**FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University**

BACHELOR THESIS

David Podivín

Spinor-helicity formalism and amplitudes in different dimensions

Institute of Particle and Nuclear Physics

Supervisor of the bachelor thesis: doc. RNDr. Karol Kampf, Ph.D.
Study programme: Physics
Study branch: Physics

Prague 2022

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like to extend my deepest gratitude to doc. RNDr. Karol Kampf, Ph.D. and Christoph Bartsch, MSc. for their guidance and patience. I am thankful for their advice and discussions that allowed this thesis to exist.

Lastly my thanks belong to my parents for their undying support.

Title: Spinor-helicity formalism and amplitudes in different dimensions

Author: David Podivín

Department: Institute of Particle and Nuclear Physics

Supervisor: doc. RNDr. Karol Kampf, Ph.D., Institute of Particle and Nuclear Physics

Consultant: Christoph Bartsch, MSc., Institute of Particle and Nuclear Physics

Abstract: This thesis focuses on spinor-helicity formalism its extensions to different dimensions and its use in constructing scattering amplitudes. We begin with an outline of this formalism in various dimensions and follow up with a brief introduction to constructing amplitudes from a bottom up approach. Following these ideas we calculate possible scattering amplitudes in a theory with a single massless scalar. We further discuss the influence of spacetime dimension on allowed interactions. Lastly we focus on reducing the degrees of freedom in allowed amplitudes by adding properties that interactions should satisfy. These properties are derived from behaviour of known theories.

Keywords: spinor-helicity formalism, amplitudes, soft limits, Adler zero

Contents

Introduction

Quantum field theory is currently our best description of the microscopic universe. It finds uses in numerous areas of physics as a powerful quantitative description. The standard method of calculating physical quantities starts with a Lagrangian, works out Feynman rules and finally arrives at physically relevant results. Despite its tremendous success over the past 70 years there are reasons to doubt whether this truly is the fundamental approach. From practical point of view the number of terms needed to be summed suffers from combinatorial explosion. In addition the starting point and result tend to be very simple expression but this simplicity is often lost in the intermediate steps. The most famous example of such simplicity is the Parke-Taylor formula. Which reduces dozens of pages filled with calculations to just a one line expression.

Many of these issues is what lead up to the conception of modern amplitudes methods. This program puts S-matrix at its centre point as it seeks to find new ways of looking at scattering amplitudes. Over the past two decades these methods have found an extensive use in many parts of particle physics, ranging from simplifying difficult calculations to finding entirely new objects hidden in plain sight.

The original methods were only applicable to Yang-Mills theory, but as time passed they have been extended to many others, including effective field theories. These extensions have been used in exploring space of all possible theories and classifying them [1]. Furthermore they have been applied to some physically relevant theories for example the non-linear σ-model or chiral perturbation theory [2].

This thesis focuses on the simplest case of constructing amplitudes directly from a simple set of properties. We follow an existing construction [2], which we partially redo and partially extend. We further focus on extending our construction to different dimensions.

We begin the first chapter with an introduction into the arguably most important part of modern amplitude methods the spinor-helicity formalism. We introduce the standard 4 dimensional formalism and then focus on extensions to momentum twistors and to other dimensions. In the second chapter we outline the ideas that will allow us to calculate amplitudes without having to ever consider Lagrangians or fields. We further describe processes by which it is possible to reduce degrees of freedom in constructed amplitudes. This is followed by the calculations of amplitudes in the third chapter. We focus on several different aspects and observe how all possible interactions behave. The fourth chapter focuses on an attempt to extend some of the ideas and methods to particles with mass. Lastly in the fifth chapter we give a short discussion on our results and mention some further possibilities.

1. Spinor-helicity formalism

Vital part of any theory is having proper language to express its data. Over the past two decades it became obvious that such language for massless scattering processes are spinor-helicity variables. This formalism naturally incorporates spin and on shell conditions while reducing computational difficulty connected with more traditional methods of parametrization.

Before constructing the formalism itself we describe several ideas that while not necessary will lay out some terminology that is going to prove useful in later sections. The following parts are standard in any quantum field theory textbook [3, 4] and as such will be presented only briefly.

1.1 Introductory remarks

1.1.1 Helicity and little group

From quantum mechanics we know that the state of a particle $|p^\mu, \lambda\rangle$ is not fully specified by particles momentum but requires additional labels. Analysis of these labels was first introduced by Wigner [5]. This analysis consist of asking what Lorentz transformations leave particles momentum unchanged. The group of all such transformations is called the little group.

For massive particles this additional label is just a spin. Massless particles require slightly more care. Their label should have 2 parts continuous and discrete. However the continuous part has not been observed experimentally and such tends to be considered unphysical. The discrete part is called helicity. Whereas particles with spin s have $2s+1$ possible states, helicity of a massless particle has only 2 permissible values $\pm h$.

Physically helicity corresponds to projection of spin into momentum. Helicity operator is usually defined through the spin operators $\widehat{\boldsymbol{S}}$ and momentum vector $\boldsymbol{p}$ as

$$\hat{h} = \frac{\widehat{\boldsymbol{S}} \cdot \boldsymbol{p}}{|\boldsymbol{p}|}. \tag{1.1}$$

Since parity transformation switches the sign of momentum but not of angular momentum it changes helicity h into $-h$ and vice versa.

During the following sections we will return to little group and helicity because they are important parts of spinor-helicity formalism. However most of our interest will lie in massless particles with spin zero. Which means we will not be taking full advantage of this formalism. Even so this discussion will help us with technical and foundational parts of spinor-helicity formalism.

1.1.2 Notation and conventions

Spinor-helicity formalism is very technical when it comes to convention and notation. Various choices lead to different representations of identical physical results. In this section we outline some of our general choices, conventions and other oddities that might be encountered in the following chapters.

We will work with mostly minus metric extended to arbitrary dimensions $\eta_{\mu\nu} = (1, -1, \ldots, -1)$. Corresponding Lorentz indices are denoted by greek letters $\mu, \nu = 0, \ldots, D-1$. Accordingly in this text D is going to stand for spacetime dimension.

As a rule we choose lower case indices both dotted and undotted to take on values $a, \dot{a} = 1, 2$. In addition they will always start at the beginning of the alphabet. Capital indices then take on the values $A, B = 1, 2, 3, 4$. For labels of momenta or anything else we use $i, j, \ldots = 1, 2, \ldots, n$.

Lastly we set $c = \hbar = 1$ and consider all momenta to be incoming. These choices will simplify many of our results.

Over the years there have been many introductory texts written with various entry points. While 4D spinors have received the most attention [6, 7], extensions to several other dimensions have been found as well. We will be focusing on 3D and 6D cases which are both discussed at length in [8].

1.2 4 dimensions

1.2.1 Spinors

The starting point of our construction will be the Clifford algebra

$$\{\gamma^\nu, \gamma^\mu\} = 2\eta^{\mu\nu} I. \tag{1.2}$$

Where I stands for identity and curly brackets for anticommutation. We consider the following solution

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{1.3}$$

With σ^i being Pauli matrices defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

From here it is possible to construct a four-vector of Pauli matrices as

$$(\sigma^\mu)_{a\dot{b}} = \left(I, \sigma^i\right)_{a\dot{b}}, \quad (\overline{\sigma}^\mu)^{\dot{a}b} = \left(I, -\sigma^i\right)^{\dot{a}b}. \tag{1.5}$$

The indices $a, \dot{b}$ are referred to as the spinor indices.

Arbitrary four-vector can be contracted with Lorentz indices of Pauli vector to get bispinors

$$(p_\mu \sigma^\mu)_{a\dot{b}} = p_{a\dot{b}} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}, \tag{1.6}$$

and

$$(p_\mu \bar{\sigma}^\mu)^{a\dot{b}} = p^{a\dot{b}} = \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}. \tag{1.7}$$

Now we observe that determinant of these 2×2 matrices is

$$\det p_{a\dot{b}} = \det p^{a\dot{b}} = p_0^2 - p_1^2 - p_2^2 - p_3^2 = p_\mu p^\mu = m^2. \tag{1.8}$$

Thus if p^μ is light-like its corresponding bispinor is a singular matrix. We can thus decompose it into 2-component variables

$$p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}}. \tag{1.9}$$

These new variables are called spinors. Spinor indices $a, \dot{b}$ can be raised and lowered using Levi-Civita tensors $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$. We make the choice $\epsilon^{12} = -\epsilon_{12} = 1$ and follow the convention for raising indices

$$\lambda^a = \epsilon^{ab} \lambda_b. \tag{1.10}$$

First let us note that the decomposition (1.9) is not unique. This is realized by rescaling

$$\lambda_a \to z\lambda_a \quad \text{and} \quad \tilde{\lambda}_{\dot{b}} \to \frac{1}{z} \tilde{\lambda}_{\dot{b}}, \tag{1.11}$$

which does not change the original momentum. This rescaling corresponds to little group transformations. Strictly speaking spinors have two indices, one spinor a and one little group index $\pm$, but since λ_a^+ is always going to have index $+$ it is usually omitted.[1] As such spinors transform nontrivially under little group unlike four-momenta.

The parameter z can generally be any non-zero complex number. However if we want to keep momenta real it is required to enforce $\lambda = \pm \tilde{\lambda}^*$. Resulting in z being just a phase. For practical reasons we will be considering complex momenta allowing us to work with unrelated spinors.

Because spinor indices can be raised and lowered they can also be contracted. This allows us to define spinor brackets as

$$\langle ij \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b, \qquad [ij] = \epsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{i\dot{a}} \tilde{\lambda}_{j\dot{b}}. \tag{1.12}$$

Both of these brackets are antisymmetric.

For the following section there are two identities with Pauli matrices that will prove useful [9]. The first is

$$2\eta^{\mu\nu} = (\sigma^\mu)_{a\dot{b}} (\bar{\sigma}^\nu)^{a\dot{b}}. \tag{1.13}$$

And the second is the Fierz identity

$$(\sigma^\mu)_{a\dot{a}} (\sigma_\mu)_{b\dot{b}} = 2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}. \tag{1.14}$$

[1] This index will be omitted in 4D and 3D, but in 6D it will become rather important.

Using the first identity (1.13) we can construct four-momentum from spinors as follows

$$p^\mu = \eta^{\mu\nu} p_\nu = \frac{1}{2} \left(\sigma^\mu\right)_{a\dot{b}} \left(\bar{\sigma}^\nu\right)^{a\dot{b}} p_\nu = \frac{1}{2} \left(\sigma^\mu\right)_{a\dot{b}} \lambda^a \tilde{\lambda}^{\dot{b}}. \tag{1.15}$$

From here we can construct the simplest Lorentz invariant object using the Fierz identity (1.14).

$$\begin{aligned} s_{ij} = \left(p_i + p_j\right)^2 = 2 p_i \cdot p_j &= \frac{1}{2} \left(\sigma^\mu\right)_{a\dot{b}} \lambda_i^a \tilde{\lambda}_i^{\dot{b}} \left(\sigma_\mu\right)_{c\dot{d}} \lambda_j^c \tilde{\lambda}_j^{\dot{d}} \\ &= \epsilon_{ac} \epsilon_{\dot{b}\dot{d}} \, \lambda_i^a \tilde{\lambda}_i^{\dot{b}} \lambda_j^c \tilde{\lambda}_j^{\dot{d}} = \langle ij \rangle \left[ji\right]. \end{aligned} \tag{1.16}$$

Where we used the notation $p \cdot k = p^\mu k_\mu$ and $\left(p_i + p_j\right)^2 = \eta_{\mu\nu} \left(p_i^\mu + p_j^\mu\right) \left(p_i^\nu + p_j^\nu\right)$. This relation results in $s_{ii} = 0$ and $s_{ij} = s_{ji}$, both of which are expected behaviour of s_{ij}.

As it stands spinor-helicity formalism encodes dimensional and on shell conditions. To get valid kinematics configuration we will require momentum conservation

$$\sum_{i=1}^{n} p_i^\mu = 0. \tag{1.17}$$

For simplicity all momenta were chosen to be incoming. This equation can be rewritten into several useful equivalent forms

$$\sum_{i=1}^{n} \lambda_{ia} \tilde{\lambda}_{i\dot{b}} = 0, \qquad \sum_{i=1}^{n} \langle ij \rangle \left[ik\right] = 0, \qquad \sum_{i=1}^{n} s_{ij} = 0. \tag{1.18}$$

Where the first equality is simply using definition (1.9), second one follows from contracting with λ_j^a and $\tilde{\lambda}_k^{\dot{b}}$ while for the last we chose $j = k$.

Finally it is simple to derive

$$p_{ia\dot{b}} \lambda_i^a = \lambda_{ia} \tilde{\lambda}_{i\dot{b}} \lambda_i^a = \langle ii \rangle \tilde{\lambda}_{i\dot{b}} = 0 \tag{1.19}$$

and

$$p_{ia\dot{b}} \tilde{\lambda}_i^{\dot{b}} = \lambda_{ia} \tilde{\lambda}_{i\dot{b}} \tilde{\lambda}_i^{\dot{b}} = \left[ii\right] \lambda_{ia} = 0. \tag{1.20}$$

The equations (1.19) and (1.20) are massless Dirac or Weyl equations in momentum space [8]. They will prove useful later, but we will not be using them directly in our calculations.

1.2.2 Momentum twistors

For methods used later in this text it is paramount that we can generate valid momenta configurations. To achieve this we would have to solve momentum conservation as nontrivial restriction. But on shell and dimensional conditions can be trivialized by changing variables. In 4D we can go one step further and arrive at variables that actually satisfy momentum conservation. These new variables are called momentum twistors.

Twistors are very difficult subject and describing them in any general form is well beyond the scope of this text. We thus follow the construction given in [6] with focus on momentum conservation and relating twistors to spinors.

We start by considering ordered set of n light-like momenta p_i^μ satisfying momentum conservation (1.17). Followed by defining four-vectors y_j and an arbitrary four-vector Q by

$$y_j^\mu = Q^\mu + \sum_{i=1}^{j} p_i^\mu \implies p_i^\mu = y_i^\mu - y_{i-1}^\mu \quad \text{and} \quad y_n^\mu = Q^\mu. \tag{1.21}$$

First of all the labels are considered to be mod n. Secondly the existence of y_j^μ is only possible if p_i^μ satisfy momentum conservation. Equivalently starting with y_j^μ we arrive at momenta p_i^μ that satisfy momentum conservation. These new four-vectors y_j^μ are often called dual coordinates.

Now we define the so called incidence relation

$$\mu_{i\dot{a}} = y_{ia\dot{a}}\lambda_i^a = y_{i-1a\dot{a}}\lambda_i^a. \tag{1.22}$$

The second equality follows from definition of y_j^μ and the massless Weyl equation (1.19). There are only two constraints on $y_{ia\dot{a}}$

$$\mu_{i\dot{a}} = y_{ia\dot{a}}\lambda_i^a, \quad \mu_{i+1\dot{a}} = y_{ia\dot{a}}\lambda_{i+1}^a. \tag{1.23}$$

Both of them are solved by choosing

$$y_{a\dot{a}}^i = \frac{\lambda_a^{i+1}\mu_{\dot{a}}^i - \lambda_a^i\mu_{\dot{a}}^{i+1}}{\langle ii+1\rangle}. \tag{1.24}$$

Now we use this relations to return reconstruct momenta $p_{a\dot{a}}^i$ from (1.21), break them into spinors and contract the result with λ^{ai+1}. After all these steps we arrive

$$\tilde{\lambda}_{\dot{a}}^i = \frac{\langle ii+1\rangle\mu_{\dot{a}}^{i-1} + \langle i+1i-1\rangle\mu_{\dot{a}}^i + \langle i-1i\rangle\mu_{\dot{a}}^{i+1}}{\langle ii+1\rangle\langle ii-1\rangle}. \tag{1.25}$$

Twistor itself is then defined as a four component vector

$$\mathcal{Z}^i = \begin{pmatrix}\lambda_a^i \\ \mu_{\dot{a}}^i\end{pmatrix}. \tag{1.26}$$

So for a given twistor we can derive corresponding spinors from (1.25) and (1.26). The fact that twistors satisfy momentum conservation should follow from construction, but we also provide direct proof in appendix A.

We now have several ways at our disposal to easily generate valid kinematics configurations. These configurations can later be used in numerical evaluations that we will be performing. In 4D twistors are the simpler and for us the preferred choice of variables. Mainly because they can be generated freely without the need to solve any constraints. However as they directly build on top of spinor we will be using both to some capacity.

1.3 3 dimensions

The first extension we will look at are going to be spinors in three dimensions. Spinor-helicity formalism in 3D can be build directly from 4D. For this section we follow construction in [8].

1.3.1 Spinors

Our starting point is once again going to be the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I. \tag{1.27}$$

For our purposes it is advantageous to take the 4D solution (1.3) and simply remove γ^2. Afterwards the remaining Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.28}$$

This removes imaginary numbers from bispinors. We once more follow this with defining Lorentz vector of Pauli matrices as

$$(\sigma^\mu)_{ab} = \left(I, \sigma^i\right)_{ab}, \quad (\overline{\sigma}^\mu)^{ab} = \left(I, -\sigma^i\right)^{ab}. \tag{1.29}$$

Contracting Lorentz indices with arbitrary momentum we get

$$(p_\mu \sigma^\mu)_{ab} = p_{ab} = \begin{pmatrix} p_0 + p_2 & p_1 \\ p_1 & p_0 - p_2 \end{pmatrix} = \begin{pmatrix} p^0 - p^2 & -p^1 \\ -p^1 & p^0 + p^2 \end{pmatrix}. \tag{1.30}$$

Which again results in

$$\det p_{ab} = p_0^2 - p_1^2 - p_2^2 = m^2 = 0. \tag{1.31}$$

So for light-like momenta bispinor is a singular matrix and can be broken into two component vectors. However bispinor is also symmetric which means that can be viewed as a "square" of a single spinor

$$p_{ab} = \lambda_a \lambda_b. \tag{1.32}$$

As such there is no $\tilde{\lambda}$ this time. Limiting ourselves to real momenta forces λ to be purely real or imaginary. Furthermore the only ambiguity left in the definition is just multiplication by sign. Little group this time is discrete $\mathbb{Z}_2$.

Since we took Pauli matrices directly from 4D many identities still hold with the added caveat of undotting indices. Adding the absence of $\tilde{\lambda}$ directly gives us

$$p^\mu = \frac{1}{2} (\sigma^\mu)_{ab} \lambda^a \lambda^b. \tag{1.33}$$

And further enforces

$$\langle ij \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b \implies s_{ij} = \langle ij \rangle^2. \tag{1.34}$$

Spinor bracket still transforms nontrivially under little group making s_{ij} the simplest Lorentz invariant object.

1.3.2 Momentum twistors

Working out momentum twistors in 3D requires quite a bit technical work. We will outline the derivation and summarize results that have been arrived at in [10] within our needs.

The starting point will be the dual coordinates and incidence relation as defined in 4D (1.21) and (1.22). Following by the consideration that bispinors in 3D are symmetric, which means that after we undot indices in y^i_{ab} we can decompose it into symmetric and antisymmetric parts

$$y^i_{ab} = \frac{(y^i_{ab} + y^i_{ba})}{2} + \frac{(y^i_{ab} - y^i_{ba})}{2}. \tag{1.35}$$

Followed by a simple demand that the second fraction is zero. For this section it is convenient to introduce the notation

$$\left[\tilde{i}\tilde{j}\right] = \lambda_{ia}\mu^a_j - \lambda_{ja}\mu^a_i. \tag{1.36}$$

Where μ_a is defined through incidence relation in (1.22). This allows us to write the symmetry conditions as

$$\left[\tilde{i}\tilde{i} - 1\right] = 0. \tag{1.37}$$

With steps similar to those in 4D and applying the conditions (1.37) it is possible to derive the formula

$$p^i_{ab} = \frac{\left[\tilde{i} - 1\tilde{i} + 1\right]}{\langle ii - 1\rangle\langle ii + 1\rangle}\lambda^i_a\lambda^i_b. \tag{1.38}$$

Now we can define shifted spinor $\tilde{\lambda}$ as

$$\tilde{\lambda}^i_a = \sqrt{\frac{\left[\tilde{i} - 1\tilde{i} + 1\right]}{\langle ii - 1\rangle\langle ii + 1\rangle}}\,\lambda^i_a. \tag{1.39}$$

This translates into

$$p^i_{ab} = \tilde{\lambda}^i_a\tilde{\lambda}^i_b, \tag{1.40}$$

which satisfies momentum conservation. Now we can build all relevant variables from $\tilde{\lambda}$.

Unfortunately conditions (1.37) aren't satisfied by (1.40) trivially and we will have to solve them, removing the main appeal of twistors as an unconstrained parametrization in our calculations. As such twistor parametrization can be useful tool for checking results but we will be primarily using spinors in 3D.

1.4 6 dimensions

The 6D spinor-helicity formalism was first introduced in [11]. We once again outline the main points of the constructions that will be required for our purposes.

The Clifford algebra we focus on this time will to be

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}I. \tag{1.41}$$

The standard choice of Pauli matrices is

$$\begin{aligned}
\sigma^0 &= i\sigma_1 \otimes \sigma_2 & \bar{\sigma}^0 &= -\, i\sigma_1 \otimes \sigma_2 \\
\sigma^1 &= i\sigma_2 \otimes \sigma_3 & \bar{\sigma}^1 &= i\sigma_2 \otimes \sigma_3 \\
\sigma^2 &= -\, \sigma_2 \otimes \sigma_0 & \bar{\sigma}^2 &= \sigma_2 \otimes \sigma_0 \\
\sigma^3 &= i\sigma_2 \otimes \sigma_1 & \bar{\sigma}^3 &= -\, i\sigma_2 \otimes \sigma_1 \\
\sigma^4 &= -\, \sigma_3 \otimes \sigma_2 & \bar{\sigma}^4 &= \sigma_3 \otimes \sigma_2 \\
\sigma^5 &= i\sigma_0 \otimes \sigma_2 & \bar{\sigma}^5 &= i\sigma_0 \otimes \sigma_2 .
\end{aligned} \tag{1.42}$$

Where the Pauli matrices on both side of tensor products are 4D Pauli matrices defined in (1.4). This choice has several advantages. Firstly all Pauli matrices are antisymmetric. Secondly it has clear parallels to 4D formalism. Bispinors are once again defined as contractions of Lorentz indices

$$p_{AB} = p_\mu \sigma^\mu_{AB} = \begin{pmatrix} 0 & p_5 + ip_4 & p_1 + ip_2 & p_0 - p_3 \\ -p_5 - ip_4 & 0 & -p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 & 0 & p_5 - ip_4 \\ -p_0 + p_3 & p_1 - ip_2 & -p_5 + ip_4 & 0 \end{pmatrix} \tag{1.43}$$

and

$$p^{AB} = p_\mu \bar{\sigma}^{\mu AB} = \begin{pmatrix} 0 & p_5 - ip_4 & p_1 - ip_2 & -p_0 - p_3 \\ -p_5 + ip_4 & 0 & p_0 - p_3 & -p_1 - ip_2 \\ -p_1 + ip_2 & -p_0 + p_3 & 0 & p_5 + ip_4 \\ p_0 + p_3 & p_1 + ip_2 & -p_5 - ip_4 & 0 \end{pmatrix}. \tag{1.44}$$

It is clear that $A, B = 1, 2, 3, 4$. These indices cannot be lowered or raised. Furthermore bispinors are antisymmetric and related by

$$p^{AB} = \frac{1}{2}\epsilon^{ABCD} p_{CD}. \tag{1.45}$$

Where we chose $\epsilon^{1234} = \epsilon_{1234} = 1$. Both bispinors are still singular matrices. After imposing condition $m^2 = 0$ their matrix ranks become 2. As such we should be able to break them into four 4-component vectors or two 4×2 spinors.

We thus define

$$p_i^{AB} = \lambda_i^{Aa} \lambda_{ia}^B, \qquad p_{iAB} = \lambda_{iA\dot{a}} \lambda_{iB}^{\dot{a}}. \tag{1.46}$$

First of all we do not use $\tilde{\lambda}$ as we cannot raise indices A, B, making λ^A and λ_A completely separate objects. Secondly indices $a, \dot{a}$ transform under little group. They can be raised and lowered using Levi-Civita tensor with $\epsilon^{12} = -\epsilon_{12} = 1$, so the final object p_i^{AB} is little group invariant. We can observe that helicity of a 6D massless particle is given by 2 numbers $a, \dot{a} = 1, 2$ and as such has 4 possible states. The leftover freedom in definition of spinors is now much larger as a result. It is possible to multiply the first spinor by any 2×2 matrix with unit determinant and the second by its inverse matrix.

Again let us note useful identities [11].

$$\mathrm{Tr}\left(\sigma^\mu \bar{\sigma}^\nu\right) = 4\eta^{\mu\nu}, \tag{1.47}$$

and

$$\sigma^{\mu}_{AB}\bar{\sigma}^{CD}_{\mu} = -2\left(\delta^C_A\delta^D_B - \delta^D_A\delta^C_B\right). \tag{1.48}$$

We immediately use them in deriving momenta reconstruction

$$p^{\mu} = \eta^{\mu\nu}p_{\nu} = \frac{1}{4}\sigma^{\mu}_{AB}\bar{\sigma}^{\nu BA}p_{\nu} = \frac{1}{4}\sigma^{\mu}_{AB}\lambda^{Ba}\lambda^{A}_{a} = -\frac{1}{4}\sigma^{\mu}_{AB}\lambda^{Aa}\lambda^{B}_{a}. \tag{1.49}$$

Followed by

$$\begin{aligned} s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j &= \frac{1}{8}\sigma^{\mu}_{AB}\lambda^{Aa}_{i}\lambda^{B}_{ia}\bar{\sigma}^{CD}_{\mu}\lambda_{jC\dot{a}}\lambda^{\dot{a}}_{jD} \\ &= -\frac{1}{4}\left(\delta^C_A\delta^D_B - \delta^D_A\delta^C_B\right)\lambda^{Aa}_{i}\lambda^{B}_{ia}\lambda_{jC\dot{a}}\lambda^{\dot{a}}_{jD} \end{aligned} \tag{1.50}$$

Now we return to our convention of raising lower case indices. The difference between raising first and second index is just a minus sign. Looking at the last form we see that the expressions inside brackets will be identical. This leads to

$$s_{ij} = \frac{1}{2}\lambda^{A}_{ib}\lambda^{B}_{ia}\lambda_{jB\dot{a}}\lambda_{jA\dot{b}}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} = \det\lambda^{A}_{ia}\lambda_{jA\dot{a}}. \tag{1.51}$$

Where the last equality holds by similar argument as in the previous step.

In 6D there are potentially more Lorentz invariant objects, but for our purposes s_{ij} will suffice. For more nuanced discussion we refer reader to the original source [11].

Twistor construction exists in 6D [12], however it strives far from simplifying our calculations so we will not be mentioning it.

2. Amplitudes

In this chapter we give an outline how to use previously introduced kinematic variables to construct scattering amplitudes. We begin by listing properties any amplitude should satisfy and follow with how to enforce them.

The approach presented here has seen an extensive use. We will focus on parts and ideas relevant to our purposes while we refer reader interested in more general discussion to [7, 13] for introductory texts and to [8] for more in depth exploration.

From this point onward we consider only identical particles with zero mass and spin. For later use we also define $s_{ijk} = (p_i+p_j+p_k)^2$ and $s_{ijkl} = (p_i+p_j+p_k+p_l)^2$.

2.1 General properties

We start by considering properties that should be satisfied by all possible amplitudes. Because we know how the resulting physics should behave we can demand quite strict conditions and evaluate all possibilities that satisfy them. Lastly by amplitude we exclusively understand tree-level amplitude.

More formally amplitude A is defined as

$$\langle p_f, h_f \mid \widehat{S} - \widehat{I} \mid p_i, h_i \rangle = i\,(2\pi)^4\,\delta\,(p_f - p_i)\,A_{fi}. \tag{2.1}$$

Where labels i, f correspond to initial and final states of a given interaction. From this definition it follows that amplitude is a function of four-momenta and helicity.

Furthermore we demand that any amplitude has the following properties.

1. Lorentz invariance

   Our specific constraint employed will be more strict. We demand that amplitude is a function of Lorentz invariant quantities. For purposes of this text we specifically mean s_{ij}. This can be expressed symbolically as

   $$A_n\left(\{p_i, h_i\}_{i=1}^n\right) = A_n\left(\{s_{ij}\}_{i,j=1}^n\right). \tag{2.2}$$

2. Analytical structure

   Specifically we require amplitudes to be rational functions of external momenta. All singularities are going to be simple poles corresponding to propagator going on shell. Residues located at these poles will be equal to products of lower-point amplitudes.

3. Symmetry

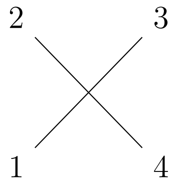
   We will be considering amplitudes that are invariant under certain permutations of external particles. The first case will be amplitudes invariant under arbitrary permutations of particles

   $$A_n\left(p_1, p_2, \ldots, p_n\right) = A_n\left(p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)}\right). \tag{2.3}$$

Which holds for all $\pi \in S_n$. The second case will be amplitudes invariant under cyclic permutations by which we understand

$$A_n\left(p_1, p_2, \ldots, p_n\right) = A_n\left(p_n, p_1, \ldots, p_{n-1}\right) = \cdots = A_n\left(p_2, p_3, \ldots, p_1\right). \tag{2.4}$$

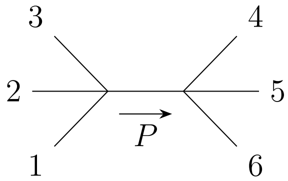
The first two points lead us to considering that amplitudes should have polynomial and rational parts. The polynomial part will belong to vertices. One such contribution to 4-point amplitude is shown in (2.5).



$$= iV_4(1,2,3,4) = i \text{ Polynomial in } s_{ij}. \tag{2.5}$$

Since this diagram features no internal lines there cannot be a propagator. Without propagator there are no poles in this contribution either.

We can now glue vertices together with a propagator to arrive at possible rational parts. Example of such contribution to 6-point amplitude is shown in (2.6).

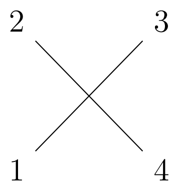


$$= iV_4(1,2,3,P)\frac{i}{s_{123}}iV_4(4,5,6,-P). \tag{2.6}$$

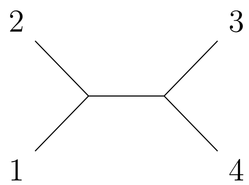
Where P stands for propagator which is determined by momentum conservation $P = -p_1 - p_2 - p_3 = p_4 + p_5 + p_6$. The minus sign in the right vertex is simply a direction of momentum, which can be chosen arbitrarily, but different on each side.

Now in order to satisfy given symmetry we need to sum over all relevant graph permutations. So for example the 4-point amplitude with Bose symmetry is shown in (2.7) and (2.8).

$$iA_4(1,2,3,4) =$$



$$+$$



$$+ \text{permutations}. \tag{2.7}$$

Leading to

$$A_4(1,2,3,4) = V_4(1,2,3,4) - \frac{V_3(1,2,P)V_3(3,4,-P)}{s_{12}} + \text{permutations}. \tag{2.8}$$

In order to calculate actual amplitudes we have to take care when summing over symmetries. Naturally we will want to avoid double counting any contributions. Problems arise when gluing identical vertices. Since summing over symmetry will always double count we will be diving such contributions by 2 in our numerics. The rest of prefactors will be absorbed into constants in front of amplitudes since for a given number of particles they will be identical. As such all of our results should be trusted up to two multiplicative constants. One in front of the polynomial part and the second in front of the rational part.

Finally in this description we obviously do not know any coupling constants, so there are many free parameters in all vertices. To reduce their numbers it is possible to impose further restrictions on amplitudes themselves.

2.2 Kinematics

As explained in the last section the main building blocks of amplitudes will be s_{ij}. Altogether for n-point scattering there are n^2 choices for s_{ij}. However n are trivially zero by on-shell conditions $s_{ii} = 0$. Further $n(n-1)/2$ are related by symmetry $s_{ij} = s_{ji}$. Finally n more are dependent by momentum conservation (1.18). Leaving us with $n(n-3)/2$ variables to consider. From (2.6) we can observe that it is useful to use momentum conservation to completely remove the dependence on the last momentum.

The list above does not include dimensional constraints. In D dimensions any $D+1$ momenta have to be linearly dependent. These constraints can be described in terms of Gram matrix G. The standard definition is $G_{ij} = s_{ij}$. We demand that this matrix has rank D or that every $(D+1) \times (D+1)$ minor is zero. These constraints are clearly highly nonlinear and in this text will be solved by spinor-helicity formalism. Dimensional constraints are referred to as Gram conditions.

Lastly if we consider n-point scattering in D dimensions we would expect that Gram conditions will play role starting at $n = D+1$. This however just implies that the entire Gram matrix is singular which is equivalent to momentum conservation. As such we expect Gram conditions to appear from $n = D+2$.

At 3-point interaction we see that there should be no variables left. This is reflected by momentum conservation, that gives us $s_{12} = s_{13} = s_{23} = 0$. The only possibility left is that 3-point interaction is just a constant, which satisfies all our previous demands. However for simplicity we consider this constant to be zero. This simplification will prove very useful for many of our later calculations. In a similar fashion we could consider constant terms for pretty much any vertex. But all of them will be turned to zero for simplicity.

2.3 Further relations

In order to reduce the number of free constants in amplitudes we need to impose additional constrains. These will allow us to both reduce degrees of freedom and potentially gain control over what theories we might construct. The choices and methods employed in this text are directly inspired by [2].

First and the most important constraint is going to be Adler zero. We take arbitrary momentum, shift it to $p_i \to zp_i$ and demand that amplitude vanishes for z going to 0. This can be written symbolically as

$$\lim_{z \to 0} A_n(zp_i) \propto z^\sigma. \tag{2.9}$$

The procedure of taking limit of single momentum to zero is called soft limit. While σ is referred to as the soft degree. We will be mostly interested in $\sigma = 1$ but will look at higher ones as well. Adler zero has been shown to be very powerful tool for categorizing effective field theories and can be used in constructing many theories that satisfy it recursively [1, 14].

Our implementation of Adler zero is going to be very straightforward. In our numerics we will be putting a factor z in front of the relevant variable and create momenta configurations with z as a parameter. Afterwards we will be able to explore behaviour of amplitudes with $zp_i \to 0$.

As was stated before, one of our interest lies in cyclically symmetric amplitudes. These amplitudes appear in many theories with additional structures like colour or flavour. In some important cases the actual tree-level amplitude can be written as [8]

$$\mathcal{A}_n = g^{n-2} \sum_{\pi \in S_n/Z_n} \mathrm{Tr}\left(T^{\pi(a_1)} \dots T^{\pi(a_n)}\right) A_n\left(\pi\left(1^{h_1}\right), \dots, \pi\left(n^{h_n}\right)\right). \tag{2.10}$$

Where the momenta were suppressed for their labels $p_i \to i$ and h_i is helicity of given particle. The first part contains all of the colour information, with T^a being generators of $SU(N)$. The second part, the so called partial amplitude, contains all of the kinematical information. The sum runs over all permutations S_n without cyclic Z_n to avoid double counting, as traces are cyclic by nature.

This behaviour allows us to calculate partial amplitudes with only kinematical information. These partial amplitudes satisfy our definition of cyclicity (2.4).

There are still some redundancies in the sum (2.10). They are realized through additional relations amongst partial amplitudes. First are the Kleiss-Kuijf (KK) relations [15]. They are given by the equation

$$A_n\left(1, \{\alpha\}, n, \{\beta\}\right) = (-1)^{|\beta|} \sum_{\pi \subset S_{n-2}} A_n\left(1, \{\pi\}, n\right). \tag{2.11}$$

Where α, β are disjoint subsets of momenta $2, \dots, n-2$. While π is subset of permutations that keeps the ordering of α and reverses β independently. The number of elements in set β is $|\beta|$.

The sum over S_n in (2.10) can be reduced to S_{n-1} by cyclicity and to S_{n-2} by KK [16].

To give out explicit example at 4-point the only nontrivial equation reads

$$A_4(1,2,4,3) + A_4(1,2,3,4) + A_4(1,3,2,4) = 0. \tag{2.12}$$

For higher points the number of equations grows very quickly, however not all of them are independent.

Second set of relations are the Bern-Carraso-Johansson (BCJ). They are given by

$$\sum_{i=2}^{n-1} \left(\sum_{j=1}^{i} s_{2j}\right) A_n(1,3,...,i,2,i+1,...,n) = 0. \tag{2.13}$$

These equations are called the fundamental BCJ. Unlike KK these relations are nonlinear in amplitudes. They were originally derived from observation that kinematical structure is related to colour structure [17].

To again give an example at 4-point the equation reads

$$s_{12}A_4\left(1,2,3,4\right)+\left(s_{12}+s_{23}\right)A_4\left(1,3,2,4\right)=0. \tag{2.14}$$

Last idea we will add is the so called chiral anomaly. Its meaning is discussed for example in [2], but for our purposes it will suffice to demand that at odd-point interactions all terms should include additional Lorentz invariant object

$$\epsilon^{\mu\nu\rho\sigma}p^i_\mu p^j_\nu p^k_\rho p^l_\sigma \equiv \epsilon\left(ijkl\right). \tag{2.15}$$

In spinor-helicity formalism we can rewrite it as

$$\epsilon\left(ijkl\right)=i\Bigg(\langle ik\rangle\langle jl\rangle\left[il\right]\left[jk\right]-\langle il\rangle\langle jk\rangle\left[ik\right]\left|jl\right|\Bigg). \tag{2.16}$$

Where we directly used the decomposition [9]

$$\epsilon^{\mu\nu\rho\lambda}=i\sigma^{\mu a\dot{a}}\sigma^{\nu b\dot{b}}\sigma^{\rho c\dot{c}}\sigma^{\lambda d\dot{d}}\Bigg(\epsilon_{ac}\epsilon_{bd}\epsilon_{\dot{a}\dot{d}}\epsilon_{\dot{b}\dot{c}}-\epsilon_{ad}\epsilon_{bc}\epsilon_{\dot{a}\dot{c}}\epsilon_{\dot{b}\dot{d}}\Bigg). \tag{2.17}$$

This object is Lorentz invariant as both parts feature λ and $\tilde{\lambda}$ for all momenta, so transformation under little group becomes trivial. Secondly this object does not scale to other dimensions trivially.

3. Results

Having established formalism and properties we can finally turn to constructing amplitudes. We will start with a vertex and work out basis of polynomials from all possible allowed options. Following that we move onto adding restrictions to reduce the number of free parameters. The calculations will be separated by power counting, number of interacting particles and dimension.

Because finding linear independence of high degree multivariable polynomials is a nontrivial task we will use numerical approach. To determine whether m polynomials are linearly independent is it enough to evaluate them at m points and then check for determinant or rank of matrix created this way.

Numerical evaluation can be run with 3 possible setups, directly and through spinors or twistors, where possible. Direct evaluation is done by directly generating s_{ij} with zero mass and momentum conservation, thus not including Gram conditions.

Spinors and twistors should be equivalent, but we will be using both methods. Throughout our calculations we will be checking all kinematical restriction to ensure that they are applied correctly.

The entire procedure can be summarized as follows. Since vertex should be a polynomial in s_{ij} we can simply generate all possible monomials in s_{ij} at a given power. In general for n variables and power m, which translates to s_{ij}^m, the number of possible monomials is $\binom{n+m-1}{n-1}$. Afterwards we sum over desired subset of permutations to get polynomials invariant under given symmetry and construct basis. That can be done by starting with arbitrary polynomial as a basis. We then go over all the remaining polynomials and check whether they are independent of the current basis. If a linearly independent one is found, it can be simply added to the basis. Having constructed basis we can impose further restriction and see how many elements remain.

At higher points this is accompanied by rational terms that are contributed by lower point interaction.

Lastly let us introduce notation to clarify results. We define

$$ {}_{\sigma}A^{(p)}_{n,s/c} \tag{3.1} $$

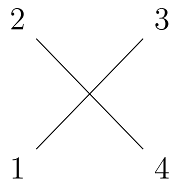
to denote amplitude at n-point and power $O(p^p)$. This amplitude is either cyclic c or fully symmetric s and has soft degree σ. If the amplitude satisfies KK or BCJ the lower c will be replaced by K/B respectively. Let us stress that both KK or BCJ are only sensible for cyclic amplitudes. Lastly if any of these are omitted general case is to be considered. As an example 5-point amplitude at $O(p^2)$ that is invariant under arbitrary permutation of particles and has soft degree 1 would be

$$ {}_{1}A^{(2)}_{5,s}. \tag{3.2} $$

3.1 4-Point

The only contribution to 4-point amplitude is the 4-point vertex alone. Accordingly we will not be making distinction between the two. This thus corresponds to

$$iA_4\left(1,2,3,4\right) = iV_4\left(1,2,3,4\right) = \tag{3.3}$$



The 4-point amplitude is by far the simplest case. This can be easily seen by number of remaining variables, which should be 2. However by using standard Mandelstam variables $s = s_{12} = s_{34},\ t = s_{13} = s_{24},\ u = s_{14} = s_{23}$, it is possible to simplify symmetry demands.

We begin by observing that polynomial being invariant under arbitrary permutation of momenta is equivalent to it being symmetric under permutations of s, t, u. This can be checked directly by writing down every possible permutation. As such it is easy to start writing down all possibilities at given order.

By similar observation we can see that cyclically invariant amplitude has to be invariant under exchange of s, u, while t remains untouched.

At the end one of the variables can be eliminated by momentum conservation which now reads $s + t + u = 0$.

Lastly Gram conditions do not appear here, so there is no need to separate calculations by dimension.

3.1.1 Fully symmetric

The previous discussion allows us to quickly evaluate all possible amplitudes to an arbitrary order. Results are summarized in table 3.1. The c-s in the table 3.1 represent arbitrary coefficients.

The length of bases as given in the last column of 3.1 can be directly extended to $O(p^{2n})$ as $\lfloor \frac{n+4}{4} \rfloor - \lfloor \frac{n+4}{6} \rfloor$. We have verified this formula up to order $O(p^{100})$.[1]

Also following from definition of Mandelstam variables it is clear that all variables disappear under Adler zero. The behaviour at $O(p^{2n})$ is thus

$$\lim_{z \to 0} A_{4,s}^{(2n)}(zp_i) \propto z^n. \tag{3.4}$$

Now the lowest order amplitude that has soft degree $\sigma = 2$ is

$${}_2A_{4,s}^{(4)}(1,2,3,4) = c_4^{(4)} \left(s^2 + t^2 + u^2\right), \tag{3.5}$$

or $\sigma = 3$

$${}_3A_{4,s}^{(6)}(1,2,3,4) = c_4^{(6)} \left(s^3 + t^3 + u^3\right). \tag{3.6}$$

[1] This sequence can be derived analytically from Hilbert series as done in [18].

Order	Independent terms	Size
$O(p^2)$	$c_4^{(2)}\,(s+t+u)=0$	0
$O(p^4)$	$c_4^{(4)}\,(s^2+t^2+u^2)$	1
$O(p^6)$	$c_4^{(6)}\,(s^3+t^3+u^3)$	1
$O(p^8)$	$c_4^{(8)}\,(s^4+t^4+u^4)$	1
$O(p^{10})$	$c_4^{(10)}\,(s^5+t^5+u^5)$	1
$O(p^{12})$	$c_{41}^{(12)}\,(s^6+t^6+u^6)+c_{42}^{(12)}\,(stu)^2$	2
$O(p^{14})$	$c_4^{(14)}\,(s^7+t^7+u^7)$	1
$O(p^{16})$	$c_{41}^{(16)}\,(s^8+t^8+u^8)+c_{41}^{(16)}\,(s^6t^2+\text{permutations})$	2
$O(p^{18})$	$c_{41}^{(18)}\,(s^9+t^9+u^9)+c_{41}^{(18)}\,(s^7t^2+\text{permutations})$	2
$O(p^{20})$	$c_{41}^{(20)}\,(s^{10}+t^{10}+u^{10})+c_{41}^{(20)}\,(s^8t^2+\text{permutations})$	2

Table 3.1: 4-point fully symmetric amplitudes

3.1.2 Cyclic

In a similar fashion to fully symmetric case we can quite quickly evaluate all possible polynomials.

We start at $O(p^2)$

$$A_{4,c}^{(2)}=c_4^{(2)}t=c_4^{(2)}s_{13}=-c_4^{(2)}\,(s+u)=-c_4^{(2)}\,(s_{12}+s_{23})\,. \tag{3.7}$$

The first form is the simplest and the one that we will be using in this text. The last one is the form usually used while discussing these calculations for example in [14].

The results are summarized in the table 3.2. The sequence of free constants as presented in last column of 3.2 can be generated higher with the formula $\lfloor n/4\rfloor+1$ at $O(p^{2n})$. We have again verified this up to $O(p^{100})$.

From previous discussion we see that all amplitudes satisfy Adler zero with the same soft degree as in the fully symmetric case (3.4).

Main difference from fully symmetric is that interaction at $O(p^2)$ becomes nontrivial and satisfies Adler zero which will cause difference at higher points.

Order	Independent terms	Size
$O(p^2)$	$c_4^{(2)}t$	1
$O(p^4)$	$c_{41}^{(4)}\left(s^2+u^2\right)+c_{42}^{(4)}t^2$	2
$O(p^6)$	$c_{41}^{(4)}\left(s^3+u^3\right)+c_{42}^{(4)}t^3$	2
$O(p^8)$	$c_{41}^{(8)}t^4+c_{42}^{(8)}\left(s^4+u^4\right)+c_{43}^{(8)}st^2u$	3
$O(p^{10})$	$c_{41}^{(10)}t^5+c_{42}^{(10)}\left(s^5+u^5\right)+c_{43}^{(10)}st^3u$	3
$O(p^{12})$	$c_{41}^{(12)}t^6+c_{42}^{(12)}\left(s^6+u^6\right)+c_{43}^{(12)}st^4u+c_{44}^{(12)}s^3u^3$	4
$O(p^{14})$	$c_{41}^{(14)}t^7+c_{42}^{(14)}\left(s^7+u^7\right)+c_{43}^{(14)}st^5u+c_{44}^{(14)}s^3tu^3$	4
$O(p^{16})$	$c_{41}^{(16)}t^8+c_{42}^{(16)}\left(s^8+u^8\right)+c_{43}^{(16)}st^6u+c_{44}^{(16)}s^4u^4+c_{45}^{(16)}s^3t^2u^3$	5
$O(p^{18})$	$c_{41}^{(18)}t^9+c_{42}^{(18)}\left(s^9+u^9\right)+c_{43}^{(18)}st^7u+c_{44}^{(18)}s^4tu^4+c_{45}^{(18)}s^3t^3u^3$	5
$O(p^{20})$	$c_{41}^{(20)}t^{10}+c_{42}^{(20)}\left(s^{10}+u^{10}\right)+c_{43}^{(20)}st^8u+c_{44}^{(20)}s^4t^2u^4+$ $c_{45}^{(20)}s^3t^4u^3+c_{46}^{(20)}s^5u^5$	6

Table 3.2: 4-point cyclic amplitudes

KK

We now move to other relations starting with KK. In this section we run calculations up to $O(p^{20})$. The equations will be applied by orders so explicitly

$$A_4^{(2n)}(1,2,4,3)+A_4^{(2n)}(1,2,3,4)+A_4^{(2n)}(1,3,2,4)=0. \tag{3.8}$$

The simplest case is obviously $O(p^2)$ with amplitude (3.7) which can be put into equation (3.8) directly to arrive at

$$c_4^{(2)}\left(s_{14}+s_{13}+s_{12}\right)=0, \tag{3.9}$$

which holds by momentum conservation and is thus trivial. Resulting in the amplitude

$$A_{4,K}^{(4)}(1,2,3,4)=c_4^{(2)}t. \tag{3.10}$$

At $O(p^4)$ we take the result from 3.2. The equation (3.8) now reads

$$c_{41}^{(4)}\left(s_{14}^2+s_{13}^2+s_{12}^2\right)+2c_{42}^{(4)}\left(s_{12}^2+s_{13}^2+s_{14}^2\right)=0. \tag{3.11}$$

Which gives the condition $c_{41}^{(4)}=-2c_{42}^{(4)}$. Giving us the full amplitude

$$A_{4,K}^{(4)}=c_{42}^{(4)}\left(s^2-2t^2+u^2\right) \tag{3.12}$$

This procedure is repeated for higher orders, where it is simpler to run evaluation numerically with sufficiently high number of randomly generated momenta.

Order	Independent terms	Size
$O(p^2)$	$c_4^{(2)}t$	1
$O(p^4)$	$c_{42}^{(4)}\left(s^2-2t^2+u^2\right)$	1
$O(p^6)$	$c_{42}^{(6)}\left(s^3-2t^3+u^3\right)$	1
$O(p^8)$	$c_{42}^{(8)}\left(s^4-2t^4+u^4\right)+c_{43}^{(8)}st^2u$	2
$O(p^{10})$	$c_{42}^{(10)}\left(s^5-2t^5+u^5\right)+c_{43}^{(10)}\left(st^3u-\frac{2}{5}t^5\right)$	2
$O(p^{12})$	$c_{41}^{(12)}\left(t^6-3st^4u+2s^3u^3\right)+c_{42}^{(12)}\left(s^6+u^6-6st^4u+4s^3u^3\right)$	2
$O(p^{14})$	$c_{41}^{(14)}\left(t^7-7s^3tu^3\right)+c_{42}^{(14)}\left(s^7+u^7-14s^3tu^3\right)+$ $c_{43}^{(14)}\left(st^5u-2s^3tu^3\right)$	3
$O(p^{16})$	$c_{41}^{(16)}\left(t^8-2s^4u^4+16s^3t^2u^3\right)+$ $c_{42}^{(16)}\left(s^8+u^8-4s^4u^4+32s^3t^2u^3\right)+c_{43}^{(16)}\left(st^6u+5s^3t^2u^3\right)$	3
$O(p^{18})$	$c_{42}^{(18)}\left(s^9-2t^9+u^9\right)+c_{44}^{(18)}\left(s^4tu^4+\frac{3}{7}t^9-\frac{10}{7}st^7u\right)+$ $c_{45}^{(18)}\left(s^3t^3u^3+\frac{2}{7}t^9-\frac{9}{7}st^7u\right)$	3
$O(p^{20})$	$c_{42}^{(20)}\left(s^{10}-2t^{10}+u^{10}\right)+c_{44}^{(20)}\left(s^4t^2u^4-\frac{1}{7}st^8u\right)+c_{45}^{(20)}s^3t^4u^3+$ $c_{46}^{(20)}\left(s^5u^5+\frac{1}{2}t^{10}-\frac{25}{14}st^8u\right)$	4

Table 3.3: 4-point amplitudes with KK implemented

The solutions for higher orders are given in table 3.3. The choices of remaining polynomials are rather arbitrary. In our case they were chosen to make results a bit more compact.

BCJ

BCJ relations are applied in a similar fashion to KK by order. So the relation now reads

$$sA_4^{(2n)}(1,2,3,4)+(s+u)\,A_4^{(2n)}(1,3,2,4)=0. \tag{3.13}$$

To give explicit example we take the $O(p^2)$ amplitude from (3.7) and directly substitute it into equation (3.13) to arrive at

$$c_4^{(2)}s\,(s+t+u)=0. \tag{3.14}$$

This equality is again trivial by momentum conservation.

$$A_{4,B}^{(2)}=c_4^{(2)}t \tag{3.15}$$

At $O(p^4)$ this is followed by taking the amplitude from 3.2 which results into

$$c_{41}^{(4)}\left(st^2+s^3+s^2u\right)+c_{42}^{(4)}\left(s^3+2su^2+st^2+t^2u+u^3\right)=0, \tag{3.16}$$

with the only solution being $c_{41}^{(4)} = c_{42}^{(4)} = 0$ which leads to

$$A_{4,B}^{(4)} = 0. \tag{3.17}$$

Results are summarized in table 3.4. Comparing our results it is possible to check that 3.4 is a subcase of 3.3. This means that our results would be the same if we started with KK compliant amplitude and then imposed BCJ or demanded BCJ directly.

Order	Independent terms	Size
$O(p^2)$	$c_4^{(2)} t$	1
$O(p^4)$	0	0
$O(p^6)$	$c_{42}^{(6)} \left(s^3 - 2t^3 + u^3\right)$	1
$O(p^8)$	$c_{43}^{(8)} st^2u$	1
$O(p^{10})$	$\frac{c_{43}^{(10)}}{5} \left(s^5 - 4t^5 + 5st^3u + u^5\right)$	1
$O(p^{12})$	$c_{41}^{(12)} \left(t^6 - s^6 - u^6 + 3st^4u - 2s^3u^3\right)$	1
$O(p^{14})$	$c_{41}^{(14)} \left(t^7 - 3st^5u - s^3tu^3\right) + c_{42}^{(14)} \left(s^7 + u^7 - 4st^5u - 6s^3tu^3\right)$	2
$O(p^{16})$	$c_{41}^{(16)} \left(t^8 - s^8 - u^8 + 2st^6u + 2s^4u^4 - 6s^3t^2u^3\right)$	1
$O(p^{18})$	$c_{41}^{(18)} \left(t^9 - 4st^7u + s^4tu^4 + 2s^3t^3u^3\right) +$ $c_{42}^{(18)} \left(s^9 + u^9 - 5st^7u + 8s^4tu^4 - 5s^3t^3u^3\right)$	2
$O(p^{20})$	$c_{42}^{(20)} \left(s^{10} - t^{10} + u^{10} - \frac{5}{3}st^8u - \frac{40}{3}s^4t^2u^4 + 2s^5u^5\right) + c_{45}^{(20)} s^3t^4u^3$	2

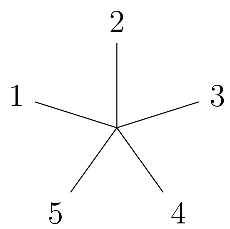
Table 3.4: 4-Point amplitudes with BCJ implemented

This concludes our discussion around 4-point vertex. We will return to these results when talking about 6 and 7-point interactions as they will include dependence on 4-point scattering.

3.2 5-Point

The only contribution is once again the 5-point vertex itself.

$$iA_5(1,2,3,4,5) = iV_5\,(1,2,3,4,5) = \tag{3.18}$$



Whereas at 4-point we were able to recast symmetry demands to permutations of variables, from now on we will have to rely on permutations of momenta labels.

Our choice of 5 independent variables is $s_{12}, s_{13}, s_{14}, s_{23}, s_{24}$. For simplicity or clarity we might use others, but those can be always expressed as linear combination of these.

Lastly the dimensional constraints become non-trivial. Specifically 3D splits while 4D and 6D remain identical to solutions without Gram conditions.

3.2.1 Fully symmetric

Starting at the lowest order we have

$$A_{5,s}^{(2)} = c_5^{(2)} \left(s_{12} + \text{perms}\right) = 0, \tag{3.19}$$

which again disappears by momentum conservation. In this section perms will stand for summing over all permutations of external momenta labels.

We can immediately go to higher powers and arrive at the table 3.5.

Order	Independent terms	Size
$O(p^2)$	$c_5^{(2)} \left(s_{12} + \text{perms}\right) = 0$	0
$O(p^4)$	$c_5^{(4)} \left(s_{12}^2 + \text{perms}\right)$	1
$O(p^6)$	$c_5^{(6)} \left(s_{12}^3 + \text{perms}\right)$	1
$O(p^8)$	$c_{51}^{(8)} \left(s_{12}^4 + \text{perms}\right) + c_{52}^{(8)} \left(s_{12}^2 s_{13}^2 + \text{perms}\right)$	2
$O(p^{10})$	$c_{51}^{(10)} \left(s_{12}^5 + \text{perms}\right) + c_{52}^{(10)} \left(s_{12}^3 s_{13} s_{23} + \text{perms}\right)$	2

Table 3.5: 5-point fully symmetric amplitudes up to $O(p^{10})$

These results do resemble 4-point rather closely. The first large difference comes at $O(p^{12})$. The amplitude now has 5 independent terms as

$$\begin{aligned} A_{5,s}^{(12)} &= c_{51}^{(12)} \left(s_{12}^6 + \text{perms}\right) + c_{52}^{(12)} \left(s_{12}^4 s_{13}^2 + \text{perms}\right) \\ &+ c_{53}^{(12)} \left(s_{12}^3 s_{13}^3 + \text{perms}\right) + c_{54}^{(12)} \left(s_{12}^4 s_{13} s_{23} + \text{perms}\right) \\ &+ c_{55}^{(12)} \left(s_{12}^2 s_{14}^2 s_{23}^2 + \text{perms}\right). \end{aligned} \tag{3.20}$$

The sequence of free parameters is $0, 1, 1, 2, 2, 5, 4, 8, 9, 13, 15$ up to $O(p^{22})$.

The 3D amplitudes start to differ from $O(p^8)$, where we get $c_{52}^{(8)} = 0$. This means that adding dimensional constraints of 3D makes the polynomials in the fourth row of 3.5 become multiples of each other. The results further match at $O(p^{10})$ and at $O(p^{12})$ we have $c_{55}^{(12)} = 0$. This statement again means that the last polynomial is just sum of the others. This is how dimensional constrains will always workout and such we will not be mentioning it explicitly further.

The 3D sequence is $0, 1, 1, 1, 2, 4, 3, 6, 7, 8, 11$ up to $O(p^{22})$. We shall return to why do these conditions appear only at higher powers later.

If the last two sequences are subtracted we get $0,0,0,1,0,1,1,2,2,5,4$, which is the result for 5-point symmetric amplitudes 3.1. This seems to be just a coincide albeit very intriguing one.

Adler zero

Demands for Adler zero become much less trivial at 5-point when compared to 4-point. Unlike 4-point where all variables went to zero, here only 10 out of the 20 possible do. This however means that any amplitude with only 1 free constant can never satisfy Adler zero. We might thus expect that the first nontrivial results will appear at $O(p^8)$.

This can be checked directly and we truly get

$$ {}_1A_{5,s}^{(2,4,6)} = 0 \tag{3.21} $$

as the only solution to Adler zero.

For higher orders nontrivial solutions do indeed exist starting at $O(p^8)$ with the condition $c_{52}^{(8)} = -3c_{51}^{(8)}$, leaving us with

$$ {}_1A_{5,s}^{(8)} = c_{51}^{(8)} \left(s_{12}^4 - 3s_{12}^2 s_{13}^2 + \text{perms} \right). \tag{3.22} $$

To again give examples at higher powers the amplitude at $O(p^{10})$ is

$$ {}_1A_{5,s}^{(10)} = c_{51}^{(10)} \left(s_{12}^5 - \frac{15}{4} s_{12}^3 s_{13} s_{23} + \text{perms} \right) \tag{3.23} $$

and at $O(p^{12})$

$$ \begin{aligned} {}_1A_{5,s}^{(12)} &= c_{51}^{(12)} \left(s_{12}^6 - 3s_{12}^4 s_{13} s_{23} - 6s_{12}^2 s_{14}^2 s_{23}^2 + \text{perms} \right) \\ &+ c_{52}^{(12)} \left(s_{12}^4 s_{13}^2 - 2s_{12}^2 s_{14}^2 s_{23}^2 + \text{perms} \right) \\ &+ c_{53}^{(12)} \left(s_{12}^3 s_{13}^3 - \frac{1}{2} s_{12}^4 s_{13} s_{23} + 2s_{12}^2 s_{14}^2 s_{23}^2 + \text{perms} \right). \end{aligned} \tag{3.24} $$

The number of free parameters in this case is $0,0,0,1,1,3,3,6,7,11$. Interestingly all of these results automatically have soft degree $\sigma = 2$.

For soft degree $\sigma = 3$ solutions can be found at $O(p^{16,18,20})$ with $2,3,6$ free constants. While soft degree $\sigma = 4$ produces solutions at $O(p^{16,18,20})$ with $1,1,5$ free constants. Demanding higher soft degrees yielded no solutions at our search depth but it would be reasonable to assume that they exist.

The 3D constrains reduce the numbers of free parameters after Adler zero to $0,1,2,2,4,5,6$ at $O(p^{8-20})$ with $\sigma = 2$. While $\sigma = 3$ leaves us with $1,2,3$ and $\sigma = 4$ with $0,0,2$ at powers $O(p^{16,18,20})$.

In the future sections we will be using these results only to $O(p^{10})$, the rest of these amplitudes serve as an illustration how do the dimensional constrains change the possible interactions.

3.2.2 Cyclic

We now return to cyclic amplitudes. At the lowest order the amplitude reads

$$\begin{aligned} A_{5,c}^{(2)} &= c_5^{(2)}\left(s_{12}+s_{15}+s_{23}+s_{34}+s_{45}\right) = c_5^{(2)}\left(s_{12}+\text{cyc}\right) \\ &= c_5^{(2)}\left(-s_{13}-2s_{14}+s_{23}+s_{24}\right). \end{aligned} \tag{3.25}$$

The equalities are just recasting with momentum conservation. For actual calculations it simpler to use the last form, however it is quite unclear that it is cyclically invariant, so throughout out this text we be using the shortened version.

At higher powers the amplitudes are

$$A_{5,c}^{(4)} = c_{51}^{(4)}\left(s_{12}^2+\text{cyc}\right) + c_{52}^{(4)}\left(s_{12}s_{13}+\text{cyc}\right) + c_{53}^{(4)}\left(s_{13}^2+\text{cyc}\right) \tag{3.26}$$

and

$$\begin{aligned} A_{5,c}^{(6)} &= c_{51}^{(6)}\left(s_{13}^3+\text{cyc}\right) + c_{52}^{(6)}\left(s_{12}s_{13}^2+\text{cyc}\right) + c_{53}^{(6)}\left(s_{12}^2s_{13}+\text{cyc}\right) \\ &+ c_{54}^{(6)}\left(s_{12}^3+\text{cyc}\right) + c_{55}^{(6)}\left(s_{13}s_{14}^2+\text{cyc}\right) \\ &+ c_{56}^{(6)}\left(s_{14}s_{23}s_{24}+\text{cyc}\right) + c_{57}^{(6)}\left(s_{14}s_{23}^2+\text{cyc}\right). \end{aligned} \tag{3.27}$$

The sequence of free parameters is $1, 3, 7, 14, 26, 42, 66, 99, 143, 201$ at $O(p^{2-20})$. While 3D constraints reduce it to $1, 3, 7, 13, 25, 39, 59, 85, 117, 159$ at $O(p^{2-20})$.

From these results it becomes clear that number of free parameters in this case explodes rapidly. As such it quickly becomes impractical to list amplitudes even in simplified forms.

Adler zero

Demanding Adler zero again becomes nontrivial constraint. It turns out that

$${}_1A_{5,c}^{(2,4)} = 0. \tag{3.28}$$

The first nonzero amplitude is thus $O(p^6)$ with constraints lowering the number of free constants in (3.27) from 7 to 3

$$c_{54}^{(6)} = c_{51}^{(6)} - \frac{c_{52}^{(6)}}{2} + \frac{c_{53}^{(6)}}{2},\ c_{55}^{(6)} = 6c_{51}^{(6)} - 3c_{52}^{(6)},\ c_{56}^{(6)} = 0,\ c_{57}^{(6)} = -\frac{c_{52}^{(6)}}{2} + \frac{c_{53}^{(6)}}{2}. \tag{3.29}$$

The amplitude is thus

$$\begin{aligned} {}_1A_{5,c}^{(6)} &= c_{51}^{(6)}\left(s_{13}^3+s_{12}^3+6s_{13}s_{14}^2+\text{cyc}\right) \\ &+ c_{52}^{(6)}\left(s_{12}s_{13}^2-\frac{s_{12}^3}{2}-3s_{13}s_{14}^2-\frac{s_{14}s_{23}^2}{2}+\text{cyc}\right) \\ &+ c_{53}^{(6)}\left(s_{12}^2s_{13}+\frac{s_{12}^3}{2}+s_{13}s_{14}^2+\frac{s_{14}s_{23}^2}{2}+\text{cyc}\right). \end{aligned} \tag{3.30}$$

The sequence of free parameters grows as $0, 0, 3, 9, 20, 35, 58, 90, 133, 190,$ after demanding Adler zero.
While restricting ourselves to 3D reduces it to $0, 0, 3, 8, 19, 32, 51, 76, 107, 148$.

KK

The KK relations now give 18 equations. One of which reads

$$A_5(1,2,3,5,4) + A_5(1,2,3,4,5) + A_5(1,2,4,3,5) + A_5(1,4,2,3,5) = 0, \quad (3.31)$$

which will again be applied by orders. Not all of these equations are independent, but rather than finding the basic set of them it is simpler to just solve them all.

First solution appears at $O(p^6)$ with the extra conditions on the amplitude (3.30)

$$c_{51}^{(6)} = 0, \qquad c_{53}^{(6)} = 2c_{52}^{(6)}. \quad (3.32)$$

Giving us

$$A_{5,K}^{(6)} = c_{52}^{(6)} \left(\frac{s_{12}^3}{2} - s_{13}s_{14}^2 + s_{12}s_{13}^2 + \frac{s_{14}s_{23}^2}{2} + 2s_{12}^2 s_{13} + \text{cyc} \right). \quad (3.33)$$

The number of free parameters changes to $0,0,1,2,5,8,14,21,32,45$. In 3D the sequence is reduced to $0,0,1,2,5,8,13,19,27,37$.

BCJ

Formula for BCJ now becomes

$$s_{12}A_5(1,2,3,4,5) + (s_{12}+s_{23})A_5(1,3,2,4,5) + (s_{12}+s_{23}+s_{24})\,A_5(1,3,4,2,5) = 0. \quad (3.34)$$

Solving this equation is rather nontrivial. The first solution appears at $O(p^{14})$ and is followed at higher powers with the number of free parameters $1,2,3,6$. These results are rather long and will not be written explicitly as we will not need them later.

Interestingly 3D conditions add solutions this time. With the first appearing at $O(p^{12})$ and then the sequence of free parameters grows as $1,2,3,4,8$. This goes directly against all previous results as 3D should be adding constraints and reducing degrees of freedom. Possible explanation of the phenomenon might be nonlinearity of BCJ, where the equations are just much simpler to solve in 3D.

Lastly let us note that results satisfy relation

$$\text{BCJ} \subset \text{KK} \subset \text{Adler zero}. \quad (3.35)$$

Meaning that anything satisfying BCJ automatically satisfy KK and that satisfies Adler zero.

ϵ - invariant

We now focus on adding the ϵ invariant to our results. We begin with simple observation

$$\epsilon\,(1234) = -\sum_{i=2}^{5} \epsilon\,(i234) = -\epsilon\,(5234)\,. \quad (3.36)$$

The first equality is just momentum conservation and the second follows from antisymmetry.

The equation (3.36) implies that $\epsilon\,(ijkl)$ goes to zero if we send any momentum to zero. Amplitude with a single ϵ thus immediately satisfies Adler zero.

It is simple to construct amplitudes from our results in the previous cyclic section as just the corresponding amplitude multiplied by $\epsilon(1234)$. These new results will always satisfy Adler zero. To explicitly separate these amplitudes we will be using lower index E.

Amplitudes start at $O(p^4)$ with

$$A_{5,E}^{(4)}(1234) = c_5^{(4)}\epsilon(1234)\,. \tag{3.37}$$

Following at higher powers with

$$\begin{aligned} A_{5,E}^{(6)}(1234) &= c_5^{(6)}\epsilon(1234)\,(s_{12} + \text{cyc})\,, \\ A_{5,E}^{(8)}(1234) &= \epsilon(1234)\left(c_{51}^{(8)}s_{12}^2 + c_{52}^{(8)}s_{12}s_{13} + c_{53}^{(8)}s_{13}^2 + \text{cyc}\right) \end{aligned} \tag{3.38}$$

and so on.

To close this section let us mention that we have found no solutions to either KK or BCJ with ϵ inserted.

3.3 6-Point

In the previous sections we focused on vertices alone. At 6-point in order to discuss amplitudes we need to include terms with propagators that will introduce dependence on lower points. For the purposes of this text we give a short overview of the 6-point vertex with the most of our attention being directed towards amplitudes and reducing the number of free constants in them.

Lastly we choose independent variables as $s_{12}, s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25}, s_{34}, s_{35}$.

Vertex

The process here is identical to those presented in previous sections. However let us outline few important points. First of all 3D constraints are still in effect and appear from $O(p^8)$. While 4D constraints now become nontrivial, but only at $O(p^{10})$, which also acts as an upper limit for calculations. Summary of results is given in the form of tables 3.6 and 3.7 as most of them are too long to be presented in their entirety.

It is also possible to solve KK for some powers, but as noted before that would only matter to us if we lacked 4-point interactions so these results are mostly presented to illustrate that 4D constrains truly appear at 6-point $O(p^{10})$.

Power	Free constants			Adler zero		
	3D	4D	6D	3D	4D	6D
2	0	0	0	0	0	0
4	1	1	1	0	0	0
6	2	2	2	1	1	1
8	3	4	4	2	2	2
10	5	5	6	3	3	4

Table 3.6: Number of free constants at 6-point for fully symmetric vertex

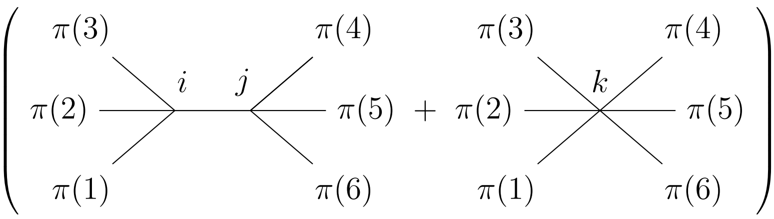
Power	Free constants			Adler zero		
	3D	4D	6D	3D	4D	6D
2	2	2	2	0	0	0
4	9	9	9	0	0	0
6	32	32	32	5	5	5
8	86	89	89	27	29	29
10	206	225	226	97	111	112

Table 3.7: Number of free constants at 6-point for cyclic vertex

3.3.1 Fully symmetric

We finally turn to 6-point amplitudes. The relevant contributions are given by (3.39).

$$iA^{(k)}_{6,s} = \sum_{\pi \in S_6}$$

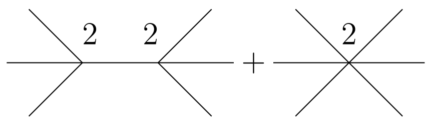


. (3.39)

Where i, j, k stand for $O(p^i), O(p^j), O\left(p^k\right)$ and they must satisfy $i + j - 2 = k$. For simplicity we will be dropping the momenta labels and all diagrams are going to be implicitly summed over relevant symmetry. We mention cases where we have to avoid double counting in results explicitly.

At $O(p^2)$ all relevant contributions are

$$iA^{(2)}_{6,s} =$$

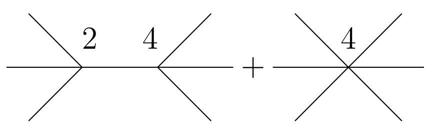


$= 0.$ (3.40)

However both 4 and 6-point interactions are zero by momentum conservation so we move onto higher powers.

The contributions to $O(p^4)$ amplitude are

$$iA^{(4)}_{6,s} =$$



. (3.41)

The interaction becomes nontrivial but only the 6-point vertex is non-zero, so the amplitude is

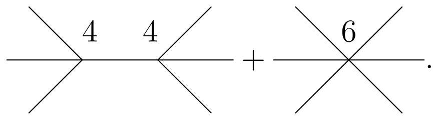
$$A^{(4)}_{6,s} = c^{(4)}_6 \left(s^2_{12} + \text{perms}\right), \tag{3.42}$$

which does not satisfy Adler zero so we carry on.

From this point onward we will not be drawing diagrams that are trivially zero.

At $O(p^6)$ the contributions to amplitude are

$$iA^{(6)}_{6,s} =$$



(3.43)

From previous section we know that the sub-amplitudes in left diagram have soft degree 2, so a similar behaviour is expected here. Secondly since both vertices are identical we will divide the left contribution by 2 in numerical results. This gives the amplitude

$$\begin{aligned} A^{(6)}_{6,s} =& -c^{(6)}_{61}\left(\frac{(s^2_{12}+s^2_{13}+s^2_{23})\,(s^2_{45}+s^2_{46}+s^2_{56})}{s_{123}} + \text{perms}\right) \\ &+ c^{(6)}_{62}\left(s^3_{12} + \text{perms}\right) + c^{(6)}_{63}\left(s_{12}s_{15}s_{24} + \text{perms}\right). \end{aligned} \tag{3.44}$$

With constants being related to lower point as $c^{(6)}_{61} = c^{(4)}_4 \times c^{(4)}_4$. Demanding Adler zero leads to

$$c^{(6)}_{63} = 16c^{(6)}_{61} + 12c^{(6)}_{62}, \tag{3.45}$$

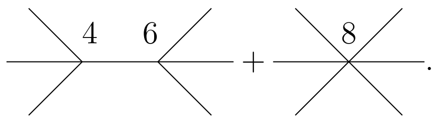
while demanding soft degree 2 results in

$$c^{(6)}_{62} = -\frac{1}{2}c^{(6)}_{61}, \quad c^{(6)}_{63} = 10c^{(6)}_{61}, \tag{3.46}$$

making this amplitude fully determined by lower point interactions with only one parameter left. Demanding any higher soft degree leads to a trivial result.

At $O(p^8)$ the amplitude is

$$iA^{(8)}_{6,s} =$$



(3.47)

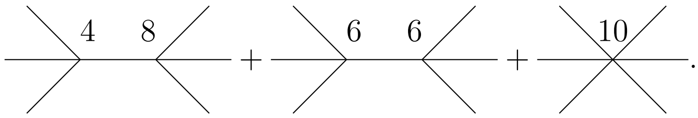
This amplitude has 1 constant given from 4-point and 3/4/4 from 6-point vertex in 3/4/6D. Demanding Adler zero reduces them to 1 + 2/3/3. Enforcing soft degree 2 leaves only the 4-point constant undetermined. The final amplitude is

$$\begin{aligned} {}_2A^{(8)}_{6,s} = c^{(8)}_{61}\Bigg(&-\frac{(s^2_{12}+s^2_{13}+s^2_{23})\,(s^3_{45}+s^3_{46}+s^3_{56})}{s_{123}} \\ &+ 5s^4_{12} + 18s^2_{16}s^2_{23} + 312s_{12}s^2_{15}s_{24} + 72s^2_{14}s^2_{15} + \text{perms}\Bigg). \end{aligned} \tag{3.48}$$

Where $c^{(8)}_{61} = c^{(4)}_4 c^{(6)}_4$.

At $O(p^{10})$ the amplitude is

$$iA^{(10)}_{6,s} = \qquad (3.49)$$



At the beginning there are 2 constants from 4-point and 5/5/6 from 6-point in 3/4/6D. Since the middle digram features identical vertices its contribution is divided by 2 in numerics. Imposing Adler zero reduces the number of independent terms to 2 + 3/3/4. Soft degree 2 makes the amplitude fully determined by 4-point, leading to

$$\begin{aligned}
{}_2A^{(10)}_{6,s} = c^{(10)}_{61} \Bigg(& -\frac{\left(s_{12}^2 + s_{13}^2 + s_{23}^2\right)\left(s_{45}^4 + s_{46}^4 + s_{56}^4\right)}{s_{123}} + \frac{68}{5} s_{12}^2 - 160 s_{14}^3 s_{23}^2 \\
& + 352 s_{12} s_{14}^3 s_{23} - 336 s_{12} s_{14}^2 s_{23}^2 - \frac{416}{3} s_{12}^3 s_{13}^2 + \text{perms} \Bigg) \\
+ c^{(10)}_{62} \Bigg(& -\frac{\left(s_{12}^3 + s_{13}^3 + s_{23}^3\right)\left(s_{45}^3 + s_{46}^3 + s_{56}^3\right)}{s_{123}} + \frac{216}{5} s_{12}^2 - 432 s_{14}^3 s_{23}^2 \\
& + 576 s_{12} s_{14}^3 s_{23} - 1152 s_{12} s_{14}^2 s_{23}^2 - 384 s_{12}^3 s_{13}^2 + \text{perms} \Bigg).
\end{aligned} \qquad (3.50)$$

The constants relate to lower points as $c^{(10)}_{61} = c^{(4)}_4 c^{(8)}_4$ and $c^{(10)}_{62} = c^{(6)}_4 c^{(6)}_4$. Soft degree 3 can be satisfied as well. Such solution coincides with the previous and adds the condition $c^{(10)}_{61} = 0$. This result is again determined by 4-point scattering. The reduction to soft degree 3 solution seems rather obvious as the 4-point $O(p^4)$ vertex on the left does not have soft degree 3 by itself. This demand gives us $c^{(4)}_4 = 0$, resulting in the leftmost diagram in 3.49 being zero.

As we saw demanding Adler zero can lead to 6-point amplitudes being fully determined by lower point interactions. We will carry this notion further and see how we can restrict cyclic amplitudes which feature larger number of free constants.

3.3.2 Cyclic

From tables 3.6 and 3.7 we see that cyclic amplitudes have much higher number of free constants then symmetric ones. We might also expect that $O(p^2), O(p^4)$ with Adler zero will be fully determined from lower point amplitudes as there are no free parameters left in the 6-point vertex. Diagrams in this section are considered to be summed over all cyclic permutations.

The KK relations now result in 72 equations. Example of which is

$$\begin{aligned}
A_6(1,2,6,5,4,3) + A_6(1,2,3,4,5,6) + A_6(1,3,2,4,5,6) \\
+ A_6(1,3,4,2,5,6) + A_6(1,3,4,5,2,6) = 0.
\end{aligned} \qquad (3.51)$$

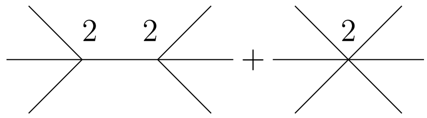
While BCJ relations are now given by

$$\begin{aligned}
s_{12} A_6(1,2,3,4,5,6) + (s_{12} + s_{23}) A_6(1,3,2,4,5,6) \\
+ (s_{12} + s_{23} + s_{24}) A_6(1,3,4,2,5,6) \\
+ (s_{12} + s_{23} + s_{24} + s_{25}) A_6(1,3,4,5,2,6) = 0.
\end{aligned} \qquad (3.52)$$

We again move by power-counting.

At $O(p^2)$ the amplitude is made out of the diagrams

$$iA^{(2)}_{6,c} =$$



(3.53)

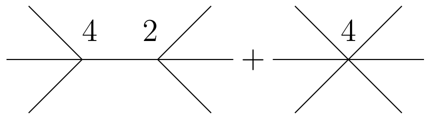
We are starting with only 1 constant from 4-point interaction and 3 from 6-point. Since the left diagram is made out of identical vertices it will be divided by 2 during numerics. Demanding Adler zero makes this amplitude fully determined by lower point interaction. The result is

$$ {}_1A^{(2)}_{6,c} = c^{(2)}_6 \left(-\frac{s_{13}s_{46}}{s_{123}} - \frac{s_{26}s_{35}}{s_{126}} - \frac{s_{15}s_{24}}{s_{156}} + s_{13} + s_{15} + s_{35} \right). \tag{3.54}$$

This result satisfies Adler zero, but interestingly it also satisfies both KK and BCJ relations. Lastly the constant relates to 4-point as $c^{(2)}_6 = c^{(2)}_4 c^{(2)}_4$.

At $O(p^4)$ the contributions are

$$iA^{(4)}_{6,c} =$$



(3.55)

They result in 2 constants from 4-point and 9 from contact 6-point. Demanding Adler zero leaves only the 4-point constants not determined, thus this amplitude is fully given by 4-point scattering. The resulting amplitude reads

$$\begin{aligned} {}_1A^{(4)}_{6,c} = c^{(4)}_{61} \Bigg(&- \frac{\left(s^2_{12} + s^2_{23}\right) s_{46}}{s_{123}} - s^2_{14} + s_{15}s_{16} - s_{13}s_{14} + s^2_{13} \\ &- s_{12}s_{16} - s_{12}s_{14} - s_{16}s_{24} + \text{cyc} \Bigg) \\ + c^{(4)}_{62} \Bigg(&- \frac{s^2_{13}s_{46}}{s_{123}} + s^2_{12} + s_{15}s_{16} - s_{13}s_{14} + s_{12}s_{16} \\ &+ s_{12}s_{14} - s_{16}s_{24} + \text{cyc} \Bigg). \end{aligned} \tag{3.56}$$

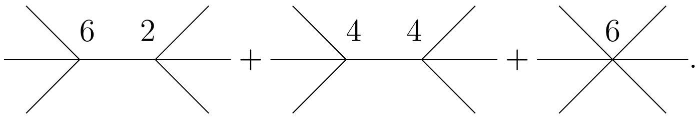
Applying KK relations forces $c^{(4)}_{62} = -2c^{(4)}_{61}$. This is exactly the results from (3.12) which means if we take the 4-point vertex that satisfies KK, this amplitude will be fully determined by a single constant. Imposing BCJ demands $c^{(4)}_{62} = c^{(4)}_{61} = 0$. Such result coincides with the result from (3.17) which tells us that there is no 4-point $O(p^4)$ satisfying BCJ, leading to zero naturally

$$A^{(6)}_{6,B} = 0. \tag{3.57}$$

Because we will not be giving any further results explicitly we are not going to be mentioning double counting either as it won't change results as presented.

The contributions to amplitude at $O(p^6)$ are

$$iA_{6,c}^{(6)} =$$

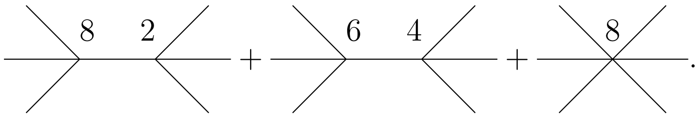


(3.58)

They add up to give 5 constants from 4-point scattering and 32 from 6-point contact terms. Demanding Adler zero leaves us with 5+5 remaining constants. While KK reduces both the 4 and 6-point contributions to only 2 constants. Adding BCJ reduces the whole amplitude to just a single constant from 4-point scattering.

At $O(p^8)$ the amplitude is

$$iA_{6,c}^{(8)} =$$

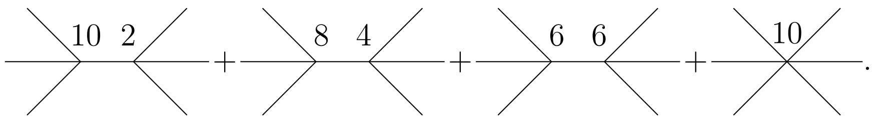


(3.59)

At the begging there are 7 constants from 4-point and 86/89/89 from 6-point contact terms in 3/4/6D. Adler zero reduces them to 7 + 27/29/29 and KK leaves only 3 + 7/8/8 undetermined. After applying BCJ we are once again left with only a single constant from 4-point.

At $O(p^{10})$ the contributions are

$$iA_{6,c}^{(10)} =$$



(3.60)

The amplitude starts with 12 free constants from 4-point and 206/225/226 from 6-point contact terms in 3/4/6D. Demanding Adler zero reduces them further to 12 + 97/111/112 and imposing KK leaves 5 + 23/26/27 free constants. After applying BCJ there are only 2 constants from 4-point left, making this amplitude fully determined by them.

We once again saw that adding restrictions to 6-point amplitude can make it fully determined by lower point scattering.

There are two methods of applying conditions which give the same results. The physically correct one respects that if we apply condition to amplitude, it should be satisfied by any point and power. As such after applying any condition to a 6-point amplitude we should only consider contribution from 4-point that themselves satisfy given condition.

However we could also take a less rigorous path and just apply these conditions directly, without first reducing the 4-point. Both of these methods produce identical results. Which means that 6-point amplitude is able to enforce proper behaviour to lower points. This works only for 4-point vertices as will be shown in the next section about 7-point amplitude.

This behaviour has been observed in the previous section on fully symmetric amplitudes where certain parts were forced to become zero, because they did not have high enough soft degree.

Lastly let us note that 4D constraints appear only at $O(p^{10})$. We shall return to these later and discuss them in broader terms.

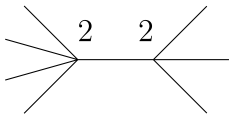
3.4 7-Point

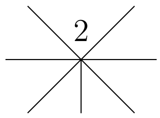
Whereas before our exploration was methodical at 7-point we start being computationally limited. Most of our methods are pretty much brute force techniques, so the rapid growth in number of terms makes many of our calculations impractical. This also makes 7-point the highest number of particles we will be looking into.

Additionally we will not be highlighting the 7-point vertex itself. The relevant results can be found in the following text. Lastly there are 14 independent variables, which are chosen identically to 6-point with $s_{16}, s_{26}, s_{36}, s_{46}, s_{56}$ added.

3.4.1 Fully symmetric

We again begin at $O(p^2)$, this time with combinations of 4 and 5-points to arrive at

$iA^{(2)}_{7,s} =$ 

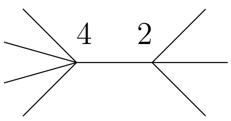
$+$ 

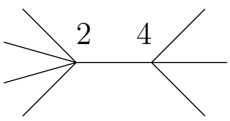
$= 0.$ (3.61)

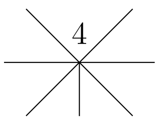
This amplitude is trivial by momentum conservation.

$$A^{(2)}_{7,s} = 0. \tag{3.62}$$

At $O(p^4)$ the amplitude is

$iA^{(4)}_{7,s} =$ 

$+$ 

$+$ 

. (3.63)

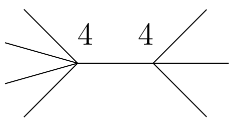
Then only non-zero contribution is the 7-point vertex itself resulting in

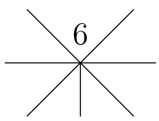
$$A^{(4)}_{7,s} = c^{(4)}_7 \left(s^2_{12} + \text{perms}_7\right). \tag{3.64}$$

This amplitude clearly does not satisfy Adler zero, so we continue to higher powers.

We again omit anything that goes to zero trivially.

At $O(p^6)$ the contributions are

$iA^{(6)}_{7,s} =$ 

$+$ 

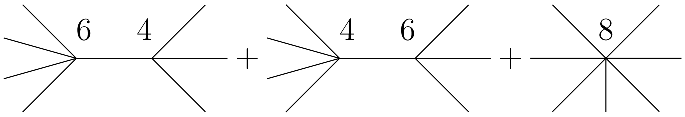
. (3.65)

They result in an amplitude

$$\begin{aligned} A^{(6)}_{7,s} = c^{(6)}_{71} &\left(-\frac{\left(s^2_{12} + \text{perms}_5\right)\left(s^2_{56} + s^2_{57} + s^2_{57}\right)}{s_{1234}} + \text{perms}_7\right) \\ &+ c^{(6)}_{72}\left(s^3_{12} + \text{perms}_7\right) + c^{(6)}_{73}\left(s_{12}s_{14}s_{23} + \text{perms}_7\right). \end{aligned} \tag{3.66}$$

Where perms_5 denotes permutations of the left 4 external legs and the propagator momentum. While perms_7 corresponds to permutations of external legs. Because the 5-point vertex does not satisfy Adler zero it is not surprising that neither does this amplitude, so we move onto higher orders.

At $O(p^8)$ the contributions are

$iA^{(8)}_{7,s} =$  . (3.67)

The amplitude thus reads

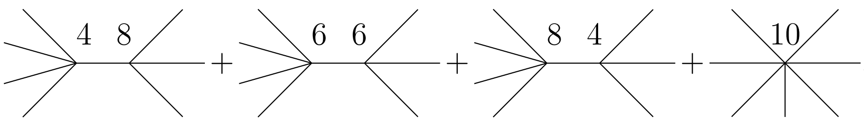
$$\begin{aligned} A^{(8)}_{7,s} = c^{(8)}_{71} & \left(-\frac{(s_{12}^3 + \text{perms}_5)\,(s_{56}^2 + s_{57}^2 + s_{58}^2)}{s_{1234}} + \text{perms}_7 \right) \\ + c^{(8)}_{72} & \left(-\frac{(s_{12}^2 + \text{perms}_5)\,(s_{56}^3 + s_{57}^3 + s_{58}^3)}{s_{1234}} + \text{perms}_7 \right) \\ + & \left(c^{(8)}_{73} s_{12}^4 + c^{(8)}_{74} s_{12}^2 s_{13}^2 + c^{(8)}_{74} s_{12} s_{14}^2 s_{23} + c^{(8)}_{75} s_{14}^2 s_{23}^2 + \text{perms}_7 \right). \end{aligned} \tag{3.68}$$

Demanding Adler zero gives the following relations

$$\begin{aligned} c^{(8)}_{72} = -\frac{1}{8} c^{(8)}_{71},\ c^{(8)}_{73} = \frac{192}{5} c^{(8)}_{71},\ c^{(8)}_{74} = 288 c^{(8)}_{71}, \\ c^{(8)}_{75} = 2016 c^{(8)}_{71},\ c^{(8)}_{76} = 48 c^{(8)}_{71}. \end{aligned} \tag{3.69}$$

We see that in order to satisfy Adler zero this amplitude needs the $O(p^4)$ 5-point vertex to be non-zero. However this vertex cannot satisfy Adler zero by itself, so in a theory with Adler zero this amplitude has to be zero regardless. Same argument can be applied to $O(p^6)$ 5-point. Higher soft degrees are automatically zero. The 3D dimensional constraints appear here, but they only reduce the number of free constants in the 7-point vertex by 1.

At $O(p^{10})$ the contributions are

$iA^{(10)}_{7,s} =$  . (3.70)

They result in an amplitude

$$\begin{aligned} A^{(10)}_{7,s} = c^{(10)}_{71} & \left(-\frac{(s_{12}^2 + \text{perms}_5)\,(s_{56}^4 + s_{57}^4 + s_{67}^4)}{s_{1234}} + \text{perms}_7 \right) \\ + c^{(10)}_{72} & \left(-\frac{(s_{12}^3 + \text{perms}_5)\,(s_{56}^3 + s_{57}^3 + s_{67}^3)}{s_{1234}} + \text{perms}_7 \right) \\ + c^{(10)}_{73} & \left(-\frac{(s_{12}^4 + \text{perms}_5)\,(s_{56}^2 + s_{57}^2 + s_{67}^2)}{s_{1234}} + \text{perms}_7 \right) \\ + c^{(10)}_{74} & \left(-\frac{(s_{12}^2 s_{13}^2 + \text{perms}_5)\,(s_{56}^2 + s_{57}^2 + s_{67}^2)}{s_{1234}} + \text{perms}_7 \right) \\ + & \Big(c^{(10)}_{75} s_{12}^5 + c^{(10)}_{76} s_{12}^3 s_{13}^2 + c^{(10)}_{77} s_{12} s_{14}^3 s_{23} + c^{(10)}_{78} s_{12}^2 s_{14}^2 s_{23} \\ + & \ c^{(10)}_{79} s_{12} s_{13} s_{14}^2 s_{23} + c^{(10)}_{710} s_{14}^3 s_{23}^2 + c^{(10)}_{711} s_{12} s_{14}^2 s_{23}^2 + \text{perms}_7 \Big). \end{aligned} \tag{3.71}$$

While demanding Adler zero only forces $c_{71} = 0$, since there is no 5-point $O(p^6)$ vertex satisfying Adler zero we have to set $c_{72} = 0$ and continue. This results in

$$\begin{aligned}
c_{71}^{(10)} &= 0,\ c_{72}^{(10)} = 0,\ c_{74}^{(10)} = -3c_{73}^{(10)}, \\
c_{77}^{(10)} &= -\frac{464}{7}c_{73}^{(10)} + \frac{375}{7}c_{75}^{(10)} + \frac{9}{7}c_{76}^{(10)}, \\
c_{78}^{(10)} &= -\frac{1536}{7}c_{73}^{(10)} + \frac{300}{7}c_{75}^{(10)} + \frac{24}{7}c_{76}^{(10)}, \\
c_{79}^{(10)} &= -\frac{456}{7}c_{73}^{(10)} + \frac{375}{14}c_{75}^{(10)} + \frac{9}{14}c_{76}^{(10)}, \\
c_{710}^{(10)} &= -\frac{2448}{7}c_{73}^{(10)} - \frac{25}{7}c_{75}^{(10)} + \frac{5}{7}c_{76}^{(10)}, \\
c_{711}^{(10)} &= -\frac{7792}{7}c_{73}^{(10)} - \frac{75}{7}c_{75}^{(10)} + \frac{15}{7}c_{76}^{(10)}.
\end{aligned} \tag{3.72}$$

We see that only 3 constants are left. After demanding soft degree 2 we get the extra conditions

$$c_{75}^{(10)} = -\frac{2768}{225}c_{73}^{(10)},\ c_{76}^{(10)} = -\frac{3136}{9}c_{73}^{(10)}. \tag{3.73}$$

They reduce the amplitude to a single constant, that is given from lower point scattering by

$$c_{73}^{(10)} = c_{51}^{(8)}c_4^{(4)}. \tag{3.74}$$

This is thus the first amplitude that can appear in theory with Adler zero at 7-point.

Lastly these results holds in 4D. The unconstrained version is identical to 6D and contains an extra term with $s_{13}^2 s_{14} s_{23} s_{24} + perms_7$. This additional term stops being independent after Adler zero. On the other hand 3D now leads to only one constants begin left in the 7-point vertex after Adler zero. Higher soft degrees are zero immediately as in 3D the corresponding 5-point vertices are zero.

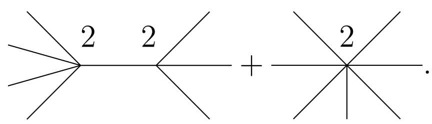
3.4.2 Cyclic

Here we will separate two cases, first without ϵ insertion and second with.

Standard

We start at $O(p^2)$ with the contributions being

$$iA_{7,c}^{(2)} =$$



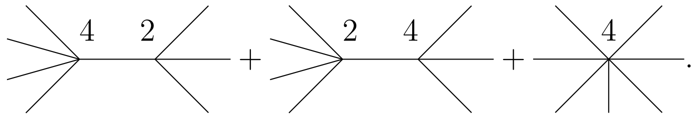
(3.75)

The amplitude thus reads

$$A_7^{(2)} = c_{71}^{(2)} \left(-\frac{(s_{12} + \text{cyc}_5)\, s_{57}}{s_{1234}} + \text{cyc}_7 \right) + \left(c_{72}^{(2)} s_{13} + c_{73}^{(2)} s_{12} + \text{cyc}_7 \right). \tag{3.76}$$

However as 5-point vertex cannot satisfy Adler zero we have to move higher. In similar fashion neither KK or BCJ can be solved.

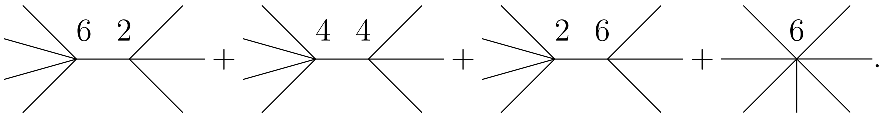
The contributions at $O(p^4)$ are

$$iA^{(4)}_{7,c} =$$ 

(3.77)

This amplitude starts with 5 constants from lower point scattering and 15 from 7-point contact terms. Demanding Adler zero makes this amplitude trivially zero.

We thus move onto $O(p^6)$ with the contributions

$$iA^{(6)}_{7,c} =$$ 

(3.78)

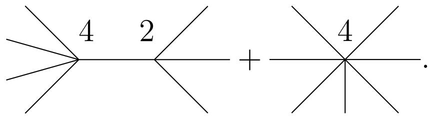
This amplitudes starts with 15 + 80 constants from lower points and contact terms respectively. After demanding Adler zero we see that the two middle diagrams contribute nothing. Afterwards there are only 3 constants left. All of which originate from the leftmost diagram. This makes amplitude fully determined by lower point scattering. Lastly we have found no solutions for either KK or BCJ.

ϵ invariant

We now require every term in an amplitude to include an ϵ invariant. Whereas in the case of 5-point this demand was trivial to meet, here it is not clear how to insert it. We begin by observing that there are $\binom{7}{4} = 35$ choices for 4 momenta to create $\epsilon(ijkl)$, but by momentum conservation only $\binom{6}{4} = 15$ are independent.

As such we will simply create all monomials out of s_{ij}, multiply them each by all 15 $\epsilon(ijkl)$ and then sum over symmetry. This method is very general, but it also makes calculations that much more difficult. Propagator terms will include their own $\epsilon(ijkl)$ from 5-point scattering.

Now the lowest amplitude is $O(p^4)$ with the contributions

$$iA^{(4)}_{7,E} =$$ 

(3.79)

It turns out that after cyclic symmetry only 3 independent $\epsilon(ijkl)$ exist. This gives the amplitude form

$$\begin{aligned} A^{(4)}_{7,E} = c^{(4)}_{71}&\left(-\frac{\epsilon(1234)s_{57}}{s_{1234}} + \text{cyc}_7\right) \\ &+ \left(c^{(4)}_{72}\epsilon(1234) + c^{(4)}_{73}\epsilon(1234) + c^{(4)}_{74}\epsilon(1245) + \text{cyc}_7\right). \end{aligned} \tag{3.80}$$

Demanding Adler zero gives the following constraints

$$c^{(4)}_{72} = \frac{2}{7}c^{(4)}_{71},\ c^{(4)}_{73} = -\frac{2}{7}c^{(4)}_{71},\ c^{(4)}_{72} = \frac{1}{7}c^{(4)}_{71}. \tag{3.81}$$

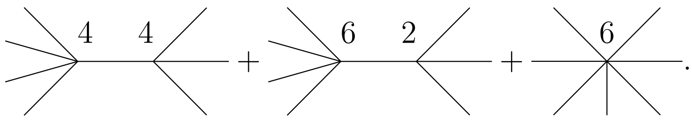
So the final amplitude reads

$${}_1A^{(4)}_7 = \frac{c^{(4)}_{71}}{7}\left(-7\frac{\epsilon(1234)s_{57}}{s_{1234}} + 2\epsilon(1234) - 2\epsilon(1235) + \epsilon(1245) + \text{cyc}_7\right). \tag{3.82}$$

We have confirmed that this result matches with the one obtained from Lagrangian [2].

At $O(p^6)$ the contributions are

$$iA^{(6)}_{7,E} =$$



(3.83)

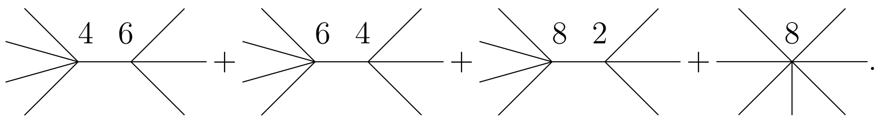
The amplitude is starting with 5 terms from lower points and 25 from 7-point contact terms. Demanding Adler zero makes amplitude fully determined by lower point scattering. The resulting amplitude has the form

$$\begin{aligned} {}_1A^{(6)}_{7,E} = &\left(-\frac{\epsilon(1234)\left(c^{(6)}_{71}s^2_{57} + c^{(6)}_{72}\left[s^2_{56} + s^2_{67}\right]\right)}{s_{1234}} + \mathrm{cyc}_7\right) \\ &+ c^{(6)}_{73}\left(-\frac{\epsilon(1234)\left(s_{12} + \mathrm{cyc}_5\right)s_{57}}{s_{1234}} + \mathrm{cyc}_7\right) + V^{(6)}_{7,E} \end{aligned} \tag{3.84}$$

with all constants in the vertex being determined by the first three.

At $O(p^8)$ the contributions to amplitude are

$$iA^{(8)}_{7,E} =$$



(3.85)

The amplitude begins with 6 terms from lower points and 245 constants from 7-point contact terms. Demanding Adler zero reduces the number of free constants in 7-point vertex to 110.

Lastly we have found no solutions to either KK or BCJ with ϵ inserted.

4. Massive particles

As we have seen in the previous chapters Adler zero is a powerful tool that allowed us to restrict amplitudes and even make them fully determined by lower points. But Adler zero and soft limit in general are only sensible for massless particles. We can try to avoid this problem and define Adler zero for massive particles through 6 dimensional spinor-helicity formalism. This way we will make Adler zero a natural demand.

The relation between 4D massive and 6D massless spinor-helicity formalisms has been worked out in [19]. We give short outline of this formalism and introduce our own notation in order to solve conditions outlined in previous sections.

We begin by generalizing 4D bispinor decomposition.

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \rho\ \mu_a \tilde{\mu}_{\dot{a}} \tag{4.1}$$

Where

$$\rho = \kappa\tilde{\kappa}, \quad \kappa = \frac{m}{\langle\lambda\mu\rangle}, \quad \tilde{\kappa} = \frac{\tilde{m}}{[\mu\lambda]}. \tag{4.2}$$

This notation can be related to 6D spinors as

$$\lambda_a^A = \begin{pmatrix} -\kappa\mu_b & \lambda_b \\ \tilde{\lambda}^{\dot{b}} & \tilde{\kappa}\tilde{\mu}^{\dot{b}} \end{pmatrix} = \begin{pmatrix} \nu_b & \lambda_b \\ \tilde{\lambda}^{\dot{b}} & \tilde{\nu}^{\dot{b}} \end{pmatrix}. \tag{4.3}$$

Where ν is just recasting into more convenient notation. They satisfy the equations

$$\langle\lambda\nu\rangle = -m, \qquad [\lambda\nu] = -\tilde{m}. \tag{4.4}$$

We can use this form and earlier derived formula (1.49) to construct 6D momenta

$$p^\mu = -\frac{1}{4}\sigma^\mu_{AB}\lambda^A_b\lambda^{Bb}. \tag{4.5}$$

By direct calculation we can see that

$$p_4 = \frac{m-\tilde{m}}{2i}, \qquad p_5 = \frac{m+\tilde{m}}{2}. \tag{4.6}$$

Further enforcing

$$M^2 = m\tilde{m} \implies p_4^2 + p_5^2 = M^2, \tag{4.7}$$

where M is physical mass. From here we can see that the if the 6D momentum is massless, the first 4 elements of the six-vector satisfy

$$p_{6D}^2 = p_{4D}^2 - M^2 = 0. \tag{4.8}$$

And such we can describe massive 4D momentum, though 6D spinors. Lastly momentum conservation in 6D for n-point scattering dictates

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \tilde{m}_i = 0. \tag{4.9}$$

Now we require a formula for converting variables between 4D and 6D. This is most easily done with s_{ij}. Direct calculation yields

$$ {}^{6D}s_{ij} = {}^{4D}s_{ij} - (m_i + m_j)(\tilde{m}_i + \tilde{m}_j) \,. \tag{4.10}$$

Where we used standard definitions $s_{ij} = (p_i + p_j)^2$. For simplicity we have assumed that all particles have identical mass M. We can check the validity of formula (4.10) by starting with

$$0 = {}^{6D}s_{ii} = {}^{4D}s_{ii} - 4M^2 \tag{4.11}$$

which is correct form of massive s_{ii} in 4D. Following that at n-point momentum conservation in 6D gives

$$0 = \sum_{i=1}^{n} {}^{6D}s_{ij} = \sum_{i=1}^{n} {}^{4D}s_{ij} - \sum_{i=1}^{n} \left(2M^2 + m_i\tilde{m}_j + m_j\tilde{m}_i\right) . \tag{4.12}$$

Momentum conservation (4.9) demands that the last two terms sum to 0 and using previous equality for s_{11} it is possible to arrive at

$$\sum_{i=1}^{n} {}^{4D}s_{ij} = 2nM^2 \tag{4.13}$$

or

$$\sum_{i\neq j}^{n} {}^{4D}s_{ij} = 2nM^2 - 4M^2. \tag{4.14}$$

This results is correct momentum conservation of massive 4D kinematics.

From (4.10) we see that the proposed dimensional reduction is not trivial as it depends on parametrization of mass through the choice of $m_i, \tilde{m}_i$. But since parameters have no direct physical meaning there seems to be no contradiction. Lastly if we want momenta to be real it is required to enforce $m_i = \tilde{m}_i^*$.

We now make assumption that if we calculate 6D amplitude it will be equal to 4D massive one. In order to fix parametrization we will require that at least 4-point amplitude is given to us. For this we can use result derived in [20], which gives the massive 4-point amplitude to be

$$A_4^{(2)} = c_4^{(2)} \left(s_{12} + s_{14} - 2M^2\right) . \tag{4.15}$$

In our formalism this can be achieved by choosing for example

$$m_1 = M,\ m_2 = iM,\ m_3 = -iM,\ m_4 = -M, \tag{4.16}$$

and taking amplitude from (3.7).

The method presented in [20] is to define soft limit as $p_i = 0$ and $p_i^2 = 0$. By this method is it clear that (4.15) satisfies Adler zero. This method is quite artificial as it requires to set one mass to zero. From our point of view this amplitudes satisfies 6D Adler zero and therefore it satisfies 4D massive one as well.

We now want to extend this formalism to higher point scattering. Let us start at 6-point. Since the result is known [20], we can easily build it from gluing vertices together

$$A_6^{(2)} = c_6^{(2)} \Bigg(\frac{(s_{12}+s_{23}-2M^2)(s_{46}+s_{56}-2M^2)}{s_{123}-M^2} \\ \frac{(s_{16}+s_{56}-2M^2)(s_{23}+s_{34}-2M^2)}{s_{156}-M^2} \\ \frac{(s_{16}+s_{12}-2M^2)(s_{34}+s_{45}-2M^2)}{s_{126}-M^2} \Bigg) + V_6. \tag{4.17}$$

Here we however run into problems. There seems to be no parametrization that satisfies this form. As it turns our if we want all vertices to have the correct form we need to have ${}^{6D}s_{ijk} = {}^{4D}s_{ijk}$. This leads to contradiction as we know that propagators are different. On the other hand if we want to have all propagators correct, our parametrization will not allow us to get correct form of vertices. To give the most obvious example we can choose

$$m_1 = M,\ m_2 = -M,\ m_3 = M,\ m_4 = -M,\ m_5 = M,\ m_6 = -M. \tag{4.18}$$

This produces correct propagators, but it also leads to

$${}^{6D}s_{ii+1} = {}^{4D}s_{ii+1}. \tag{4.19}$$

Which obviously cannot reproduce (4.17). On the other hand demanding for example

$${}^{6}s_{ii+1} = {}^{4}s_{ii+1} - M^2, \tag{4.20}$$

leads to propagators without M^2 or with $3M^2$, neither of which is correct.

We can however go against the better judgment of the article [20] and choose the 4-point amplitude to be

$$A_4^{(2)} = c_4^{(2)} \left(s_{13} - M^2\right). \tag{4.21}$$

This is achievable with parametrization

$$m_1 = Me^{i\frac{2\pi}{3}}, m_2 = M, m_3 = -M, m_4 = -Me^{i\frac{2\pi}{3}}. \tag{4.22}$$

This amplitude again satisfies Adler zero, so we can try to glue 4-point vertices together, this time resulting in (3.54)

$$A_6^{(2)} = c_6^{(2)} \Bigg(-\frac{(s_{13}-M^2)(s_{46}-M^2)}{s_{123}-M^2} - \frac{(s_{26}-M^2)(s_{35}-M^2)}{s_{126}-M^2} \\ -\frac{(s_{15}-M^2)(s_{24}-M^2)}{s_{156}-M^2} + s_{13}+s_{15}+s_{35}-3M^2 \Bigg). \tag{4.23}$$

Which is achievable with the parametrization

$$m_1 = -M,\ m_2 = M,\ m_3 = Me^{i\frac{\pi}{3}}, \\ m_4 = Me^{-i\frac{2\pi}{3}},\ m_5 = Me^{-i\frac{\pi}{3}},\ m_6 = Me^{i\frac{2\pi}{3}}. \tag{4.24}$$

As a side note the phase in front of m_2 is actually a free parameter, so it could be chosen differently, but the result remains identical.

To summarize the procedure, we begin with 6D massless kinematics. Run standard calculation to determine amplitudes that satisfy Adler zero and then recast them into 4D massive kinematics by using (4.10). Last step is not given uniquely so the parametrization has to be determined by known lower point scattering.

As we have seen this procedure can lead to reasonably looking results. However the results themselves still depend on choice of non-physical quantities $m, \tilde{m}$. In addition we rely on the assumption that 6D and 4D amplitudes are identical. It is clearly not universal as shown in (4.17). This is rather extensive issue as we cannot reproduce known results.

Overall it seems like an interesting way to implement Adler zero for massive particles. It is more formally correct than simply choosing one of the masses to be zero. But since we were unable to reproduce known results it cannot be considered correct.

5. Discussion

5.1 Gram conditions

In the previous sections we have observed that Gram conditions become nontrivial in 3D from 5-point $O(p^8)$. While in 4D they appeared from 6-point $O(p^{10})$. The 4D part has been noticed in [2]. The explanation given there follows from non-linearity of these conditions. In 4D Gram conditions are linear relations in s_{ij}^5. So in order to apply them to our linear combinations of polynomials we are forced to work at $O(p^{10})$.

We have confirmed that this argument works in 3D. It is thus possible to conjecture that for n-point scattering in D dimensions Gram conditions will become nontrivial starting with $n = D + 2$ at $O\left(p^{2D+2}\right)$.

We now know this to be correct in 3 and 4 dimensions. Based on this conjecture we would expect that in 6D Gram conditions will appear at 8-point $O(p^{14})$. Which lies firmly beyond our search depth, so the only confirmation we can provide is that the 6D results match those without Gram conditions implemented.

5.2 Recursion and validation of results

Recursion

In this text we often encountered cases where higher point amplitudes were fully determined by their lower point contributions. Natural question would thus be whether it is possible to construct these results directly. The answer turns out to be positive in some cases. For our purposes the relevant procedure is called soft recursion [14]. We will first summarize results relevant to our discussion. Let us consider n-point amplitude at $O(p^{2m})$, with soft degree σ and in D dimensions. Then this amplitude is constructible from lower point scattering by soft recursion if it satisfies

$$\frac{m}{n} < \sigma, \qquad n \geq D + 2. \tag{5.1}$$

In 4D this implies that 6-point amplitude should be fully given by 4-point scattering to $O(p^4)$ with soft degree $\sigma = 1$ and to $O(p^{10})$ for $\sigma = 2$. While 7-point amplitude should be determined up to $O(p^6)$ with $\sigma = 1$ and to $O(p^{12})$ with $\sigma = 2$. All of our results confirm the inequalities (5.1).

Interestingly if we now turn to 3D we see that 5-point should be determined up to $O(p^4)$ with $\sigma = 1$ and up to $O(p^8)$ with $\sigma = 2$. Otherwise it should correspond to 4D case. Once again our results do confirm this.

On the other hand in 6D the amplitude should be fully determined from 8-point and $O(p^6)$. Such results are beyond our search depth. We have however found cases where the amplitude was determined earlier. This is most likely caused by the simple fact that results there match with those from lower dimensions.

Constructing bases

Our procedure of generating bases has been numerical in nature. There are several analytical methods that could be employed. For example after constructing all relevant polynomials it is possible to create minimal Gröbner basis. While basis in the linear combination sense can be constructed from Gröbner basis quickly. We ran this procedure for 4 and 5-point vertices. This method starts as a slightly slower version of our numerical approach, however it scales very poorly to higher number of polynomials. As such we did not employ it further.

If our goal was not to construct the basis itself but cared only about the number of free constants we could employ Hilbert series. This was done in [18]. The results presented there match with our own for 4 and 5-point symmetric amplitudes.

5.3 Odd sector KK/BCJ

Both KK and BCJ relations allowed us to reduce the number of free parameters. With a single exception the ϵ invariant where we were unable to find anything. It is however possible to "strip" the ϵ from 5-point amplitude and work with it further. Since in this form we were able to find nontrivial solution to previous constrains it could be possible to repeat this procedure at 7-point.

The approach presented in this text does not give us any natural way to strip the ϵ from 7-point amplitude. As such it is difficult to guess the correct form of 7-point vertex.

Method of sidestepping this issue might be recursion. We notice that KK can be solved at 5-point $O(p^6)$. Afterwards we could add ϵ and have amplitude at $O(p^{10})$ and use it as a seed for 7-point soft recursion. Problem that immediately arises is that in order to have this amplitude fully determined by lower points we require soft degree 2. This procedure is quite artificial and requires going to very high powers in order to find nontrivial solutions. So at least for any practical usage it remains useless.

5.4 Adler zero for massive particles

Throughout this text the main condition that allowed us to construct amplitudes has been Adler zero. We proposed a method for extending it to massive particles through 6D spinor-helicity formalism. As we noted before the method is not universal but can lead to reasonable results. The main problem lies in formula 4.10. Some generalizations have been attempted, but none could recreate all demanded properties.

To conclude the idea of defining Adler zero through higher dimensions seems to hold some ground. We attempted to construct it directly with a rather naive method. This construction did not work, which implies that more sophisticated methods might be required to realize this connection.

Conclusion

At the beginning of this thesis we have introduced the spinor-helicity formalism. We summarized and presented its construction in 3,4 and 6 dimensions. Our main goal was to arrive at efficient approach of generating momenta for numerical methods employed later in this text.

In the following sections we gave an outline on how scattering amplitudes can be constructed from bottom up. Our sole focus have been theories with a single massless scalar. In this regime we explored all possible interactions allowed by a simple set of conditions. Lastly by systematically applying some properties, namely Adler zero, KK and BCJ relations, we were able to reduce the degrees of freedom in amplitudes. The predictions made by [14] matched our results for how Adler zero restricts theories. Further imposing BCJ relations resulted in both 6 and 7-point amplitudes being fully determined by their lower point contributions at our search depth.

Throughout this text we paid attention to how different dimensions change the possible interactions. As expected the 3 dimensional scattering has been much more constrained when compared to 4 dimensional. Following that the 6 dimensional scattering matched results without Gram conditions.

Lastly we attempted to extend some of the conditions mentioned above beyond their natural range. At first we attempted to apply KK and BCJ relations to physically relevant theories with the ϵ invariant. Afterwards we proposed method for defining Adler zero for amplitudes with massive particles. None of these methods succeeded in their goals, but more work would be required to rule them out entirely.

Bibliography

[1] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *A Periodic Table of Effective Field Theories*, *Journal of High Energy Physics* **2017** (2017) [1611.03137].

[2] K. Kampf, *The ChPT: top-down and bottom-up*, *Journal of High Energy Physics* **2021** (2021) [2109.11574].

[3] A. Duncan, *The Conceptual Framework of Quantum Field Theory*, Oxford University Press (Aug., 2012), 10.1093/acprof:oso/9780199573264.001.0001.

[4] A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, Princeton, second edition ed. (Apr., 2010).

[5] E. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, *The Annals of Mathematics* **40** (1939) 149.

[6] S. Weinzierl, *Tales of 1001 Gluons*, 1610.05318.

[7] C. Cheung, *TASI Lectures on Scattering Amplitudes*, 1708.03872.

[8] H. Elvang and Y.-t. Huang, *Scattering Amplitudes in Gauge Theory and Gravity*, Cambridge University Press (2015), 10.1017/CBO9781107706620.

[9] V.I. Borodulin, R.N. Rogalyov and S.R. Slabospitskii, *CORE 3.1 (compendium of relations, version 3.1)*, 1702.08246.

[10] A.E. Lipstein and L. Mason, *Amplitudes of 3d Yang Mills Theory*, *Journal of High Energy Physics* **2013** (2013) [1207.6176].

[11] C. Cheung and D. O'Connell, *Amplitudes and Spinor-Helicity in Six Dimensions*, *Journal of High Energy Physics* **2009** (2009) 075 [0902.0981].

[12] W. Siegel, *Fields*, hep-th/9912205.

[13] H. Elvang, *Bootstrap and Amplitudes: A Hike in the Landscape of Quantum Field Theory*, 2007.08436.

[14] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *On-Shell Recursion Relations for Effective Field Theories*, *Physical Review Letters* **116** (2015) 041601 [1509.03309].

[15] R. Kleiss and H. Kuijf, *Multigluon cross sections and 5-jet production at hadron colliders*, *Nuclear Physics B* **312** (1989) 616.

[16] V.D. Duca, L. Dixon and F. Maltoni, *New Color Decompositions for Gauge Amplitudes at Tree and Loop Level*, *Nuclear Physics B* **571** (1999) 51 [hep-ph/9910563].

[17] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Physical Review D* **78** (2008) 085011 [0805.3993].

[18] B. Henning, X. Lu, T. Melia and H. Murayama, *Operator bases, S-matrices, and their partition functions*, *Journal of High Energy Physics* **2017** (2017) [1706.08520].

[19] Z. Bern, J.J. Carrasco, T. Dennen, Y. tin Huang and H. Ita, *Generalized Unitarity and Six-Dimensional Helicity*, *Physical Review D* **83** (2010) 085022 [1010.0494].

[20] L. Susskind and G. Frye, *Algebraic Aspects of Pionic Duality Diagrams*, *Physical Review D* **1** (1970) 1682.

A. Proof that twistors satisfy momentum conservation

We begin by taking (1.25), and writing momentum conservation explicitly

$$\sum_{i=1}^{N} \lambda_a^i \tilde{\lambda}_{\dot{a}}^i = \sum_{i=1}^{N} \frac{\langle i i+1 \rangle \mu_{\dot{a}}^{i-1} + \langle i+1 i-1 \rangle \mu_{\dot{a}}^{i} + \langle i-1 i \rangle \mu_{\dot{a}}^{i+1}}{\langle i i+1 \rangle \langle i i-1 \rangle} \lambda_a^i \tag{A.1}$$

Now we focus on $\mu_{\dot{a}}^j$, there are going to be 3 terms that give us the sum

$$\left(\frac{1}{\langle j j-1 \rangle} \lambda_a^{j-1} + \frac{1}{\langle j+1 j \rangle} \lambda_a^{j+1} + \frac{\langle j+1 j-1 \rangle}{\langle j+1 j \rangle \langle j-1 j \rangle} \lambda_a^j \right) \mu_{\dot{a}}^j \tag{A.2}$$

Now we want to show that this coefficient is zero, which is equivalent to

$$\langle j-1 j \rangle \lambda_a^{j+1} + \langle j j+1 \rangle \lambda_a^{j-1} + \langle j+1 j-1 \rangle \lambda_a^j = 0 \tag{A.3}$$

We now take a sidestep and consider that spinors are two component objects and thus every three of them are linearly dependent, which means that there exists a solution to

$$\alpha \lambda_a^i + \beta \lambda_a^j + \gamma \lambda_a^k = 0, \tag{A.4}$$

for some real $\alpha, \beta.\gamma$.
We can sum this equation over all three spinors to get three equations.

$$\beta \langle ij \rangle + \gamma \langle ik \rangle = 0 \quad \alpha \langle ji \rangle + \gamma \langle jk \rangle = 0 \quad \alpha \langle ki \rangle + \beta \langle kj \rangle = 0. \tag{A.5}$$

This set of equations has solution

$$\alpha = \langle jk \rangle \quad \beta = \langle ki \rangle \quad \gamma = \langle ij \rangle, \tag{A.6}$$

so we arrive at

$$\langle jk \rangle \lambda_a^i + \langle ki \rangle \lambda_a^j + \langle ij \rangle \lambda_a^k = 0 \tag{A.7}$$

which is known as the Schouten identity.
Now by making the choice that

$$k = j+1 \qquad i = j-1 \tag{A.8}$$

we get

$$\langle j-1 j \rangle \lambda_a^{j+1} + \langle j j+1 \rangle \lambda_a^{j-1} + \langle j+1 j-1 \rangle \lambda_a^j = 0. \tag{A.9}$$

Which is exactly equation (A.3) which implies that (A.2) is zero and from here follows that (A.1) is also zero, so we get

$$\sum_{i=1}^{N} \lambda_a^i \tilde{\lambda}_{\dot{a}}^i = 0 \tag{A.10}$$

if we take the definition (1.25).