \cdot 2^{-i}$ for every $1 \leq i \leq n$. If moreover $x \in L_1$, $y \in L_n$, then we say that $L_1, \ldots, L_n$ is an $S(\varepsilon)$–chain from x to y. For any $A \subseteq X$ we define $S(A, \varepsilon)$ by

$$S(A, \varepsilon) := \{x \in X;\ \text{there exists an } S(\varepsilon)\text{-chain from some point of } A \text{ to } x\}.$$

Let $\varepsilon > 0$, by 2 there exists $\mathscr{F}$ a finite cover of X by connected sets with diameters less than $\varepsilon/3$ and we may suppose that $\mathscr{F}$ does not contain the emptyset. Then $\{\overline{S(F, \varepsilon/3)};\, F \in \mathscr{F}\}$ is a finite cover of X by nonempty sets since $A \subseteq S(A, \delta)$ for every $A \subseteq X$ and every $\delta > 0$; for arbitrary $a \in A$ consider the $S(\delta)$–chain $L_1 = \{a\}$.

Let $F \in \mathscr{F}$ and we will show that $\overline{S(F, \varepsilon/3)}$ is a Peano continuum of diameter less than ε. Clearly it is closed and thus compact since X is compact. To show connectedness it suffices to prove that $S(F, \varepsilon/3)$ is connected since a closure of a connected set is connected. Let $x, y \in S(F, \varepsilon/3)$, then there exists chains

$L_1, \dots L_n$ and $K_1, \dots, K_m$ such that $L_1 \cap F \neq \emptyset$, $K_1 \cap F \neq \emptyset$, $x \in L_n$, $y \in K_m$. Then

$$L_n \cup \cdots \cup L_2 \cup L_1 \cup F \cup K_1 \cup K_2 \cup \cdots \cup K_m$$

is a connected set containing both x and y. Thus all points of $S(F, \varepsilon/3)$ lie in a single connected component which means that the set is connected. This shows that $\overline{S(F, \varepsilon/3)}$ is a continuum.

Similarly, to show that $diam(\overline{S(F, \varepsilon/3)})$ is less than ε it suffices to prove that $diam(S(F, \varepsilon/3))$ is less than ε. Let $x, y \in S(F, \varepsilon/3)$, then there exists $S(\varepsilon/3)$–chains $L_1, \dots L_n$ and $K_1, \dots, K_m$ such that $L_1 \cap F \neq \emptyset$, $K_1 \cap F \neq \emptyset$, $x \in L_n$, $y \in K_m$. For every $2 \leq i \leq n$ and $2 \leq j \leq m$ fix some $l_i \in L_{i-1} \cap L_i$ and $k_j \in K_{j-1} \cap L_j$. Further, fix some $l_1 \in L_1 \cap F$ and $k_1 \in K \cap F$. Then

$$\begin{aligned} d(x, y) \leq d(x, l_n) + \sum_{i=1}^{n-1} d(l_{n+1}, l_n) + d(l_1, k_1) + \sum_{j=1}^{m-1} d(k_j, k_{j+1}) + d(k_m, y) \leq \\ \leq \sum_{i=1}^{n} diam(L_i) + diam(F) + \sum_{j=1}^{m} diam(K_j) \leq \\ \leq \sum_{i=1}^{n} \varepsilon/3 \cdot 2^{-i} + \varepsilon/3 + \sum_{j=1}^{m} \varepsilon/3 \cdot 2^{-j} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Thus we proved that $diam(S(F, \varepsilon/3)) \leq \varepsilon$.

To complete the proof that $\overline{S(F, \varepsilon/3)}$ is a Peano continuum, we will show that $\overline{S(F, \varepsilon/3)}$ has the property S and then the implication 2⇒1 we have already proved will give us the result. Let $\delta > 0$, we first want to find connected sets $C_1, \dots, C_n$, each of diameter less than δ, such that $S(F, \varepsilon/3) = C_1 \cup \cdots \cup C_n$.

Fix some $k \in \mathbb{N}$ satisfying

$$\sum_{i=k}^{\infty} \varepsilon/3 \cdot 2^{-i} = \varepsilon/3 \cdot 2^{-k+1} < \delta/3$$

and let $K \subseteq (F, \varepsilon/3)$ be the set of those points $x \in S(F, \varepsilon/3)$ such that there exists an $S(\varepsilon/3)$–chain from some point of F to x consisting of at most k members. By 2 there exists $\mathscr{H}$ a finite cover of X by connected sets with diameter less than $\varepsilon/3 \cdot 2^{-k-1}$. Let $D_1, \dots, D_n$ be a list of those members of $\mathscr{H}$ that intersects K. Then $K \subseteq D_1 \cup \cdots \cup D_n$ and thus, in particular, $\emptyset \neq F \subseteq K \subseteq D_1 \cup \cdots \cup D_n$ implies that there is at least one set in $\mathscr{H}$ intersecting K.

Fix some $1 \leq i \leq n$. Then there exists some $x \in K \cap D_i$ and thus there is, by the definiton of K, an $S(\varepsilon/3)$–chain $L_1, \dots, L_t$, $t \leq k$, such that $L_1 \cap F \neq \emptyset$ and $x \in L_t$. Thus, since $diam(D_i)$ is less than $\varepsilon/3 \cdot 2^{-k-1} \leq \varepsilon/3 \cdot 2^{-t-1}$, we obtain an $S(\varepsilon/3)$–chain $L_1, \dots, L_t, D_i$ and therefore $D_i \subseteq S(F, \varepsilon/3)$. Let

$$\mathscr{C}_i := \{C \subseteq S(F, \varepsilon/3);\, C \cap D_i \neq \emptyset,\, diam(C) < \delta/3,\, C \text{ is connected}\}$$

and $C_i := \bigcup \mathscr{C}_i$. Note we have already proved that $D_i \subseteq S(F, \varepsilon/3)$, $D_i \cap D_i \neq \emptyset$ since D_i intersect K and thus it is nonempty, $diam(D_i) < \varepsilon/3 \cdot 2^{-k-1} < \delta/3$ and D_i is connected. Thus $D_i \in \mathscr{C}_i$ and therefore $D_i \subseteq C_i$.

Let $x, y \in C_i$, then there exist connected C, C' satisfying $x \in C$, $y \in C'$, $C \cap D_i \neq \emptyset \neq C' \cap D_i$ with diameters less $\delta/3$. Then $x, y \in C \cup D_i \cup C'$ is connected since D_i is connected and therefore x and y lie in the same connected component of C_i. It also follows that $d(x,\, y) \leq diam(C)+diam(D_i)+diam(C') < \delta/3 + \delta/3 + \delta/3 = \delta$. Since $x,\, y$ were arbitrary, this entails that $diam(C_i) \leq \delta$ and further C_i is connected because all points of C_i lie in a single connected component.

We claim that $S(F,\, \varepsilon/3) = C_1 \cup \cdots \cup C_n$. The inclusion $C_1 \cup \cdots \cup C_n \subseteq S(F,\, \varepsilon/3)$ follows immediately from the definition of $\mathscr{C}_1, \ldots \mathscr{C}_n$. To prove the reverse inclusion, first note that we already know that $K \subseteq D_1 \cup \cdots \cup D_n \subseteq C_1 \cup \cdots \cup C_n$. Let $x \in S(F,\, \varepsilon/3)$, $x \notin K$, and we will show that $x \in C_1 \cup \cdots \cup C_n$.

By the definition, there exists an $S(\varepsilon/3)$–chain $L_1, \ldots, L_t$ from some point of F to x and since $x \notin K$, there must be $t > k$. By the definition again, $\emptyset \neq L_k \subseteq K$, and since $K \subseteq D_1 \cup \cdots \cup D_n$, there exists $1 \leq i \leq n$ such that $L_k \cap D_i \neq \emptyset$.

We will show that $L := L_k \cup L_{k+1} \cup \cdots \cup L_t \subseteq C_i$. Firstly $L \subseteq S(F,\, \varepsilon/3)$ just by the definition, secondly it is connected by the definition of a chain and thirdly $L \cap D_i \neq \emptyset$ since $L_k \cap D_i \neq \emptyset$. Fourthly, using an argument analogical to the one in the proof of $diam(S(F,\, \varepsilon/3)) \leq \varepsilon$, it follows that

$$diam(L) \leq diam(L_k) + diam(L_{k+1}) + \cdots + diam(L_t) \leq \sum_{j=k}^{t} \varepsilon/3 \cdot 2^{-j} < \delta/3.$$

Thus $L \in \mathscr{C}_i$, hence $L \subseteq C_i$ and in particular $x \in L_t \subseteq L \subseteq C_i$. This concludes the proof that $C_1, \ldots, C_n$ are connected sets, each of diameter less than δ, and $S(F,\, \varepsilon/3) = C_1 \cup \cdots \cup C_n$.

Finally, note that $\overline{C_1}, \ldots, \overline{C_n}$ are connected sets with diameter less than δ too. Moreover

$$\overline{S(F,\, \varepsilon/3)} = \overline{C_1 \cup \cdots \cup C_n} = \overline{C_1} \cup \cdots \cup \overline{C_n},$$

thus $\overline{S(F,\, \varepsilon/3)}$ is indeed a continuum with the property S and thus by 2⇒1 we obtain that $\overline{S(F,\, \varepsilon/3)}$ is a Peano continuum.

3⇒4: let $x,\, y \in X$ and $\varepsilon > 0$. By 3 there exists $\mathscr{F}$ a finite cover of X by Peano subcontinua of X with diameters less than ε and we may suppose that $\emptyset \notin \mathscr{F}$. We will find a chain $A_1, \ldots, A_n$ satisfying $x \in A_1$ and $\{A_1, \ldots, A_n\} = \mathscr{F}$ by induction on $|\{A_1, \ldots, A_n\}|$. Let $A_1 \in \mathscr{F}$ satisfy $x \in A_1$.

Further, suppose we have constructed a chain $A_1, \ldots, A_k$ for some $k \geq 1$ such that $x \in A_1$ and $A_1, \ldots, A_k \in \mathscr{F}$. If $\mathscr{F} \setminus \{A_1, \ldots, A_n\} \neq \emptyset$, then $\bigcup(\mathscr{F} \setminus \{A_1, \ldots, A_n\})$ and $\bigcup\{A_1, \ldots, A_n\}$ are nonempty closed subsets whose union is $\bigcup \mathscr{F} = X$. Since by 3 X is a continuum and thus connected, the sets $\bigcup(\mathscr{F} \setminus \{A_1, \ldots, A_n\})$ and $\bigcup\{A_1, \ldots, A_n\}$ are not disjoint.

Hence there exists $1 \leq i \leq k$ and $A_{2k-i+1} \in \mathscr{F} \setminus \{A_1, \ldots, A_n\}$ such that $A_i \cap A_{2k-i+1} \neq \emptyset$. Then $A_1, \ldots, A_k, A_{k+1} := A_{k-1}, A_{k+2} := A_{k-2}, \ldots, A_{2k-i} := A_i, A_{2k-i+1}$ is a chain satisfying $x \in A_1$, $A_1, \ldots, A_{2k-i+1} \in \mathscr{F}$ and moreover $|\{A_1, \ldots, A_k\}| < |\{A_1, \ldots, A_{2k-i+1}\}|$. This proves the existence of a chain $A_1, \ldots, A_n$ satisfying $x \in A_1$ and $\{A_1, \ldots, A_n\} = \mathscr{F}$. Analogically there exists a chain $B_1, \ldots B_m$ satisfying $y \in B_1$ and $\{B_1, \ldots, B_m\} = \mathscr{F}$.

In particular, since $\{B_1, \ldots, B_m\} = \mathscr{F} = \{A_1, \ldots, A_n\}$, there exists $1 \leq i \leq m$ such that $A_n = B_i$. Then the chain

$$A_1, \ldots, A_{n-1}, A_n = B_i, B_{i-1}, B_{i-2}, \ldots, B_1$$

has the desired properties.

4⇒5: we will use the General Mapping Theorem 4. By 4 there exists a chain $A^1_1, \ldots, A^1_{k_1}$ consisting of Peano subcontinua with diameters less than 1/2 such that $A^1_1 \cup A^1_2 \cup \cdots \cup A^1_{k_1} = X$. Let $n \in \mathbb{N}$ and suppose that we have already found a chain $A^n_1, \ldots, A^n_{k_n}$ consisting of Peano subcontinua with diameters less than 2^{-n} such that $A^n_1 \cup A^n_2 \cup \cdots \cup A^n_{k_n} = X$. For every $1 \leq i \leq k_n - 1$ there is some $x_i \in A^n_i \cap A^n_{i+1}$ by the definition of a chain.

Using the implication 1⇒4 that we have already proved we obtain for each Peano continuum A^n_i a chain $B^i_1, \ldots, B^i_{m_i}$ such $\{B^i_1, \ldots, B^i_{m_i}\} = A^n_i$ and $x_1 \in B^1_{m_1}$, $x_{k_n-1} \in B^{k_n}_1$ and $x_{i-1} \in B^i_1$, $x_i \in B^i_{m_i}$ for every $2 \leq i \leq k_n - 1$. We may assume that $m_1 = m_2 = \cdots = m_{k_n}$, because we can prolong the shorter chains by repeating some of its members if necessary.

Define functions F_n, $n \in \mathbb{N}$, from I to the space of subcontinua of X as follows:

$$F_n(x) := \begin{cases} A^n_1, & x = 0, \\ A^n_i, & (i-1)/k_n < x < i/k_n,\ i = 1, 2, \ldots, k_n \\ A^n_i \cup A^n_{i+1}, & x = i/k_n,\ i = 1, 2, \ldots, k_{n-1} \\ A^n_{k_n}, & x = 1. \end{cases}$$

By the construction, for all $n \in \mathbb{N}$ and $x \in X$ it holds $F_{n+1}(x) \subseteq F_n(x)$, for every $x \in X$ it holds that $\lim\limits_{n\to\infty} diam(F_n(x)) = 0$ (use that $diam(A^n_i \cup A^n_{i+1}) \leq diam(A^n_i) + diam(A^n_{i+1})$ since these sets always intersect by the definition of a chain) and for every $n \in \mathbb{N}$ it holds that $\bigcup\{F_n(x),\ x \in X\} = Y$.

Further, note then for every $n \in \mathbb{N}$ the function F_n is usc. This is because for every $x \in I \setminus \{i/k_n,\ i = 1, 2, \ldots, k_{n-1}\}$ the function F_n is constant on some (relatively) open interval containing x and for every $i = 1, 2, \ldots, k_{n-1}$ it holds that $F_n(y) \subseteq F_n(i/k_n)$ for every $y \in ((i-1)/k_n, (i+1)/k_n)$. Thus the hypotheses of the General Mapping Theorem 4 are satisfied and we obtain a continuous onto map $f : I \to X$.

5⇒2: by 5 there exists a continuous onto map $f : I \to X$. Thus since I is nonempty, connected and compact, its continuous image $f(I) = X$ is nonempty, connected and compact; that is, a continuum.

Note that since I is compact, the map f is not just continuous but uniformly continuous. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that for every $x, y \in I$, $|x - y| \leq \delta$, it holds that $d(x, y) \leq \varepsilon$. Let $\mathscr{I}$ be a finite set of subintervals of I of length less than δ such that $\bigcup \mathscr{I} = I$. Then $\{f(D),\ D \in \mathscr{I}\}$ is a finite cover of X, since f is onto, and it is formed by sets of diameter less than ε.

□

In particular, we immediately obtain:

Corollary 9. *Every Peano continuum is path–connected.*

Proof. By the previous theorem 8 there exists an onto map $f : I \to X$. Let $x, y \in X$, then there exists $s, t \in I$ such that $f(s) = x$, $f(t) = y$. We may suppose that $s \leq t$. Let $g : I \to [s, t]$ be continuous, onto and satisfying $g(0) = s$, $g(1) = t$, then $f \circ g : I \to X$ is a path from x to y in X. □

As we stated in the previous section, path components are already arcwise connected in metric spaces. However, the hard part of the proof of this fact is to prove that every Peano continuum is arcwise connected. Neverless, Theorem 8 together with the fact that Peano continua are arcwise connected entails easily that Peano continua are locally arcwise connected. Anyway, we will state this fact without a proof.

Theorem 10. *[Nadler, 1992, Theorem 8.25] Every Peano continuum is locally arcwise connected, that is, for every Peano contiuum X, every $x \in X$ and N a neighborhood of x there exists $M \subseteq N$ an arcwise connected neighborhooh of x.*

Finally using this fact, we may easily show that in Peano continua not just path components are arcwise connected, but even connected components are arcwise connected, and what is more, this does not hold only in Peano continua, but also in all open subsets of Peano continua.

Corollary 11. *In every Peano continuum, connected and arcwise connected components of open sets coincide.*

Proof. Let X be a Peano continuum, $U \subseteq X$ open and C an arcwise connected component of U and we only need to prove that C is open. Let $x \in C$, by the previous theorem there exists $V \subseteq U$ an arcwise connected neighborhood of x. Then $V \cup C \subseteq U$ is arcwise connected and thus $V \subseteq C$. □

Observation 12. *Let (X, d) be a continuum, Y a Peano continuum, $a \neq b \in X$ and $c, d \in Y$. Then there exists $f : X \to Y$ continuous, onto and satisfying $f(a) = c$, $f(b) = d$.*

Proof. Define $h : X \to [0, 2d(a, b)]$ by

$$h(x) := \min\{d(a, b); d(a, x)\} + \max\{0; d(a, b) - d(x, b)\}$$

for $x \in X$. It is easy to observe that h is well–defined, continuous, $h(a) = 0$ and $h(b) = 2d(a, b)$. Hence, by connectedness of X, the map h is onto. Since Y is Peano, there exists $g : [d(a, b)/2; d(a, b)] \to Y$ continuous and onto. We can extend this map to a map defined on the whole interval $[0; 2d(a, b)]$ by letting $g|_{[0, d(a,b)/2]}$ be any arc from c to $g(d(a, b)/2)$ and $g|_{[d(a,b); 2d(a,b)]}$ be any arc from $g(d(a, b))$ to d. Then $f := g \circ h : X \to Y$ has the required properties.

Note that we know that Y is arcwise connected, since it is an open subset of a Peano continuum Y and it is connected. □

2. Transitive maps

After preliminaries, we are prepared to follow up on the Introduction in the study of dynamical systems. As was mentioned there, in this chapter we will describe some “inflating” maps, and at the same time we will investigate which properties should have something called “chaos”.

In the first section we concentrate on unfolding the definitions, while in the second section we generalize a result of Agronsky and Ceder [1991/92] by showing that for every Peano continuum there exists a LEO self–map of that Peano space with dense set of periodic points.

2.1 The features of chaos

Let us present here the most widely known and accepted definition of chaos introduced in Devaney [1986].

Definition 13 (Devaney’s definition of chaos)**.** *Let (X, d) be a metric space. A function $f : X \to X$ is called chaotic if it satisfies the following three conditions:*

D1 f is topologically transitive, that is, for any two nonempty open sets U and V, there exists k such that $f^k(U) \cap V \neq \emptyset$,

D2 The set of periodic points of f is dense. A point x is called periodic if $f^k(x) = x$ for some $k \geq 1$.

D3 f has sensitive dependence on initial conditions, that is, there exists $\delta > 0$ such that for any open set U and for any point $x \in U$, there exists a point $y \in U$ such that $d(f^k(x), f^k(y)) > \delta$ for some k. The positive number δ is called a sensitivity constant, it only depends on the space X and the function f.

Let us start with the sensitivity condition. It says that regardless how close are two points to each other, after some number of iteration they will be at least δ apart. Thus arbitrarily small error results in a non-negligible inaccuracy. This is certainly a feature of chaos and thus there is no wonder that the notion of sensitive dependence of initial conditions is understood to be the central idea in chaos.

In contrast to sensitivity, the other two conditions are not such a typical feature of chaos by themself. For example, the set of periodic points of the identity (on any space) is of course dense since in this case all points are periodic, but the identity is definitely not chaotic. For transitivity, let us recall the plane circle S^1 we have defined in the Introduction. If $\alpha \in \mathbb{R}$ satisfies that α/π is irrational, then the rotation of S^1 by angle α is transitive, while rotation is definitely not chaotic map.

Nevertheless, the situation is completely different when we consider the conditions D1 and D2 together.

In Silverman [1992] it was shown that if X is a continuum, then $f : X \to X$ is transitive if and only if there exists a transitive point, i.e. a point $x \in X$ such that the set $\{x, f(x), f^2(x), \dots\}$ is dense in X. Note that transitive point might be understood to be a kind of the opposite of a periodic point as it systematically travels through and through the space X, while a periodic point only visits a very small part of X — just finite. Also note that if x is a transitive point, then all the points $f(x)$, $f^2(x)$, $\dots$ are transitive and therefore the set of transitive points is dense in X.

Therefore periodic points of a chaotic self–map of a continuum form a dense subset as well as transitive points of that map form a dense subset. This after all sounds like real chaos, therefore it maybe will not be so surprising that the first two conditions imply the third as it was proved in Banks et al. [1992].

When considering self–maps of the interval I, even more is true since it was proved in Vellekoop and Berglund [1994] that in this very special case, the transitivity itself implies that the set of periodic points is dense and thus it implies also the sensitivity condition.

Hence, perhaps surprisingly, transitivity is an important feature of chaos too and therefore it makes sense to try to strengthen the transitivity condition. Natural generalization of transitive maps are topological mixing and LEO functions.

We say that a map $f : X \to X$ is a topological mixing if for every nonempty open U, $V \subseteq X$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ the intersection $f^n(U) \cap V$ is nonempty.

Definition 14. *Let X be a topological space. We say that a continuous map $f : X \to X$ is LEO (locally eventually onto), if for every nonempty open $U \subseteq X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$.*

It follows immediately from the definitions that every LEO map is a topological mixing and every topological mixing is topologically transitive. Also note that every LEO map has sensitive dependence on initial conditions with sensitivity constant $diam(X)/3$ since for any U open and $x \in U$, there exists $n \in \mathbb{N}$ such that $f^n(U) = X$. Let $z \in X$ be any point such that $d(f^n(x), z) > diam(X)/3$, then there is $y \in U$ such that $f^n(y) = z$.

Recall two maps we defined in the Introduction; the tent map f_4 and the left shift. Both these maps are LEO and both these maps has a dense set of periodic points. To realize that this is true in the case of the left shift, consider any open set of the form $\{x \in \Lambda^\omega,\, x|_{dom(s)} = s\}$, for some $s \in S^{<\omega}$. Applying to this set the left shift $|dom(s)|$–times yields the whole space, and if we define $x \in \Lambda^\omega$ by $x(m) := s(m \mod n)$, we obtain a periodic point lying inside the above set.

Consider any subset of I with nonempty interior; it must contain some nondegenerate interval. Note that when applying the tent map to some interval, we obtain either an interval twice as large or an inerval containing either 0, 1 or 1/2. Since I is bounded, after finitely many steps the second option must occur. Further interval containing 1/2 maps to an interval containing 1 and that maps to an interval containing 0.

Thus after finitely many steps we obtain an interval $[0, \varepsilon]$ for some $\varepsilon > 0$. But $f_4([0, \varepsilon]) = [0, \min\{2\varepsilon, 1\}]$. Thus after another finitely many iterations we finally obtain the whole space I. Thus also the set of periodic points of he tent map is dense by Vellekoop and Berglund [1994].

2.2 Existence of LEO maps on Peano continua

In the previous section we described two LEO maps, both with dense set of periodic points. Our goal now is to provide such a map in much more general setting. In Agronsky and Ceder [1991/92] authors constructed a LEO self–map on general Peano continuum contained in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We will generalize this result to all Peano continua and further we will construct LEO maps with dense set of periodic points.

Lemma 15. *Let X be a nondegenerate Peano continuum. Then there exist finite coverings $\mathscr{F}_n$, $n \in \mathbb{N}$, of X and a continuous map $f : X \to I$ satisfying:*

1. *For every n and every $F \in \mathscr{F}_n$, F is a Peano subcontinnum of X of diameter less than 2^{-n},*

2. *For every $n \in \mathbb{N}$ and every $H \in \mathscr{F}_n$, there is some $\mathscr{F} \subseteq \mathscr{F}_{n+1}$ such that $H = \bigcup \mathscr{F}$,*

3. *For every $n \in \mathbb{N}$ and every $F \in \mathscr{F}_n$ it holds that $|f(F)| > 1$.*

Proof. By Theorem 8 there exists $\mathscr{F}_1$ a finite cover of X by Peano subcontinua of diameter less than $1/2$. We may assume that every continuum in $\mathscr{F}_1$ is nondegenerate since X is nondegenerate. Suppose we have already defined $\mathscr{F}_n = \{F_1, \dots, F_k\}$ for some $n \in \mathbb{N}$. Then for every $1 \le i \le k$ there exists by Theorem 8 a finite covering of F_i by Peano continua of diameter less than 2^{-n-1}, say $\mathscr{F}_{n+1,i}$. Again, we may assume that all these continua are nondegenerate. Let $\mathscr{F}_{n+1} := \mathscr{F}_{n+1,1} \cup \dots \cup \mathscr{F}_{n+1,k}$.

It is clear that 1 and 2 are satisfied. It is also clear that the set $\bigcup\{\mathscr{F}_n,\, n \in \mathbb{N}\}$ is countable, so we may write $\bigcup\{\mathscr{F}_n,\, n \in \mathbb{N}\} = \{F_i,\, i \in \mathbb{N}\}$.

Next, we will define inductively a uniformly converging sequence of continuous functions $(f_i : X \to I)_{i\in\mathbb{N}}$ and $S_1 \subseteq S_2 \subseteq \dots$ finite subsets of X such that for all $i \in \mathbb{N}$ it holds that $|f_i(S_i \cap F_i)| > 1$ and $f_{i+1}|_{S_i} = f_i|_{S_i}$.

Since F_1 is nondenegenerate, we can find $x \neq y \in F_1$ and an open set G such that $x \in G$, $y \notin G$. Let $S_1 := \{x, y\}$ and $f_1(z) := d(z, X \setminus G)$ where d is a compatible metric on X. Suppose we have already defined S_i, f_i for some $i \in \mathbb{N}$. If $|f_i(F_{i+1})| > 1$ then we can choose some $x, y \in F_{i+1}$ such that $f_i(x) \neq f_i(y)$ and set $f_{i+1} := f_i$, $S_{i+1} := S_i \cup \{x, y\}$.

Assume that $|f_i(F_{i+1})| = 1$. Choose some $x \in F_{i+1}$, then since F_{i+1} is nondegenerate and hence of size continuum, we can find some $y \in F_{i+1}$, $y \notin S_i \cup \{x\}$. There exists an open set G such that $y \in G$ and $G \cap (S_i \cup \{x\}) = \emptyset$. Let $S_{i+1} := S_i \cup \{x, y\}$ and $f_{i+1}(z) := f_i(z) + \min\{2^{-i}, d(z, X \setminus G)\}$.

Now it is clear that the functions f_i, $i \in \mathbb{N}$, are continuous and uniformly converge to some function $f : X \to I$. Then f is continuous and for every $i \in \mathbb{N}$ we have

$$|f(F_i)| \ge |f(F_i \cap S_i)| = |f_i(F_i \cap S_i)| > 1.$$

□

Observation 16. *Let (X, d) be a nondegenerate Peano continuum and let $\mathscr{F}_n$, $n \in \mathbb{N}$, and $f : X \to I$ be as in Lemma 15. Then for every open $\emptyset \neq U \subseteq X$ the set $f(U)$ contains a nondegenerate interval.*

Proof. Let $x \in X$ and $\epsilon > 0$ satisfy $B(x, \epsilon) \subseteq U$. Let $n \in \mathbb{N}$ satisfy $2^{-n} < \epsilon$. Then there exists $F \in \mathscr{F}_n$ such that $x \in F$. Further, $F \subseteq B(x, \epsilon) \subseteq U$, since the diameter of F is less then $2^{-n} < \epsilon$. We also know that $|f(F)| > 1$ and since F is connected, $f(F) \subseteq f(U)$ is a nondegenerate interval. □

Observation 17. *Let (X, d) be a compact metric space. Then there exist $(x_n, \epsilon_n) \in X \times (0, 1)$; $n \in \mathbb{N}$, such that any y_n, $n \in \mathbb{N}$, satisfying $d(x_n, y_n) \leq \epsilon_n$ for all $n \in \mathbb{N}$, form a dense subset of X.*

Proof. For given $n \in \mathbb{N}$, let D_n be a finite 2^{-n-1}–net in X. Let

$$D := \bigcup_{n \in \mathbb{N}} D_n \times \{2^{-n-1}\}.$$

Clearly D is a countable subset of $X \times (0, 1)$, so we may write

$$D = \{(x_n, \epsilon_n);\, n \in \mathbb{N}\}.$$

Let y_n, $n \in \mathbb{N}$, satisfy $d(x_n, y_n) \leq \epsilon_n$ for all $n \in \mathbb{N}$, we want to show that $\{y_n, n \in \mathbb{N}\}$ is dense in X. Let $x \in X$ and $\epsilon > 0$. We can find $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$ and, since D_n is a 2^{-n-1}–net, some $c \in D_n$ satisfying $d(c, x) < 2^{-n-1}$. Then $(c, 2^{-n-1}) = (x_k, \epsilon_k) \in D$ for some $k \in \mathbb{N}$. Thus we entail $d(x, y_k) \leq d(x, x_k) + d(x_k, y_k) < 2^{-n-1} + 2^{-n-1} = 2^{-n} < \epsilon$. □

Theorem 18. *For every Peano continuum (X, d) there exists a LEO map $h : X \to X$ such that the set of periodic points of h is dense in X. Thus in particular, the map h is chaotic with respect to Devaney's definition of chaos.*

Proof. If X is a one-point space, then the assertion is clearly true. Suppose that X is nondegenerate and let $\mathscr{F}_n$, $n \in \mathbb{N}$, and $f : X \to I$ be as in Lemma 15 and $(x_n, \epsilon_n) \in X \times (0, 1)$, $n \in \mathbb{N}$, as in Observation 17. Let $\mathscr{F}_0 := \{X\}$, then clearly $\mathscr{F}_0$ is a covering of X by subcontinua of X and for every $F \in \mathscr{F}_1$ we have $F \subseteq X \in \mathscr{F}_0$. Let $\delta_n := \min\{diam(f(F)), F \in \mathscr{F}_n\} > 0$ for every $n \in \mathbb{N}$. We will find by induction sequence y_n, $n \in \mathbb{N}$ and sets $S_n \subseteq I$, maps G_n, g_n and chains $A^n_1, \ldots, A^n_{k_n}$, $n \in \mathbb{N}_0$, such that for every $n \in \mathbb{N}_0$:

1. $k_n \cdot \delta_{n+1} \geq 2$, $\{A^n_1, \ldots, A^n_{k_n}\} = \mathscr{F}_n$,
2. for every $1 \leq j \leq n$ there exists l such that $k_{n+1} = k_j \cdot l$ and for every $1 \leq i \leq k_j$:
$$\bigcup\{A^{n+1}_{l\cdot(i-1)+1}, A^{n+1}_{l\cdot(i-1)+2}, \ldots, A^{n+1}_{l\cdot i}\} = A^j_i,$$
3. $G_n : I \to 2^X$ and the collection of maps $(G_k)_{k \in \mathbb{N}_0}$ satisfy the hypotheses of the General Mapping Theorem (Theorem 4),
4. the function G_n is given by:

$$G_n(x) := \begin{cases} A^n_1, & x = 0, \\ A^n_i, & (i-1)/k_n < x < i/k_n,\ i = 1, 2, \ldots, k_n \\ A^n_i \cup A^n_{i+1}, & x = i/k_n,\ i = 1, 2, \ldots, k_{n-1} \\ A^n_{k_n}, & x = 1, \end{cases}$$

5. the set S_n is a finite subset of irrational numbers and $S_n \subseteq S_{n+1}$,

6. $g_n : S_n \to X$ and $g_n = g_{n+1}|_{S_n}$,

7. $g_n(s) \in G_n(s)$ for every $s \in S_n$,

8. if $n > 0$, then $d(x_n, y_n) \le \epsilon_n$ and $(g_n \circ f)^n(y_n) = y_n$ (we also claim that this is well–defined).

The idea is that the desired function h will be of the form $h = g \circ f$. We will obtain the map g by applying the General Mapping Theorem (Theorem 4) to the collection of maps $(G_n)_{n\in\mathbb{N}}$ and the maps g_n will helps us to control behaviour of the resulting function g on certain subset of I.

Firstly, we let $S_0 := g_0 := \emptyset$. Clearly there is $k_0 \in \mathbb{N}$ such that $k_0 \cdot \delta_1 \ge 2$. Let $A^n_1 := \cdots := A^n_{k_n} := X$ and G_0 be a constant X–valued function. Then the induction hypotheses for $n = 0$ are satisfied.

Secondly, suppose that we have already defined S_n, G_n, g_n and a chain $A^n_1, \ldots, A^n_{k_n}$ for some $n \in \mathbb{N}_0$ and we will construct S_{n+1}, G_{n+1}, g_{n+1}, $A^{n+1}_1, \ldots, A^{n+1}_{k_{n+1}}$ and y_{n+1}.

By Observation 16 the set $f(B(x_{n+1}, \epsilon_{n+1}))$ contains a nondegenerate interval, hence there is $y_{n+1} \in B(x_{n+1}, \epsilon_{n+1})$ such that $f(y_{n+1})$ is irrational and does not lie in the finite set S_n. Since $f(y_{n+1})$ is irrational, $G_n(f(y_{n+1})) = A^n_{i_n}$ for some $1 \le i_n \le k_n$, by definition of G_n. Since $A^n_{i_n} \in \mathscr{F}_n$ is a continuum and $diam(f(A^n_i)) \ge \delta_n \ge 2/k_{n-1}$, $f(A^n_i)$ is a closed interval that contains $[(i_{n-1} - 1)/k_{n-1}, i_{n-1}/k_{n-1}]$ for some $1 \le i_{n-1} \le k_{n-1}$.

Similarly there exists $1 \le i_{n-2} \le k_{n-2}$ such that

$$f(A^{n-1}_{i_{n-1}}) \supseteq [(i_{n-2} - 1)/k_{n-2}, i_{n-2}/k_{n-2}],$$

since $diam(f(A^{n-1}_{i_{n-1}})) \ge \delta_{n-1} \ge 2/k_{n-2}$. In this manner, we can find $i_n \in \{1, \ldots, k_n\}$, $i_{n-1} \in \{1, \ldots, k_{n-1}\}, \ldots, i_0 \in \{1, \ldots, k_0\}$ such that for every $1 \le j < n$ it holds that $f(A_{i^{j+1}_{j+1}}) \supseteq [(i_j - 1)/k_j, i_j/k_j]$.

By induction hypotheses(1, 2): $A^0_{i_0} \in \mathscr{F}_0 = \{X\}$, there exists l_0 such that $k_{n+1} = k_0 \cdot l_0$ and:

$$\bigcup\{A^{n+1}_{l_0\cdot(i_0-1)+1}, A^{n+1}_{l_0\cdot(i_0-1)+2}, \ldots, A^{n+1}_{l_0\cdot i_0}\} = A^0_{i_0} = X \ni y_{n+1}.$$

Hence there exists $1 \le j_0 \le l_0$ such that $y_{n+1} \in A^{n+1}_{l_0\cdot(i_0-1)+j_0}$. Clearly we can find $t_0 \in [(l_0 \cdot (i_0 - 1) + j_0 - 1)/k_{n+1}, (l_0 \cdot (i_0 - 1) + j_0)/k_{n+1}]$ that is irrational and does not lie in the finite set $S_n \cup \{f(y_{n+1})\}$. Notice that then $G_n(t_0) = A^{n+1}_{l_0\cdot(i_0-1)+j_0} \ni y_{n+1}$. Since $f(A^1_{i_1}) \supseteq [(i_0 - 1)/k_0, i_0/k_0] \supseteq [(l_0 \cdot (i_0 - 1) + j_0 - 1)/k_{n+1}, (l_0 \cdot (i_0 - 1) + j_0)/k_{n+1}] \ni t_0$, there is $z_1 \in A^1_{i_1}$ such that $f(z_1) = t_0$. Using induction hypotheses (2) again we obtain l_1 such that $k_{n+1} = k_1 \cdot l_1$ and:

$$\bigcup\{A^{n+1}_{l_1\cdot(i_1-1)+1}, A^{n+1}_{l_1\cdot(i_1-1)+2}, \ldots, A^{n+1}_{l_1\cdot i_1}\} = A^1_{i_1} \ni z_1,$$

hence there exists $1 \le j_1 \le l_1$ such that $z_1 \in A^{n+1}_{l_1\cdot(i_1-1)+j_1}$. Similarly to what was done above, we can find $t_1 \in [(l_1\cdot(i_1-1)+j_1-1)/k_{n+1}, (l_1\cdot(i_1-1)+j_1)/k_{n+1}]$ that is irrational and does not lie in the finite set $S_n \cup \{f(y_{n+1}), f(z_1) = S_n \cup \{f(y_{n+1}), t_0\}$ and show that $t_1 \in f(A^2_{i_2})$. Thus there is $z_2 \in A^2_{i_2}$ such that

$f(z_2) = t_1$. Notice that this again implies $G_n(t_1) = A^{n+1}_{l_1\cdot(i_1-1)+j_1} \ni z_1$. In this manner we can find mutually distinct $t_0, t_1, \ldots, t_n \in I$ that does not lie in S_n and $z_m \in A^m_{i_m}$, $1 \le m \le n$, such that for every $1 \le m \le n$ we have $f(z_m) = t_{m-1}$ and $z_m \in G_n(t_m)$.

Let $S_{n+1} := S_n \cup \{t_0, t_1, \ldots, t_n\}$ and define $g_{n+1}|_{S_n} := g_n$, $g_{n+1}(t_m) := z_m$ for $1 \le m \le n$ and $g_{n+1}(t_0) := y_{n+1}$. It is easy to find $A^{n+1}_1, \ldots, A^{n+1}_{k_{n+1}}$ and G_{n+1} satisfying the induction hypotheses.

Hence by General Mapping Theorem (Theorem 4) the function $g : I \to X$ defined by $\{g(x)\} = \bigcap\{G_n(x), n \in \mathbb{N}\}$ is well–defined, continuous and onto. The theorem gives even more. Let $m \in \mathbb{N} \cup \{0\}$ and $1 \le i \le k_m$. Consider the functions G'_n from $[(i-1)/k_m, i/k_m]$ to the space of subcontinua of A^m_i given by $G'_n((i-1)/k_m) := G_n((i-1)/k_m + 1/(2k_n))$, $G'_n(i/k_m) := G_n(i/k_m - 1/(2k_n))$, $G'_n(x) := G_n(x)$ for all $x \in ((i-1)/k_m, i/k_m)$ and $n \ge m$. It follows from 2 that then

$$\bigcup\{G'_n(x), x \in [(i-1)/k_m, i/k_m]\} = A^m_i$$

for all $n \ge m$ and hence the functions G'_n, $n \ge m$, also satisfy hypotheses of the General Mapping Theorem. Further, it is easy to see that $\{g(x)\} = \bigcap\{G'_n(x), n \ge m\}$ for $x \in [(i-1)/k_m, i/k_m]$. Hence, by the General Mapping Theorem, $g([(i-1)/k_m, i/k_m]) = A^m_i$ for every $m \in \mathbb{N} \cup \{0\}$ and $1 \le i \le k_m$.

Finally, we want to show that $h := g \circ f : X \to X$ is LEO and the set of its periodic points is dense in X. In order to show that h is LEO we need the following claim: for any $n \in \mathbb{N}$ and $1 \le i \le k_n$, it holds that $h(A^n_i) \supseteq F$ for some $F \in \mathscr{F}_{n-1}$. This is true because for any such n and i it follows just by definitions that $diam(f(A^n_i)) \ge \delta_n \ge 2/k_{n-1}$. Since A^n_i is a continuum, $f(A^n_i)$ is a closed interval. Therefore $f(A^n_i) \supseteq [(j-1)/k_{n-1}, j/k_{n-1}]$ for some $1 \le j \le k_{n-1}$. Thus $h(A^n_i) = gf(A^n_i) \supseteq g([(j-1)/k_{n-1}, j/k_{n-1}]) = A^{n-1}_j \in \mathscr{F}_{n-1}$. This proves the claim.

Let $U \subseteq X$ be open and nonempty. Then there exists $n \in \mathbb{N}$ and $F_n \in \mathscr{F}_n$ such that $F_n \subseteq U$. Then by preceding claim there exists $F_{n-1} \in \mathscr{F}_{n-1}$, such that $F_{n-1} \subseteq h(F_n) \subseteq h(U)$. Using the claim $n-1$ more times, we get that there exists $F_0 \in \mathscr{F}_0 = \{X\}$ such that $X = F_0 \subseteq h(F_1) \subseteq h^n(F_n) \subseteq h^n(U)$.

To ensure that the the set of periodic points of h is dense, first observe that $g(s) = g_n(s)$ for every $s \in S_n$ since for every $m \ge n$ we have $g_n(s) = g_m(s) \in G_m(s)$. Hence for every $n \in \mathbb{N}$ it holds that $h^n(y_n) = (g \circ f)^n(y_n) = (g_n \circ f)^n(y_n) = y_n$, in other words, for every $n \in \mathbb{N}$ the point y_n is a periodic point of h. By Observation 17 and 8 the set $\{y_n, n \in \mathbb{N}\}$ is dense in X and therefore the set of periodic points of h is also dense in X. □

3. Topological fractals

In the previous chapter we dealt with those selfmaps of some space that magnify every part of it so that every open set is eventually mapped onto the whole space. Now we are interested in an opposite kind of maps: we will work with maps that eventually sends the whole space to an arbitrily small piece of it. This leads to the study of self–similar sets and topological fractals.

In the first section we introduce the basic notions and present some fundamental results that determined the direction of the research of topological fractals the most. The second chapter is focused on one of the main results of this thesis, a sufficient condition for a Peano continuum to be a topological fractal. The last section is dedicated to regenerating fractals which were invented to give another sufficient condition for a Peano continuum to be a topological fractal. We apply our result to partially answer a problem posed in Nowak [2021] introductiong the notion of regenerating fractals.

3.1 Origins

Let X be a metric space. As was mentioned in the beginning of this chapter, we are now interested in selfmaps of X that eventually sends the whole space to an arbitrily small piece of it. Basic functions satisfying this are, for example constant functions or contractions. However, we of course do not want to study trivial maps like constant functions, we want the images of the studied maps to still carry some information about X. Since we can no more require our maps to be surjective, we instead consider more than one function such that the union of their images is the whole space X. This way we ensure that no essential information about the space is lost. At the same, a difficulty occurs; when we are dealing with more than one map, we do not only want each of them to send the whole space to an arbitrily small piece, but we want any sequence of our maps to send the whole space to an arbitrily small piece, if the sequence is long enough. Note that this is something a finite number of contradictions still do satisfy.

The origin of topological fractals are attractors of iterated function systems. Suppose that X is a subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n : X \to X$ are similarities, each with ratio (strictly) less than 1. It was proven in Edgar [1990] that there exists unique nonempty and compact $K \subseteq X$ satisfying $K = f_1(K) \cup \cdots \cup f_n(K)$.

Intuitively, when we say that some space X is "a self–similar set" or "a fractal", we want to express that X consists of its own copies, just smaller. If this is the case, then of course all of these copies consists of its own copies (which are also copies of the whole space X), and so do all these even smaller copies etc. "Roughly speaking, a fractal set is a set that is more "irregular" than the sets considered in classical geometry. No matter how much the set is magnified, smaller and smaller irregularities become visible," explains Gerald A. Edgar in his book Measure, topology, and fractal geometry Edgar [1990].

This leads us to the following definition: we say that a closed set $X \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, is a self–similar set if there exists similarities $f_1, f_2, \ldots, f_n : X \to X$, each with ratio (strictly) less than 1. The set of the functions $\{f_1, f_2, \ldots, f_n\}$ is usually called an iterated function system (IFS) and X an attractor of this IFS. The following sets are typical examples of self–similar sets.

Example 19. *The most basic examples are the trivial space (a point) and the interval I. The point is already small enough so there is no need to further decompose it, it is thus (the only) attractor for the IFS consisting of just one map, namely the identity. The case of the interval is simple too. It should be obvious that the interval can be decomposed in a finite number of its own arbitrily small copies. It is an attractor for the IFS $\{x \mapsto x/2;\ x \mapsto (x+1)/2\}$.*

Example 20 (Cantor set)**.** *Another example of a self–similar set is the Cantor set 2^ω introduced in the first chapter. It holds that $2^\omega = R_0(2^\omega) \cup R_1(2^\omega)$, where R_0, R_1 are right shifts defined in the Introduction.*

However, despite that the notion of similarity map ilustrates the idea very well and is sufficient for many examples, it is not quite suitable for a general metric space. The natural generalization is not to require the maps of the correspoding IFS to be similarities, but just to be contractions. It was proven in Edgar [1990] that for every complete metric space X and contractions $f_1, f_2, \dots, f_n : X \to X$ there exists a unique nonempty and compact $K \subseteq X$ satisfying $K = f_1(K) \cup \dots \cup f_n(K)$, and moreover, if A_0 is any compact subset of X, then the sequence $(A_k)_{k \in \mathbb{N}}$ converges to K in the Hausdorff metric, where A_{k+1}, $k > 1$ is defined inductively to be $A_{k+1} := \bigcup_{1 \le i \le n} f_i(A_n)$. This is a reason why such K is called an attractor of the IFS $\{f_1, f_2, \dots, f_n\}$ and connects the theory of fractal with the theory of topological dynamics. Edgar proved also that every metric fractal is of a finite topological dimension and thus there are continua that are not metric fractals. Let us note that the argument is through topological dynamics and uses semiconjugacy with the right shifts. These notions were defined in the Introduction.

It was observed by Hata in Hata [1985] that every connected metric fractal has the property S and thus it is a Peano continuum. At the same time he asked whether every finite–dimensional Peano continuum or at least every compact subset of $\mathbb{R}^d$ for every $d \in \mathbb{N}$ is a metric fractal. This question was answered negativaly later since in Banakh and Nowak [2013] it was proven that there exists a plane 1–dimensional Peano continuum that is not homeomorphic to any IFS attractor.

Therefore it is reasonable to consider further generalizations. This leads us to the notion of a weak IFS attractor where we do not even require the maps of the IFS to be contradiction, but only so–called weak contraditions. We say that a map $f : X \to X$ is a weak contradiction if $d(f(x), d(y)) < d(x, y)$ for every $x \neq y \in X$. It was showed in Banakh et al. [2015a] that metrizable weak attractors are exactly topological fractals:

Notation. Let X be a set and $\mathscr{F}$ be a set of functions from X to X. For any $n \in \mathbb{N}_0$, we denote by $\mathscr{F}^n$ the set of maps $\{f_1 \circ \dots \circ f_n;\ f_i \in \mathscr{F} \text{ for } 1 \le i \le n\}$.

Definition 21. *Let X be a metric space. We say that X is a topological fractal if there exists $n \in \mathbb{N}$ and functions $f_1, \dots, f_n : X \to X$ satisfying $f_1(X) \cup \dots \cup f_n(X) = X$ such that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $g \in \{f_1, \dots, f_n\}^k$ the diameter of $g(X)$ is less then ε.*

An argument analogical to the one above confirms that every connected topological fractal is a Peano continuum. Thus among continua, only Peano continua might happen to be topological fractals. However the question, whether every Peano continuum is a topological fractal is still open since 1985 when Hata posted it in Hata [1985]. We will pay more attention to this problem in the last section of this this chapter.

Nevertheless, some partial results were establisted since 1985. Among zero–dimensional spaces even complete characterization was provided in Banakh et al. [2015b]. Further, some sufficient conditions where a Peano continuum is a topological fractal, namely so–called free arc or self–regenerating subcontinuum with a nonempty interior, are known and in the forthcoming section we present a new one. The precise formulations are postponed to the following sections to the places where we work with these notions more so the context is most clear.

3.2 A partial solution to Hata's hypothesis

Let us start with the following useful observation. A similar result was proved indepently in Nowak [2021].

Observation 22. *Let X be a compact metric space and $f_1, \dots f_n, g_1, \dots, g_m : X \to X$ be continuous functions such that*

$$f_1(X) \cup \dots \cup f_n(X) \cup g_1(X) \cup \dots \cup g_m(X) = X$$

and for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $f \in \{f_1, \dots, f_n\}^k$ the diameter of $f(X)$ is less then ε. Suppose that for every $f \in \bigcup_{i=0}^{\infty}\{f_1, \dots, f_n\}^i$ and $1 \leq i, j \leq m$ we have that $g_i \circ f \circ g_j$ is a constant function.

Then X is a topological fractal.

Proof. Let $\varepsilon > 0$. Then there exists $k_1 \in \mathbb{N}$ such that for every $f \in \{f_1, \dots, f_n\}^{k_1}$ the diameter of $f(X)$ is less then ε. We can find $\delta > 0$ such that for every $f \in \bigcup_{i=0}^{k_1-1}\{f_1, \dots, f_n\}^i$, for every $1 \leq j \leq m$ and every $x \in X$ the diameter of $f \circ g_j(B(x, \delta))$ is less than ε. By assumption, there exists $k_2 \in \mathbb{N}$ such that for every $f \in \{f_1, \dots, f_n\}^{k_2}$ the diameter of $f(X)$ is less then δ.

Let $k := k_1 + k_2$ and $h = h_1 \circ \dots \circ h_k \in \{f_1, \dots f_n, g_1, \dots, g_m\}^k$. If there are $i \neq j$ such that $h_i, h_j \in \{g_1, \dots, g_m\}$, then h must be by assumption a constant function and hence $h(X)$ has diameter $0 < \varepsilon$.

Suppose that $h_i \in \{g_1, \dots, g_m\}$ for at most one $1 \leq i \leq k$. If $h_1, \dots, h_{k_1} \in \{f_1, \dots f_n\}$, then $diam(h(X)) \leq \varepsilon$ since $h(X) = h_1 \circ \dots \circ h_k(X) \subseteq h_1 \circ \dots \circ h_{k_1}(X)$ and $h_1 \circ \dots \circ h_{k_1} \in \{f_1, \dots, f_n\}^{k_1}$. Otherwise $h_i \in \{g_1, \dots, g_m\}$ for exactly one $1 \leq i \leq k_1$ and $h_{i+1}, \dots, h_k \in \{f_1, \dots f_n\}$. Then $diam(h_{i+1} \circ \dots \circ h_k(X)) \leq \delta$ since $k - i \geq k - k_1 = k_2$.

Thus by the definition of δ, $diam(h(X)) \leq \varepsilon$. □

Definition 23. *Let X be a (connected) topological space. A point $x \in X$ is called a cut point if $X \setminus \{x\}$ is not connected.*

Let us introduce some more notation connected to products of sets.

Definition 24. *Let* $s = (s_1, \ldots, s_n)$, $t = (t_1, \ldots, t_k) \in 2^{<\omega}$ *and* $r = (r_1, r_2, \ldots) \in 2^\omega$. *We define the concatenation by* $s^\frown t := (s_1, \ldots, s_n, t_1, \ldots, t_k) \in 2^{<\omega}$, *respectively* $s^\frown r := (s_1, \ldots, s_n, r_1, r_2, \ldots) \in 2^\omega$. *Clearly concatenation is associative, so we will omit brackets when applying cancatenation iteratively.*

Further, define $s^{op} := (1 - s_1, \ldots, 1 - s_n)$, *respectively* $r^{op} := (1 - r_1, 1 - r_2, \ldots)$.

In what follows, we will denote by $\bar{0} := (0, 0, \ldots)$, $\bar{1} := (1, 1, \ldots) \in 2^\omega$ *and by* E *the subset of* 2^ω *given by* $E := \{s^\frown\bar{0}, s \in 2^{<\omega}\} \cup \{s^\frown\bar{1}, s \in 2^{<\omega}\}$.

With this notation, we can introduce "a free Cantor set" — our most important tool in partial answering Hata' question.

Definition 25. *Let* X *be a Peano continuum. We say that* $\varphi : E \to X$ *is a free Cantor set (in* X*) if it is one–to–one and if there exist* X_s^L, X_s, X_s^R, $s \in 2^{<\omega}$, *Peano subcontinua of* X *such that* $X = X_\emptyset^L \cup X_\emptyset \cup X_\emptyset^R$, *for every* $s \in 2^{<\omega}$*:*

1. $\varphi(s^\frown\bar{0}) \in X_s^L$ *and* $\varphi(s^\frown\bar{1}) \in X_s^R$,
2. $X_s^L \cap X_s = \{\varphi(s^\frown 0^\frown\bar{1})\}$, $X_s^R \cap X_s = \{\varphi(s^\frown 1^\frown\bar{0})\}$, $X_s^L \cap X_s^R = \emptyset$,
3. $X_s^L = X_{s^\frown 0}^L \cup X_{s^\frown 0} \cup X_{s^\frown 0}^R$, $X_s^R = X_{s^\frown 1}^L \cup X_{s^\frown 1} \cup X_{s^\frown 1}^R$,

and $\lim_{|s|\to\infty} diam(X_s^L) = 0$, $\lim_{|s|\to\infty} diam(X_s^R) = 0$, *that is, for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* $s \in 2^{<\omega}$ *satisfying* $|s| \geq n_0$ *the diameters of* X_s^L *and* X_s^R *are less than* ε.

This definition might be confusing at the first sight, because what we call a free Cantor set does not seem to be complete since it correspond only to a part of the Cantor set. However, the following Lemma unsures us that the rest of it is actually exactly where it should be.

Lemma 26. *Let* (Y, d) *be a Peano continuum. Suppose that there are* X^L, X, X^R *Peano subcontinua of* Y *such that* X *contains a free Cantor set* $\varphi : E \to X$. *Suppose further that* $Y = X^L \cup X \cup X^R$, $X^L \cap X = \{\varphi(\{0\}^\omega)\}$, $X \cap X^R = \{\varphi(\{1\}^\omega)\}$ *and* $X^L \cap X^R = \emptyset$.

Then φ *can be uniquely extended to be a continuous function defined on the whole Cantor set* 2^ω, *this extension is one–to–one and all points in* $\varphi(2^\omega \setminus \{\bar{0}, \bar{1}\})$ *are cutpoints of* X.

Proof. Fix arbitrary $s \in 2^\omega$. For any $n \in \omega$ let $S_n^s := X_{s|n}^L$ if $s(n) = 0$ and $S_n^s := X_{s|n}^R$ if $s(n) = 1$. Then it follows immediately from the definitions that $S_{n+1}^s \subseteq S_n^s$ for every $n \in \omega$ and $\lim_{n\to\infty} diam(S_n^s) = 0$. Thus, by compactness of Y, $\bigcap_{n\in\omega} S_n^s$ is a one–element set.

Notice that if $s \in E$, then $\varphi(s) \in S_n^s$ for every $n \in \omega$: suppose for example that $s = t^\frown\bar{0}$, but this by definition implies that for every $n > |t|$ it holds both $S_n^s = X_{t^\frown\{0\}^{n-|t|-1}}^L$ and $\varphi(t^\frown\bar{0}) = \varphi(t^\frown\{0\}^{m-n\frown}\bar{0}) \in X_{t^\frown\{0\}^{m-n}}^L$. This allows us to let $\{\varphi(s)\} = \bigcap_{n\in\omega} S_n^s$ for every $s \in 2^\omega$ to obtain a well–defined extension $\varphi : 2^\omega \to Y$.

To show that this extension is continuous let $s \in 2^\omega$ and $\varepsilon > 0$. By assumptions there exists $n \in \mathbb{N}$ such that for every $t \in 2^{<\omega}$ satisfying $|t| \geq n$ the diameters of X_t^L and X_t^R are less than ε. In particular, $diam(S_n^s) \leq \varepsilon$.

Let $U := \{t \in 2^\omega;\ t|_{n+1} = s|_{n+1}\}$, then clearly $s \in U$, U is open in 2^ω and $\varphi(U) \subseteq S_n^s \subseteq B(\varphi(s); \varepsilon)$. Thus $\varphi : 2^\omega \to Y$ is continuous and obviously there can not be any other continuous extension since E is dense in 2^ω.

Let $s \neq t \in 2^\omega$. Let $n \in \omega$ satisfy $s|_n = t|_n$ but $s(n) \neq t(n)$. By symmetry, we may assume that $s(n) = 0$ and $t(n) = 1$. Then $\varphi(s) \in S_n^s = X_{s|_n}^L$, $\varphi(t) \in S_n^t = X_{t|_n}^R = X_{s|_n}^R$ and $X_{s|_n}^L \cap X_{s|_n}^R = \emptyset$ by assumption. Hence $\varphi(s) \neq \varphi(t)$ and therefore φ is one–to–one. Notice that this actually shows that $\varphi : 2^\omega \to Y$ is a homeomorphism onto its image, since 2^ω is compact and Y is metrizable, in particular Hausdorff.

It remains to show that all points in $\varphi(2^\omega \setminus \{\bar{0}, \bar{1}\})$ are cut points of Y. This is straightforward for points in $\varphi(E \setminus \{\bar{0}, \bar{1}\})$ and moreover, $\varphi(\bar{0})$ and $\varphi(\bar{1})$ lie in different connected components of $Y \setminus x$ for every $x \in \varphi(E \setminus \{\bar{0}, \bar{1}\})$. For example,

$$\varphi(\bar{0}) \in X^L \subseteq X^L \cup (X_\emptyset^L \setminus \{\varphi(0^\frown\bar{1})\}) = Y \setminus (X_\emptyset \cup X_\emptyset^R \cup X^R),$$
$$\varphi(\bar{1}) \in X^R \subseteq X^R \cup X_\emptyset^R \cup (X_\emptyset \setminus \{\varphi(0^\frown\bar{1})\}) = Y \setminus (X_\emptyset^L \cup X^L)$$

and the sets $Y \setminus (X_\emptyset \cup X_\emptyset^R \cup X^R)$, $Y \setminus (X_\emptyset^L \cup X^L) \subseteq Y \setminus \{\varphi(0^\frown\bar{1})\}$ are open in Y (and thus also in $Y \setminus \{\varphi(0^\frown\bar{1})\}$) since we suppose that X^L, $X_\emptyset^L$, $X_\emptyset$, $X_\emptyset^R$, X^R are (Peano) subcontinua of Y and hence they must be closed. This shows that $\varphi(0^\frown\bar{1})$ is a cut point of Y and $\varphi(\bar{0})$, $\varphi(\bar{1})$ lie in different connected components of $Y \setminus \{\varphi(0^\frown\bar{1})\}$.

Let $s \in 2^\omega \setminus E$ and suppose for contradiction that $Y \setminus \{\varphi(s)\}$ is connected. Then it is also path connected, since in Peano continua path components of open sets are open and hence are equal to connected components. Therefore there is an arc $a : I \to Y \setminus \{\varphi(s)\}$ satisfying $a(0) = \varphi(\bar{0})$, $a(1) = \varphi(\bar{1})$. For $\varepsilon := d(\varphi(s), a(I)) > 0$ we can find $n \in \omega$ such that $diam(S_n^s) < \varepsilon$. Let $t \in E \setminus \{\bar{0}, \bar{1}\}$ be such that $\varphi(t) \in S_n^s$, this is possible since $\varphi(r^\frown 0^\frown\bar{1}) \in X_r^L$ and $\varphi(r^\frown 1^\frown\bar{0}) \in X_r^R$ for every $r \in 2^{<\omega}$. But then $a(I) \subseteq Y \setminus \{\varphi(t)\}$ and therefore $\varphi(\bar{0})$, $\varphi(\bar{1})$ lie in the same connected component of $Y \setminus \{\varphi(t)\}$. This is a contradiction since it was proven above that $\varphi(\bar{0})$ and $\varphi(\bar{1})$ lie in different connected components of $Y \setminus x$ for every $x \in \varphi(E \setminus \{\bar{0}, \bar{1}\})$. □

Example 27. *It is not true in general that if X is a Peano continuum, $x_n \in X$, $n \in \mathbb{N}$, are cut points of X and $\lim_{n\to\infty} x_n = x \in X$, then x also must be a cut point of X. Consider the following example: let*

$$X := [0, 1] \times [-1, 0] \cup \bigcup_{n\in\mathbb{N}} \{1/n\} \times [0, 1/n] \subseteq \mathbb{R}^2.$$

Then X is a Peano continuum, $x_n := (1/n, 0)$ are cut points of X for every $n \in \mathbb{N}$, but $(0, 0) = \lim_{n\to\infty} x_n$ is not a cut point of X.

But it is true that if X is a Peano continuum, $x_n \in X$, $n \in \mathbb{N}$, are cut points of X such that $\lim_{n\to\infty} x_n = x \in X$ and for every $n \in \mathbb{N}$ there are $A_n \neq B_n$ connected components of $X \setminus \{x_n\}$ such that $\bigcap_{n\in\mathbb{N}} A_n \setminus \{x\} \neq \emptyset$, $\bigcap_{n\in\mathbb{N}} B_n \setminus \{x\} \neq \emptyset$, then x is a cut point of X. This is exactly the reason why the preceding observation holds.

In particular, the preceding Lemma immediately gives us the following:

Corollary 28. *Let X be a Peano continuum. Suppose that there are Y^L, Y, Y^R Peano subcontinua of X such that Y contains a free Cantor set φ. Suppose further that $X = Y^L \cup Y \cup Y^R$, $Y^L \cap Y = \{\varphi(\{0\}^\omega)\}$, $Y \cap Y^R = \{\varphi(\{1\}^\omega)\}$ and $Y^L \cap Y^R = \emptyset$. Then the set of all cutpoints of X is uncountable.*

Perhaps suprisingly, the converse holds too. However before we can prove it, we need to prove in preparation two observations.

Observation 29. *Let A be an arc and fix some homeomorphism $h : I \to A$. Let D be an uncountable subset of A. Then the following is true:*

1. *there exists $x \in (0, 1)$ such that $D \cap h([0, x])$, $D \cap h([x, 1])$ are uncountable,*
2. *there exists $y \in (0, 1)$ such that for every $0 < \varepsilon \leq 1-y$ the set $D \cap h([y, y+\varepsilon])$ is uncountable,*
3. *there exists $y \in (0, 1)$ such that for every $0 < \varepsilon \leq y$ the set $D \cap h([y-\varepsilon, y])$ is uncountable.*

Proof. For simplicity of notation we will suppose that $A = I$ and $h = id$, the general case follows immediately.

1 Let $J := \{x \in [0, 1]; D \cap [0, x] \text{ is uncountable}\}$, $S := \{x \in [0, 1]; D \cap [x, 1] \text{ is uncountable}\}$, $i := \inf J$ and $s := \sup S$. It is sufficient to prove that $i < s$. Suppose for contradiction that $s \leq i$ and let x_n, $n \in \mathbb{N}$, be a stricly increasing sequence converging to s and y_n, $n \in \mathbb{N}$, be a stricly decreasing sequence converging to s. Then for every $n \in N$:

- $D \cap [y_n, 1]$ is at most countable, since $s < y_n$ implies $y_n \notin S$,
- $D \cap [0, x_n]$ is at most countable, since $x_n < s \leq i$ implies $x_n \notin J$.

Clearly
$$D \subseteq \bigcup_{n\in\mathbb{N}} (D \cap [0, x_n]) \cup \bigcup_{n\in\mathbb{N}} (D \cap [y_n, 1]) \cup \{s\},$$
but the right side is a countable union of at most countable sets and hence an at most countable set, a contradiction.

Thus there exists $x \in [0, 1]$ such that $D \cap h([0, x])$, $D \cap h([x, 1])$ are uncountable, in particular $x \neq 0, 1$.

2 By 1 there exists $x_0 \in (0, 1)$ such that $D \cap [0, x_0]$, $D \cap [x_0, 1]$ are uncountable. Using 1 for the interval $[x_0, 1]$ and uncountable set $D \cap [x_0, 1]$ yields $x_1 \in (x_0, 1)$ such that $D \cap [x_0, x_1]$, $D \cap [x_1, 1]$ are uncountable. Similarly if x_n is defined for some $n \in \mathbb{N}$, we can find $x_{n+1} \in (x_0, x_n)$ such that $D \cap [x_0, x_{n+1}]$, $D \cap [x_{n+1}, x_n]$ are uncountable.

This gives us a strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ of points greater then x_0. Let $0 < x_0 \leq x := \lim_{n\to\infty} x_n < 1$ be its limit and let $0 < \varepsilon \leq 1 - x$. There exists $n \in \mathbb{N}$ such that $x_n < x + \varepsilon$. Then $[x_{n+1}, x_n] \subseteq [x, x + \varepsilon]$ and hence $D \cap [x, x + \varepsilon] \supseteq D \cap [x_{n+1}, x_n]$ is uncountable.

3 is analogical to 2. □

Observation 30. *Let X be a Peano continuum and $U \subseteq X$ an open connected set with finite boundary. Then the closure of U is a Peano subcontinuum of X.*

Proof. Clearly $\overline{U}$ is closed, connected and hence a subcontinuum of X, so we only need to show that it is locally connected. Let $x \in \overline{U}$ and V be an open set in $\overline{U}$ containing x. If $x \in U$, then $U \cap V$ is an open set in X containing x and hence there exists $W \subseteq U \cap V$ connected and open in X and thus also in $\overline{U}$ contaning x.

Suppose that x lies in the boundary of U. Let W be an open neighborhood of x in X that contains no point of the boundary of U except for x. Let V' be open in X such that $V = V' \cap \overline{U}$. Since X is Peano, there exists $P' \subseteq W \cap V'$ connected and open in X contaning x. Then $P := P' \cap \overline{U}$ is an open neighborhood of x in $\overline{U}$. Suppose for contradiction that it is not connected, then there exists a (nonempty) proper $F \subseteq P$ clopen in P. We may suppose that $x \notin F$, otherwise we would consider $P \setminus F$ instead of F.

Clearly F is a (nonempty) proper subset of P'. Moreover, F is closed in P' since it is closed in P and P is closed in P' because $P = P' \cap \overline{U}$. Similarly F is open in P and thus also in $P \setminus \{x\}$ since $x \notin F$. Further $P \setminus \{x\} = (P' \cap \overline{U}) \setminus \{x\} = P' \cap U$ since $P' \subseteq W$ and thus x is the only point lying both in P' and boundary of U. Finally, this implies that F is open in P', therefore F is a (nonempty) clopen proper subset of P' which contradicts the connectedness of P'. □

Let us proceed with the promised converse of Corollary 28.

Lemma 31. *Let X be a Peano continuum with uncountably many cut points. Then there are Y^L, Y, Y^R Peano subcontinua of X, Y containing a free Cantor set φ, such that $X = Y^L \cup Y \cup Y^R$, $Y^L \cap Y = \{\varphi(\{0\}^\omega)\}$, $Y \cap Y^R = \{\varphi(\{1\}^\omega)\}$ and $Y^L \cap Y^R = \emptyset$.*

Proof. Let $\mathscr{U}$ be a countable base of open sets in X formed by nonempty connected sets. For $U, V \in \mathscr{U}$ let $D(U, V) \subseteq X$ be the set of cut points of X such that there exist $A, B \subseteq X$ disjoint, open and satisfying $X \setminus \{x\} = A \cup B$, $U \subseteq A$, $V \subseteq B$. In other words, $D(U, V)$ is the set of those cut points of X that separates U and V.

Notice that for every cut point x of X there exists nonempty disjoint open sets $A, B \subseteq X$ such that $X \setminus \{x\} = A \cup B$, and hence also $U, V \in \mathscr{U}$ such that $U \subseteq A$, $V \subseteq B$. Thus $\bigcup\{D(U, V), U, V \in \mathscr{U}\}$ is the set of all cut points of X and hence uncountable by assumption. Since $\mathscr{U} \times \mathscr{U}$ is only countable, there exist $U, V \in \mathscr{U}$ such that $D(U, V)$ is uncountable.

Fix arbitrary $u \in U$, $v \in V$. Let $x \in X \setminus D(U, V)$, if $x \in U \cup V$ then $U \cup V$ is an open subset of $X \setminus D(U, V)$ containing x. Otherwise the connected component of $X \setminus \{x\}$ containing U contains also V. Since conneted and path components of open sets coincide in Peano continua, there exists an arc $a : I \to X \setminus \{x\}$ such that $a(0) = u$, $a(1) = v$. Let $\varepsilon := d(a(I), x) > 0$, then $B(x, \varepsilon) \subseteq X \setminus D(U, V)$ since for every $y \in B(x, \varepsilon)$ either $y \in U \cup V$ or the set $U \cup a(I) \cup V \subseteq X \setminus \{y\}$ is connected. Thus $D(U, V)$ is closed.

Fix some arc $h : I \to X$ such that $h(0) = u$, $h(1) = v$. Then the set $U \cup h(I) \cup V$ is connected and contains both U, V and thus, by definiton, it can not be a subset of $X \setminus \{x\}$ for any $x \in D(U, V)$. But for every $x \in D(U, V)$ we have $U, V \subseteq X \setminus \{x\}$ and therefore, by the above, $x \in h(I)$. Hence $D(U, V)$ is an uncountable and closed subset of $h(I)$. In what follows, we will denote $D := D(U, V)$ for brevity.

By Observation 29, items 2 and 3, there exists $a < b \in (0, 1)$ such that for every $0 < \varepsilon \le \min\{1-a, b\}$ the sets $D \cap h([a, a+\varepsilon])$, $D \cap h([b-\varepsilon, b])$ are uncountable. Using Observation 29, items 2 and 3 again, there exists $a' < b' \in (a, b)$ such that for every $0 < \varepsilon \le \min\{a' - a, b - b'\}$ the sets $D \cap h([a' - \varepsilon, a'])$, $D \cap h([b', b' + \varepsilon])$ are uncountable. Let $\varphi(\bar{0}) := h(a)$, $\varphi(\bar{1}) := h(b)$, $\varphi(0^\frown\bar{1}) := h(a')$, $\varphi(1^\frown\bar{0}) := h(b') \in h(I)$, then moreover $\varphi(\bar{0})$, $\varphi(\bar{1})$, $\varphi(0^\frown\bar{1})$, $\varphi(1^\frown\bar{0}) \in D$ since D is closed.

Let Y^L be the complement (in X) of the connected component of $X \setminus \{\varphi(\bar{0})\}$ that contains $\varphi(\bar{1})$, Y^R be the complement of the connected component of $X \setminus (Y^L \cup \{\varphi(\bar{1})\})$ that contains $h((a, b))$ and $Y := (X \setminus (Y^L \cup Y^R)) \cup \{\varphi(\bar{0}), \varphi(\bar{1})\}$. Further let $Y^L_\emptyset$ be the closure of the connected component of $Y \setminus \{\varphi(0^\frown\bar{1})\}$ that contains $\varphi(\bar{0})$, $Y^R_\emptyset$ be the closure of the connected component of $Y \setminus \{\varphi(1^\frown\bar{0})\}$ that contains $\varphi(\bar{1})$ and $Y_\emptyset := (Y \setminus (Y^L_\emptyset \cup Y^R_\emptyset)) \cup \{\varphi(0^\frown\bar{1}), \varphi(1^\frown\bar{0})\}$.

Since components of open sets are open in Peano continua, it follows by Observation 30 that Y^L, Y^R, Y, $Y^L_\emptyset$, $Y_\emptyset$, $Y^R_\emptyset$ are Peano subcontinua of X.

Let $s \in 2^{<\omega}$ and suppose that we have already defined Y^L_s a Peano subcontinuum of X and $\varphi|_F$ for some $s^\frown\bar{0}, s^\frown0^\frown\bar{1} \in F \subseteq E$ finite such that $a_s := h^{-1}(\varphi(s^\frown\bar{0}))$, $b_s := h^{-1}(\varphi(s^\frown0^\frown\bar{1}))$ satisfy $a_s < b_s$ and $D \cap h([a_s, a_s + \varepsilon])$, $D \cap h([b_s - \varepsilon, b_s])$ are uncountable for every $0 < \varepsilon \le b_s - a_s$. We will define $Y^L_{s^\frown0}$, $Y_{s^\frown0}$, $Y^R_{s^\frown0}$ and $\varphi(s^\frown00^\frown\bar{1})$, $\varphi(s^\frown01^\frown\bar{0})$ having similar properties.

Let $a_n \in (a_s, b_s) \cap h^{-1}(D)$, $n \in \mathbb{N}$, be a strictly decreasing sequence converging to a_s and $b_n \in (a_s, b_s) \cap h^{-1}(D)$, $n \in \mathbb{N}$, be a strictly increasing sequence converging to b_s. Suppose for contradiction that for every $n \in \mathbb{N}$ there exists x_n lying in the connected component of $X \setminus \{h(a_s), h(a_n)\}$ containing $h((a_s, a_n))$ that does not lie in $B(h(a_s), 2^{-|s|})$. By compactness of X we may suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$.

Clearly $x \notin B(h(a_s), 2^{-|s|})$, in particular $x \neq h(a_s)$. Let P, Q be disjoint connected neighborhoods of x and $h(a_s)$, respectively. Let $n_1 \in \mathbb{N}$ satisfy that $h((a_s, a_{n_1})) \subseteq Q$ and $x_n \in P$ for every $n \ge n_1$. Since x_{n_1} lies in the connected component of $X \setminus \{h(a_s), h(a_{n_1})\}$ containing $h((a_s, a_{n_1}))$ and connected and path components of open sets concide in Peano continua, there exists an arc $g_1 : I \to X \setminus \{h(a_s), h(a_{n_1})$ satisfying $g_1(0) = x_{n_1}$, $g_1(1) \in h((a_s, a_{n_1}))$.

Let $n_2 \in \mathbb{N}$ be such that $h(a_{n_2}) \notin g_1(I)$. Since x_{n_2} lies in the connected component of $X \setminus \{h(a_s), h(a_{n_2})\}$ containing $h((a_s, a_{n_2}))$ and connected and path components of open sets concide in Peano continua, there exists an arc $g_2 : I \to X \setminus \{h(a_s), h(a_{n_2})$ satisfying $g_2(0) = x_{n_2}$, $g_2(1) \in h((a_s, a_{n_2}))$. Then $n_2 > n_1$ because $g_1(1) \in h((a_s, a_{n_1}))$ and thus $g_1(0) = x_{n_1}$, $g_2(0) = x_{n_2} \in P$. Therefore

$$U \cup [u, g_2(1)] \cup g_2(I) \cup P \cup g_1(I) \cup [g_1(1), v] \cup V$$

is connected. Moreover, it is a subset of $X \setminus \{h(a_{n_2})\}$ since $h(a_{n_2}) \in h((a_s, a_{n_1})) \subseteq Q$, which finally contradicts the fact that $a_{n_2} \in D$. Thus there exists $n \in \mathbb{N}$ such that the connected component of $X \setminus \{h(a_s), h(a_n)\}$ containing $h((a_s, a_n))$ is a subset of $B(h(a_s), 2^{-|s|})$. Analogically we can show that there exists $m \in \mathbb{N}$ such that the connected component of $X \setminus \{h(b_s), h(b_m)\}$ containing $h((b_m, b_s))$ is a subset of $B(h(b_s), 2^{-|s|})$. Let $k \ge m, n$ satisfy $h((a_s, a_k) \cup (b_k, b_s)) \cap F = \emptyset$.

By Observation 29, items 2, 3, there exist $a'_s \in (a_s, a_k)$ and $b'_s \in (b_k, b_s)$ such that for every $0 < \varepsilon \le \min\{a'_s - a_s, b_s - b'_s\}$ the sets $D \cap h([a'_s - \varepsilon, a'_s])$, $D \cap h([b'_s, b'_s + \varepsilon])$ are uncountable. Let $\varphi(s^\frown00^\frown\bar{1}) := h(a'_s)$, $\varphi(s^\frown01^\frown\bar{0}) := h(b'_s) \in h(I)$, then moreover $\varphi(s^\frown00^\frown\bar{1})$, $\varphi(s^\frown01^\frown\bar{0}) \in D$ since D is closed.

Let $Y^L_{s^\frown 0}$ be the complement in Y^L_s of the connected component of $Y^L_s \setminus \{\varphi(s^\frown 00^\frown \bar{1})\}$ that contains $\varphi(s^\frown 01^\frown \bar{0})$, $Y^R_{s^\frown 0}$ be the complement of the connected component of $Y^L_s \setminus (Y^L_{s^\frown 0} \cup \{\varphi(s^\frown 01^\frown \bar{0})\})$ that contains $h((a'_s, b'_s))$ and $Y_{s^\frown 0} := (Y^L_s \setminus (Y^L_{s^\frown 0} \cup Y^R_{s^\frown 0})) \cup \{\varphi(s^\frown 00^\frown \bar{1}), \varphi(s^\frown 01^\frown \bar{0})\}$. Since components of open sets are open in Peano continua and Y^L_s is a Peano continuum by induction hypotheses, it follows by Observation 30 that $Y^L_{s^\frown 0}$, $Y^R_{s^\frown 0}$, $Y_{s^\frown 0}$ are Peano subcontinua of X.

Analogically if we have already defined for some $s \in 2^{<\omega}$ a Peano continuum Y^R_s and $\varphi|_F$ for some $s^\frown \bar{1}$, $s^\frown 1^\frown \bar{0} \in F \subseteq E$ finite such that $a_s := h^{-1}(\varphi(s^\frown 1^\frown \bar{0}))$, $b_s := h^{-1}(\varphi(s^\frown \bar{1}))$ satisfy $a_s < b_s$ and $D \cap h([a_s, a_s + \varepsilon])$, $D \cap h([b_s - \varepsilon, b_s])$ are uncountable for every $0 < \varepsilon \leq b_s - a_s$, we can find $Y^L_{s^\frown 1}$, $Y_{s^\frown 1}$, $Y^R_{s^\frown 1}$ and $\varphi(s^\frown 11^\frown \bar{0})$, $\varphi(s^\frown 10^\frown \bar{1})$ having similar properties.

This finishes the induction step. The resulting function $\varphi : E \to X$ is a free Cantor set by construction.

□

Thus we have a characterization of continua that can be equipped with the structure described in Corollary 28 and Lemma 31, namely these are exactly continua with uncountably many cut points. Now we will show how this structure naturally entails a structure of the topological fractal.

Lemma 32. *Let X be a Peano continuum. Suppose that there are Y^L, Y, Y^R Peano subcontinua of X such that Y contains a free Cantor set φ. Suppose further that $X = Y^L \cup Y \cup Y^R$, $Y^L \cap Y = \{\varphi(\{0\}^\omega)\}$, $Y \cap Y^R = \{\varphi(\{1\}^\omega)\}$ and $Y^L \cap Y^R = \emptyset$. Then X is a topological fractal.*

Proof. We aim to use the Observation 22 with $n = 2$ and $m = 1$. Thus we need to construct three functions with certain properties. All three constructions will be based on the General Mapping Theorem 4.

However before we start, let us observe that all the sets listed below are open:

- $Y^L \setminus \{\varphi(\bar{0})\}$,
- $Y \setminus \{\varphi(\bar{0}), \varphi(\bar{1})\}$,
- $Y^R \setminus \{\varphi(\bar{1})\}$,
- $Y^L \cup (Y \setminus \{\varphi(\bar{1})\})$,
- $Y^R \cup (Y \setminus \{\varphi(\bar{0})\})$

and for every $s \in 2^{<\omega}$:

- $Y^L_s \setminus \{\varphi(s^\frown \bar{0}), \varphi(s^\frown 0^\frown \bar{1})\}$,
- $Y_s \setminus \{\varphi(s^\frown 0^\frown \bar{1}), \varphi(s^\frown 1^\frown \bar{0})\}$,
- $Y^R_s \setminus \{\varphi(s^\frown 1^\frown \bar{0}), \varphi(s^\frown \bar{1})\}$,
- $Y^L_s \cup (Y_s \setminus \{\varphi(s^\frown \bar{0}), \varphi(s^\frown 1^\frown \bar{0})\})$
- $Y^R_s \cup (Y_s \setminus \{\varphi(s^\frown 0^\frown \bar{1}), \varphi(s^\frown \bar{1})\})$.

This observation makes the verification of upper–continuity trivial in what follows. Checking that all the listed sets are indeed open in X is straightforward by induction and by cases, for example

$$Y \setminus \{\varphi(\bar{0}),\, \varphi(\bar{1})\} = X \setminus (Y^L \cup Y^R),$$

$$Y^L_\emptyset \setminus \{\varphi(\bar{0}),\, \varphi(0^\frown\bar{1})\} = (Y \setminus \{\varphi(\bar{0}),\, \varphi(\bar{1})\}) \setminus (Y_\emptyset \cup Y^R_\emptyset)$$

and for every $s \in 2^{<\omega}$:

$$Y^L_{s^\frown 0} \setminus \{\varphi(s^\frown\bar{0}),\, \varphi(s^\frown 00^\frown\bar{1})\} = (Y^L_s \setminus \{\varphi(s^\frown\bar{0}),\, \varphi(s^\frown 0^\frown\bar{1})\}) \setminus (Y_{s^\frown 0} \cup Y^R_{s^\frown 0}).$$

Let us start with the construction. Firstly we will define a sequence $(C_n)_{n\in\mathbb{N}}$ of maps from X to the space of subintervals of I. For any $m \in N$ define $c_m := \frac{1}{2} + \sum_{i=0}^{m-1}(-1)^{s(i)+1} \cdot \frac{1}{2^{2+i}}$. For $n \in \mathbb{N}$ let

$$C_n(x) := \begin{cases} \{0\}; & x \in Y^L \setminus \{\varphi(\bar{0})\}, \\ \{1\}; & x \in Y^R \setminus \{\varphi(\bar{1})\}, \\ \{c_m\}; & x \in Y_s \setminus \{\varphi(s^\frown 0^\frown\bar{1}),\, \varphi(s^\frown 1^\frown\bar{0})\},\ s \in 2^m,\ m \le n, \\ [c_m - \frac{1}{2^{1+m}};\, c_m]; & x \in Y^L_s,\ s \in 2^n, \\ [c_m;\, c_m + \frac{1}{2^{1+m}}]; & x \in Y^R_s,\ s \in 2^n. \end{cases}$$

The sequence $(C_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of the General Mapping Theorem 4, namely for every $n \in \mathbb{N}$ the function C_n is upper–continuous and satisfies $\bigcup\{F_n(x),\, x \in X\} = I$, while for every $x \in X$ the sequence $(F_n(x))_{n\in\mathbb{N}}$ is a decreasing sequence formed by continua with diameters converging to zero. Hence the function $c : X \to I$ given by $c(x) := \bigcap_{n\in\mathbb{N}} C_n(x)$ is well–defined, continuous and onto I. Let $h : I \to Y^R$ be a surjective continuous map and let $g(= g_1) := h \circ c$.

Secondly we will define by induction a sequence $(F^0_n)_{n\in\mathbb{N}}$ of maps from X to the space of (Peano) subcontinua of $Y^L \cup Y^L_\emptyset \cup Y_\emptyset$. By Observation 12 there exist continuous onto maps $h_3 : Y^R \to Y^L$, $h_2 : Y^L \to Y_\emptyset$ such that $h_3(\varphi(\bar{1})) = \varphi(\bar{0})$, $h_2(\varphi(\bar{0})) = \varphi(0^\frown\bar{1})$. Define F^0_1 by:

$$F^0_1(x) := \begin{cases} \{h_3(x)\}; & x \in Y^R \setminus \{\varphi(\bar{1})\}, \\ \{h_2(x)\}; & x \in Y^L \setminus \{\varphi(\bar{0})\}, \\ Y^L_\emptyset; & x \in Y. \end{cases}$$

Suppose that we have already found F^0_n for some $n \in \mathbb{N}$ and we will define F^0_{n+1}. For every $s \in 2^{<\omega}$, $|s| = n-2$, there exists a continuous surjective map $h^0_s : Y_s \to Y_{0^\frown s^{op}}$ such that $h^0_s(\varphi(s^\frown 0^\frown\bar{1})) = \varphi(0^\frown s^{op\frown} 1^\frown\bar{0})$, $h^0_s(\varphi(s^\frown 1^\frown\bar{0})) = \varphi(0^\frown s^{op\frown} 0^\frown\bar{1})$. Let

$$F^0_{n+1}(x) := \begin{cases} \{h^0_s(x)\}; & x \in Y_s \setminus \{\varphi(s^\frown 0^\frown\bar{1}),\, \varphi(s^\frown 1^\frown\bar{0})\},\ s \in 2^{<\omega},\ |s| = n-2, \\ Y^R_{0^\frown s^{op}}; & x \in Y^L_s,\ s \in 2^{<\omega},\ |s| = n-2, \\ Y^L_{0^\frown s^{op}}; & x \in Y^R_s,\ s \in 2^{<\omega},\ |s| = n-2, \\ F^0_n(x); & \text{otherwise.} \end{cases}$$

Thirdly we will define by induction a sequence $(F^1_n)_{n\in\mathbb{N}}$ of maps from X to the space of (Peano) subcontinua of $Y^R_\emptyset$. Let

$$F^1_1(x) := \begin{cases} \{\varphi(\bar{1})\}; & x \in Y^R \setminus \{\varphi(\bar{1})\}, \\ \{\varphi(1^\frown\bar{0})\}; & x \in Y^L \setminus \{\varphi(\bar{0})\}, \\ Y^R_\emptyset; & x \in Y. \end{cases}$$

Suppose that we have already find F^1_n for some $n \in \mathbb{N}$ and we will define F^1_{n+1}. For every $s \in 2^{<\omega}$, $|s| = n-2$, there exists a continuous surjective map $h^1_s : Y_s \to Y_{1^\frown s}$ such that $h^1_s(\varphi(s^\frown 0^\frown \bar{1})) = \varphi(1^\frown s^\frown 0^\frown \bar{1})$, $h^1_s(\varphi(s^\frown 1^\frown \bar{0})) = \varphi(1^\frown s^\frown 1^\frown \bar{0})$. Let

$$F^1_{n+1}(x) := \begin{cases} \{h^1_s(x)\}; & x \in Y_s \setminus \{\varphi(s^\frown 0^\frown \bar{1}), \varphi(s^\frown 1^\frown \bar{0})\},\ s \in 2^{<\omega},\ |s| = n-2, \\ Y^L_{1^\frown s}; & x \in Y^L_s,\ s \in 2^{<\omega},\ |s| = n-2, \\ Y^R_{1^\frown s}; & x \in Y^R_s,\ s \in 2^{<\omega},\ |s| = n-2, \\ F^1_n(x); & \text{otherwise.} \end{cases}$$

Again, both sequences $(F^0_n)_{n\in\mathbb{N}}$, $(F^1_n)_{n\in\mathbb{N}}$ satisfy the hypotheses of the General Mapping Theorem 4, hence the functions $f_0 : X \to Y^L \cup Y^L_\emptyset \cup Y_\emptyset$, $f_1 : X \to Y^R_\emptyset$ given by $f_0(x) := \bigcap_{n\in\mathbb{N}} F^0_n(x)$, $f_1 := \bigcap_{n\in\mathbb{N}} F^1_n(x)$ are well–defined, continuous and onto. Thus we immediately obtain $X = f_0(X) \cup f_1(X) \cup g(X)$.

Notice that g is constant on Y' for every $Y' \in \{Y^L, Y^R\} \cup \{Y_s,\ s \in 2^{<\omega}\}$, f_1 is constant on Y^L and on Y^R, $f_1|_{Y_s} = h^1_s$ and $f_0|_{Y_s} = h^0_s$ for every $s \in 2^{<\omega}$ and $f_0|_{Y^L} = h_2$, $f_0|_{Y^R} = h_3$. Thus in particular $f_0(Y^L) = Y_\emptyset$, $f_0(Y^R) = Y^L$ and $f_0(Y_s) = Y_{0^\frown s^{op}}$, $f_1(Y_s) = Y_{1^\frown s}$ for every $s \in 2^{<\omega}$. This immediately gives us that for every $f \in \bigcup_{i=0}^\infty \{f_0, f_1\}^i$ the function $g \circ f \circ g$ is constant since $g(X) = Y^R$.

Notice that from the definition of the functions f_0, f_1 follows that $f_1(X) = Y^R_\emptyset \subseteq Y$, $f_0(Y) = Y^L_\emptyset \subseteq Y$, $(f_0)^3(X) \subseteq (f_0)^2(Y^L \cup Y) \subseteq f_0(Y^L_\emptyset \cup Y_\emptyset) \subseteq f_0(Y) \subseteq Y^L_\emptyset$. Thus $f(X)$ is a subset of $Y^L_\emptyset$ or of $Y^R_\emptyset$ for every $f \in \{f_0, f_1\}^3$.

Finally, notice that $f_0(Y^M_s) \subseteq Y^N_t$ for every $s \in 2^{<\omega}$ and $M \in \{L, R\}$, where $t \in 2^{<\omega}$ satisfies $|t| = |s| + 1$ and $N \in \{L, R\} \setminus \{M\}$. Similarly $f_1(Y^M_s) = Y^M_t$ for every $s \in 2^{<\omega}$ and $M \in \{L, R\}$, where $t \in 2^{<\omega}$ satisfies $|t| = |s| + 1$. Let $\varepsilon > 0$, then there exists $k \in \mathbb{N}$ such that for every $s \in 2^{<\omega}$ satisfying $|s| \geq k$ the diameters of X^L_s and X^R_s are less than ε. Let $f \in \{f_1, \ldots, f_n\}^{k+3}$, then we obtain by induction from the preceding that $f(X) \subseteq Y^M_s$ for some $s \in 2^{<\omega}$, $|s| = k$. Therefore the diameter of $f(X)$ is less then ε, thus all hypotheses of Observation 22 are satisfied and hence X is a topological fractal. □

Remark. Strictly speaking, we do not need the General Mapping Theorem to find the functions c, f_0, f_1. Alternatively we can define directly their restrictions to the subset $Y^L \cup Y^R \cup \bigcup\{Y^L_s \cup Y^R_s,\ s \in 2^{<\omega}\}$, which is a dense subset of X. Moreover, the functions f_0, f_1 can be quite easily defined directly on the whole space X.

The reason why we use General Mapping Theorem is thus not the construction itself, but the (proof of the) continuity of the resulting functions. In more detail, if we chose to construct the functions c, f_0, f_1 directly, we would then have to prove they are continuous, respectively their restrictions to the subset $Y^L \cup Y^R \cup \bigcup\{Y^L_s \cup Y^R_s,\ s \in 2^{<\omega}\}$ are uniformly continuous so we can extend them to be continuous maps defined on the whole space X. But it turns out that there is probably no comfortable such proof. In fact, the straightforward proof is likely to be the most easiest one, while it is technical and in principle it just copies the proof of the General Mapping Theroem.

Let us summarize what we have proved so far.

Theorem 33. *Every Peano continuum with uncountably many cut points is a topological fractal.*

Proof. The assertion folows immediately from Lemmata 31 and 32. □

Theorem 33 strengthens significantly the result of Dumitru who proved in Dumitru [2011] that every continuum of the form $X \cup A$, where $|X \cap A| = 1$ and A is an arc, is a topological fractal. The fractal structure built by Dumitru consists of three maps just like the structure we built up in the proof of Lemma 32.

Next we will further generalize Theorem 33 to so–called local cut points.

Definition 34. *Let X be a topological space. We say that $x \in X$ is a local cut point of X if there exists U a connected neighborhood of x such that $U \setminus \{x\}$ is not connected.*

It is easy to observe, that every local cut point of any connected space is also a local cut point of that space, but the converse is not true.

Lemma 35. *Let (X, d) be a Peano continuum with uncountably many local cut points. Then there either exist Y^L, Y, Y^R Peano subcontinua of X, Y containing a free Cantor set φ, such that $X = Y^L \cup Y \cup Y^R$, $Y^L \cap Y = \{\varphi(\{0\}^\omega)\}$, $Y \cap Y^R = \{\varphi(\{1\}^\omega)\}$ and $Y^L \cap Y^R = \emptyset$, or there are Y, Z Peano subcontinua of X, Y containing a free Cantor set φ, such that $X = Y \cup Z$, $Y \cap Z = \{\varphi(\bar{0}), \varphi(\bar{1})\}$.*

Proof. If the set of all cut points of X is uncountable Lemma 31 applies, so we may suppose that there are at most countably many cut points. Let $\mathscr{U}$ be a countable base of open sets in X formed by nonempty connected sets. For P, U, $V \in \mathscr{U}$ let $D(P, U, V) \subseteq X$ be the set of those points $x \in X$ such that there exist A, $B \subseteq P$ disjoint, open and satisfying $P \setminus \{x\} = A \cup B$, $U \subseteq A$, $V \subseteq B$. In other words, $D(P, U, V)$ is the set of those local cut points of X that separates U and V in P.

We will prove that $\bigcup\{D(P, U, V);\ P, U, V \in \mathscr{U}\}$ is the set of all local cut points of X. Let $x \in X$ be a local cut point of X, let $P' \subseteq X$ connected neighborhood of x such that $P' \setminus \{x\}$ is not connected and $P \in \mathscr{U}$ such that $x \in P \subseteq P'$. Suppose for contradiction that $P \setminus \{x\}$ is connected. Let A, $B \subseteq P' \setminus \{x\}$ be disjoint, open and satisfying $A \cup B = P' \setminus \{x\}$, then either $P \setminus \{x\} \subseteq A$ or $P \setminus \{x\} \subseteq B$ since it is connected. We may suppose that $P \setminus \{x\} \subseteq A$. Thus $P \cap B = \emptyset$, but then $P' = (A \cup \{x\}) \cup B = (A \cup P) \cup B$ where $A \cup P$, B are open and disjoint, which contradicts the connectedness of P'.

Therefore there are A, $B \subseteq P \setminus \{x\}$ disjoint, open and satisfying $A \cup B = P \setminus \{x\}$, and hence also U, $V \in \mathscr{U}$ such that $U \subseteq A$, $V \subseteq B$. Thus indeed $\bigcup\{D(P, U, V);\ P, U, V \in \mathscr{U}\}$ is the set of all local cut points of X and hence it is uncountable by assumption. Since $\mathscr{U} \times \mathscr{U} \times \mathscr{U}$ is only countable, there exist P, U, $V \in \mathscr{U}$ such that $D(P, U, V)$ is uncountable.

Since connected and path components coincide in Peano continua and P is an open connected set, it is a path connected. Fix arbitrary $u \in U$, $v \in V$ and some arc $h : I \to P$ such that $h(0) = u$, $h(1) = v$. Then the set $U \cup h(I) \cup V \subseteq P$ is connected and contains both U, V and thus, by definiton, it can not be a subset of $X \setminus \{x\}$ for any $x \in D(P, U, V)$. But for every $x \in D(P, U, V)$ we have $U, V \subseteq P \setminus \{x\}$ and therefore, by the above, $x \in h(I)$. Hence $D(P, U, V)$ is a subset of $h(I)$.

Let $x \in h(I) \setminus D(P, U, V)$, if $x \in U \cup V$ then $U \cup V$ is an open subset of $P \setminus D(U, V)$ containing x. Otherwise the connected component of $P \setminus \{x\}$ containing U contains also V. Since conneted and path components of open sets coincide in Peano continua, there exists an arc $a : I \to P \setminus \{x\}$ such that $a(0) = u$, $a(1) = v$. Let $\varepsilon := d(a(I), x) > 0$, then $B(x, \varepsilon) \cap P \subseteq P \setminus D(U, V)$ since for every $y \in B(x, \varepsilon) \cap P$ either $y \in U \cup V$ or the set $U \cup a(I) \cup V \subseteq P \setminus \{y\}$ is connected. Thus $D(P, U, V)$ is an uncountable closed subset of $h(I)$. In what follows, we will denote $D := D(P, U, V)$ for brevity.

By Observation 29, items 2 and 3, there exists $a < b \in (0, 1)$ such that for every $0 < \varepsilon \leq \min\{1 - a, b\}$ the sets $D \cap h([a, a + \varepsilon])$, $D \cap h([b - \varepsilon, b])$ are uncountable. Let $\varphi(\bar{0}) := h(a)$, $\varphi(\bar{1}) := h(b) \in h(I)$, then moreover $\varphi(\bar{0}), \varphi(\bar{1}) \in D$ since D is closed.

Let Y be the connected component of $X \setminus \{\varphi(\bar{0}), \varphi(\bar{1})\}$ that contains $h((a, b))$ united with $\{\varphi(\bar{0}), \varphi(\bar{1})\}$. Let $Z := (X \setminus Y) \cup \{\varphi(\bar{0}), \varphi(\bar{1})\}$, then Z is connected since if it was not, all points in the uncountable D would be cut points of X and we suppose that there are at most countably many cut points of X. By Observation 30 Y and Z are Peano subcontinua of X.

It remains to find a free Cantor set in Y and that process is completely analogical to the one in the proof of Lemma 31 so we omit it here. □

Lemma 36. *Let X be a Peano continuum and suppose that there are Y, Z Peano subcontinua of X, Y containing a free Cantor set φ, such that $X = Y \cup Z$, $Y \cap Z = \{\varphi(\bar{0}), \varphi(\bar{1})\}$. Then X is a topological fractal.*

Proof. Similarly to the proof of Lemma 32, we aim to use Observation 22, but this time with $n = 4$ and $m = 1$. We will denote the desired functions f_{00}, f_{01}, f_{10}, f_{11} and g. The constructions themself are straightforward but technical and moreover analogical to those in Lemma 32, hence we will omit technical details leading to General Mapping Theorem here and just ilustrate the idea in pictures. □

Theorem 37. *Any Peano continuum with uncountably many local cut points is a topological fractal.*

The last theorem finally strengthens significantly even the result of M. Nowak who proved in Nowak and Fernández-Martínez [2016] that every continuum containing an arc with nonempty interior or equivalently, every continuum of the form $X \cup A$, where $|X \cap A| \leq 2$ and A is an arc, is a topological fractal. The fractal structure built by Nowak consists of five maps just like the structure we built up in the proof of Lemma 32.

3.3 Regenerating fractals

The notion of a regenerating fractal was introduced in Nowak [2021]. Here are the definition and the main result from that article.

Definition 38. *A Hausdorff topological space A is called the regenerating fractal if for every nonempty, open set $U \subset A$ there exists a family $\mathscr{F}$ of continuous selfmaps on A, constant outside U, such that $(A; \mathscr{F})$ is a topological fractal.*

Theorem 39. *For every Peano continuum X which has $A \subset X$ regenerating fractal with nonempty interior, X is an underlying space for some topological fractal.*

Beside this result, the article Nowak [2021] contains some examples of regenerating fractals. Note that, since in particular an arc is a regenerating fractal, this results generalizes the fact that every Peano continuum with a free arc is a topological fractal. Yet this generalization is independent of ours.

With this result in mind, M. Nowak posed in her article the following two questions related to the old Hata's question whether every Peano continuum is a topological fractal.

Nowak [2021, Problem 4.9]: Has every Peano continuum which is underlying space for some topological fractal, a regenerating fractal as a subset with nonempty interior?

Nowak [2021, Problem 4.10] Is there a nontrivial Peano continuum without a regenerating fractal as a subset with nonempty interior?

The first question askes whether the converse of the above therorem is true or not, while the negative answer to the second question together with the above theorem would give us immediately positive answer to Hata's question. Using the main result proven in the previous section, we can show that these two questios of M. Nowak are actually equivalent. In fact, it should be clear that the second question is stronger than the first, so we are only left to justify the converse.

Lemma 40. *Let Y be a topological space, $Y = Y_L \cup A \cup Y_R$ where Y_L, A, Y_R are closed subsets of Y such that $Y_L \cap A = \{a_L\}$, $A \cap Y_R = \{a_R\}$, $Y_L \cap Y_R = \emptyset$ and $X \subseteq Y$ a regenerating fractal. Denote by B the intersection of $\{a_L, a_R\}$ and the set of all isolated points of $X \cap A$. Then $X' := (A \cap X) \setminus B$ is empty or a regenerating fractal too.*

Proof. Suppose that X' is nonempty and let $U \subseteq X'$ be an arbitrary nonempty open set in X', then there exists $U' \subseteq Y$ open such that $U = U' \cap X'$. Then $U \cap int(A) = U' \cap ((A \cap X) \setminus B) \cap int(A) = (U' \cap int(A)) \cap X$ is open in X. Moreover it is nonempty since $X' \setminus int(A)$ is finite with no isolated points of X' and therefore nowhere dense in X'.

Thus there exists $\mathscr{F}$ a set of continuous self–maps of X constant outside $U \cap int(A)$ such that $(X; \mathscr{F})$ is a topological fractal. Then $\mathscr{F}' := \{f|_{X'};\, f \in \mathscr{F}\}$ satisfies that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $g \in \mathscr{F}'^k$ the diameter of $g(X')$ is less then ε. It also satisfies $|X \setminus \bigcup\{f(X');\, f \in \mathscr{F}'\}| \leq 1$, but we may assume that $X = \bigcup\{f(X');\, f \in \mathscr{F}'\}$ since otherwise we would just add one (constant) function to $\mathscr{F}'$ and it is straightforward to check that this enriched set of maps would still satisfy the previous condition.

Fix arbitrary $z \in X'$ and for any $f \in \mathscr{F}'$ define $f' : X' \to X'$ by

$$f'(x) := \begin{cases} f(x); & f(x) \in int(A), \\ a_L; & f(x) \in Y_L \text{ and } a_L \in X', \\ a_R; & f(x) \in Y_R \text{ and } a_R \in X', \\ z; & \text{otherwise.} \end{cases}$$

Let $\mathscr{F}'' := \{f';\, f \in \mathscr{F}'\}$, then $\mathscr{F}''$ consists of continuous self–maps of X', for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for every $g \in \mathscr{F}''^k$ the diameter of $g(X')$ is less then ε and also $X' = \bigcup\{f(X');\, f \in \mathscr{F}''\}$. Thus $(X';\, \mathscr{F}'')$ is a topological fractal.

□

Remark. The space X' we have constructed satisfies that if A and X were connected, then X' is connected too, if further X was closed then X' is a continuum and thus a Peano continuum.

Theorem 41. *If every Peano continuum which is underlying space for some topological fractal contains a regenerating fractal as a subset with nonempty interior, then every Peano continuum contains a regenerating fractal as a subset with nonempty interior. In particular, if every Peano continuum which is underlying space for some topological fractal contains a regenerating fractal as a subset with nonempty interior, then every Peano continuum is a topological fractal.*

Proof. Suppose that every Peano continuum which is underlying space for some topological fractal contains a regenerating fractal as a subset with nonempty interior. Let $(X,\, d)$ be an arbitrary nontrivial Peano continuum, we may assume that it contains no point of 2^ω. Let us recall that the Cantor set 2^ω which is homeomorphic to the set $C = \{\sum_{n\in\omega} \frac{s(n)}{3^{n+1}};\, s \in 2^\omega\} \subseteq I$ via the map $\varphi : s \mapsto \sum_{n\in\omega} \frac{s(n)}{3^{n+1}}$. The idea of what we are going to do is, roughly speaking, to attach one copy of X in between every pair of consetive points of C, the "smaller" copy the closer the points are, and in this way we will obtain a connected topological fractal.

Fix some $a,\, b \in X$ such that $d(a,\, b) = diam(X)$ and for any $s \in 2^{<\omega}$ denote $(X_s,\, d_s) := (X \times \{s\},\, \frac{1}{3^{|s|+1} \cdot diam(X)} d)$, $a_s := (a,\, s)$, $b_s := (b,\, s) \in X_s$. Let Y be the set $2^\omega \cup \bigcup_{s\in 2^{<\omega}} X_s$ quotient by the smallest equivalence $\sim$ satisfying $a_s \sim s^\frown 0^\frown \bar{1}$ and $b_s \sim s^\frown 1^\frown \bar{0}$ for every $s \in 2^{<\omega}$.

We define a metric e on Y by $e(x,\, y) :=$

$$\begin{cases} d_s(x,\, y); & x,\, y \in X_s, \\ |x - y|; & x,\, y \in 2^\omega, \\ d_s(x,\, a_s) + |a_s - y|; & y \in 2^\omega,\, x \in X_s,\, s \in 2^{<\omega}, |a_s - y| < |b_s - y|, \\ d_s(x,\, b_s) + |b_s - y|; & y \in 2^\omega,\, x \in X_s,\, s \in 2^{<\omega}, |b_s - y| < |a_s - y|, \\ d_s(y,\, a_s) + |a_s - x|; & x \in 2^\omega,\, y \in X_s,\, s \in 2^{<\omega}, |a_s - x| < |b_s - x|, \\ d_s(x,\, b_s) + |b_s - y|; & x \in 2^\omega,\, y \in X_s,\, s \in 2^{<\omega}, |b_s - x| < |a_s - x|, \\ d_s(x,\, a_s) + |a_s - b_t| + d_t(b_t,\, y); & x \in X_s, y \in X_t,\, s \neq t, |a_s - b_t| < |b_s - a_t|, \\ d_s(x,\, b_s) + |b_s - a_t| + d_t(a_t,\, y); & x \in X_s, y \in X_t,\, s \neq t, |b_s - a_t| < |a_s - b_t|, \end{cases}$$

Notice that e is well–defined since $|b_s - a_s| = d_s(a_s,\, b_s)$ for every $s \in 2^{<\omega}$. Clearly e is symmetric and satisfies that $e(x,\, y) = 0$ if and only if $x = y$, so we only need to prove the triangle unequality. Let $x,\, y,\, z \in Y$ and we may assume that they are pairwise different since otherwise the unequality holds trivially. If there exists $s \in 2^{<\omega}$ such that $x,\, y,\, z \in X_s$ or $x,\, y,\, z \in C$, then the unequality holds just because d_s and $|\cdot|$ are metrics.

Suppose that $x,\, y \in X_s$ and $z \in C$ or $y \in X_t$ for some $s \neq t$. Suppose that $|a_s - z| < |b_s - z|$, respectively $|a_s - b_t| < |b_s - a_t|$; the other case is analogous.

Then there exists $\varepsilon \geq 0$ such that $e(x, z) = d_s(x, a_s) + \varepsilon$, $e(y, z) = d_s(y, a_s) + \varepsilon$. Thus

$$e(x, y) = d_s(x, y) \leq d_s(x, a_s) + d_s(y, a_s) + 2\varepsilon = e(x, z) + e(y, z),$$

$$e(x, z) = d_s(x, a_s) + \varepsilon \leq d_s(x, y) + d(y, a_s) + \varepsilon = e(x, y) + e(y, z),$$
$$e(y, z) = d_s(y, a_s) + \varepsilon \leq d_s(x, y) + d(x, a_s) + \varepsilon = e(x, y) + e(x, z).$$

Finally, suppose that $|X_s \cap \{x, y, z\}| \leq 1$ for every $s \in 2^{<\omega}$. Let $a_w := a_s$, $b_w := b_s$, $d_w := d_s$ if $w \in X_s \setminus C$ and $a_w := b_w := w$, $d_w(w, w) := 0$ if $w \in C$ for every $w \in \{x, y, z\}$. We may assume that $b_x < a_y \leq b_y < a_z$ since no point of $c \in C$ satisfies $a_x < c < b_x$, $a_y < c < b_y$ or $a_z < c < b_z$ and moreover the sests $\{a_x, b_x\}$, $\{a_y, b_y\}$, $\{a_z, b_z\}$ are pairwise disjoint. Thus

$$\begin{aligned} e(x, z) &= d_x(x, b_x) + |b_x - a_z| + d_z(a_z, z) \leq \\ &\leq d_x(x, b_x) + |b_x - a_y| + |a_y - b_y| + |b_y - a_z| + d_z(a_z, z) = \\ &= d_x(x, b_x) + |b_x - a_y| + d_y(a_y, b_y) + |b_y - a_z| + d_z(a_z, z) \leq \\ &\leq d_x(x, b_x) + |b_x - a_y| + d_y(a_y, y) + d_y(y, b_y) + |b_y - a_z| + d_z(a_z, z) = \\ &= e(x, y) + e(y, z). \end{aligned}$$

Further

$$\begin{aligned} e(x, y) &= d_x(x, b_x) + |b_x - a_y| + d_y(a_y, y) \leq d_x(x, b_x) + |b_x - a_y| + d_y(a_y, b_y) = \\ &= d_x(x, b_x) + |b_x - a_y| + |a_y - b_y| = d_x(x, b_x) + |b_x - b_y| \leq \\ &\leq d_x(x, b_x) + |b_x - a_z| \leq e(x, z) \end{aligned}$$

and similarly $e(y, z) \leq e(x, z)$. This concludes the proof that e is a metric.

To prove that (Y, e) is a Peano continuum, we will describe a continuous onto map $f : I \to Y$. Let $f|_C := \varphi^{-1}$ and for every $s \in 2^{<\omega}$ let $f|_{[s^\frown 0^\frown \bar{1},\, s^\frown 1^\frown \bar{0}]}$ be a surjective map onto Y_s satisfying $f(\varphi(s^\frown 0^\frown \bar{1})) = s^\frown 0^\frown \bar{1}$, $f(\varphi(s^\frown 1^\frown \bar{0})) = s^\frown 1^\frown \bar{0}$, there exists some by Observation 12. It is immediate that f is well–defined and surjective.

Let $s \in 2^{<\omega}$ and $x, y \in f([s^\frown \bar{0}, s^\frown 0^\frown \bar{1}])$. Let $a_x := x$ if $x \in 2^\omega$ and otherwise there exists $r \in 2^{<\omega}$ such that $x \in X_r$. Notice that in both cases $a_x \in 2^\omega \cap f([s^\frown \bar{0}, s^\frown 0^\frown \bar{1}]) = \{t \in 2^\omega;\ t_{|s|+1} = s^\frown 0\}$ and thus in particular in the latter case $|r| \geq |s|$ because $s^\frown \bar{0} \leq r^\frown 0^\frown \bar{1} \leq s^\frown 0^\frown \bar{1}$. Therefore $e(x, a_x) = d_r(x, a_x) \leq diam_{d_r}(X_r) = 1/(3^{|r|+1}) \leq 1/(3^{|s|+1})$. Similarly we find $a_y \in 2^\omega \cap f([s^\frown \bar{0}, s^\frown 0^\frown \bar{1}])$ such that $e(y, a_y) \leq 1/(3^{|s|+1})$. This gives us $e(x, y) \leq e(x, a_x) + e(a_x, a_y) + e(a_y, y) = e(x, a_x) + |a_x - a_y| + e(a_y, y) \leq 1/(3^{|s|+1}) + |s^\frown \bar{0} - s^\frown 0^\frown \bar{1}| + 1/(3^{|s|+1}) = 2/(3^{|s|+1}) + \sum_{i=|s|+2}^{\infty} = 3/(3^{|s|+1}) = 1/(3^{|s|})$. Thus $diam(f([s^\frown \bar{0}, s^\frown 0^\frown \bar{1}])) \leq 1/(3^{|s|})$ and hence $diam(f([s^\frown \bar{0}, s^\frown 0^\frown \bar{1}]))$ converges to 0 as $|s|$ tends to infinity. This implies that f is continuous since we already know from the construction that it is continuous on $[s^\frown 0^\frown \bar{1}, s^\frown 1^\frown \bar{0}]$ for every $s \in 2^{<\omega}$.

Finally, for every $s \in 2^\omega$ the sets $f([\bar{0}, s])$, $f([s, \bar{1}])$ are closed, hence the sets $f([\bar{0}, s))$, $f((s, \bar{1}])$ are open and moreover by construction they are disjoint. Therefore every point of $2^\omega \subseteq Y$ is a cutpoint of Y and thus by Corollary Y is a topological fractal. Thus by our assumption, there exists $R \subseteq Y$ a regenerating fractal with nonempty interior.

Notice that 2^ω is nowhere dense in Y; it is closed since it is an image of compact Cantor set and it is a closure of a nowhere dense set E. Thus there exists $s \in 2^{<\omega}$ such that $int(R) \cap X_s$ is nonempty. Since $Y = f([\bar{0}, s^\frown 0^\frown \bar{1}]) \cup X_s \cup f([s^\frown 1^\frown \bar{0}])$, $f([\bar{0}, s^\frown 0^\frown \bar{1}]) \cap X_s = \{s^\frown 0^\frown \bar{1}\}$, $X_s \cap f([s^\frown 1^\frown \bar{0}]) = \{s^\frown 1^\frown \bar{0}\}$, $f([\bar{0}, s^\frown 0^\frown \bar{1}]) \cap f([s^\frown 1^\frown \bar{0}]) = \emptyset$, by Lemma 40 $R' = (R \cap X_s) \setminus B$ is a topological fractal or the emptyset, where $B \subseteq X_s$, $|B| \leq 2$. In particular, $(int(R) \cap X_s) \setminus B$ is still nonempty and open in X_s. Thus $R' \subseteq X_s$ is a regenerating fractal with nonempty interior in X_s. Finally X_s is homeomorphic to X, thus the same holds for X and this concludes the proof.

□

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