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Vector-valued integral representation

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I would like to thank my supervisor Jiří Spurný for his kind help and cooperation throughout my studies.

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Abstrakt: Tato práce sestává ze sedmi odborných článků. První dva články studují vlastnosti fragmentovaných konvexních funkcí, především takzvaný princip maxima. První z těchto článků se věnuje konvexním funkcím definovaným na kompaktních konvexních množinách, druhý abstraktním konvexním funkcím na Hausdorffových kompaktních prostorech. Další čtyři články práce obsahují výsledky v duchu Banach-Stoneovy věty v kontextu podprostorů spojitých funkcí. První z těchto čtyř článků se věnuje komplexním afinním spojitým funkcím na kompaktních konvexních množinách. Druhý článek zobecňuje výsledky prvního do kontextu obecných podprostorů spojitých funkcí definovaných na lokálně kompaktních prostorech. Zbylé dva články dále zobecňují tyto výsledky pro případ funkcí s hodnotami v Banachových prostorech a Banachových svazech. Poslední článek práce zkoumá Banach-Mazurovu vzdálenost mezi podprostory spojitých vektorových funkcí, které mají hranice s jistou speciální topologickou vlastností.

Klíčová slova: princip maxima, fragmentovaná konvexní funkce, Banach-Stoneova věta, funkční prostor, Banach-Mazurova vzdálenost, hranice

Title: Vector-valued integral representation

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Abstract: The thesis consists of seven research papers. The first two papers study the properties of fragmented convex functions, mainly the so-called maximum principle. The first paper deals with convex functions defined on compact convex subsets of locally convex spaces, the second one with the abstract convex functions defined on general compact Hausdorff spaces. The next four papers present results in the spirit of the well-known Banach-Stone theorem in the area of subspaces of continuous functions. The first of those four papers deals with the spaces of affine continuous complex functions on compact convex sets. The second paper extends the results of the first one to the context of general subspaces of continuous functions defined on locally compact spaces. The other two papers further extend the previous results to the case of Banach space-valued and Banach lattice-valued functions, respectively. The last paper is devoted to the study of the Banach-Mazur distance between subspaces of vector-valued continuous functions that have scattered boundaries.

Keywords: maximum principle, fragmented convex function, Banach-Stone theorem, function space, Banach-Mazur distance, boundary

Contents

Introduction

The thesis consists of seven research papers. The first two papers deal with the so-called maximum principles for convex functions. We recall that if X is a compact convex set in a locally convex space, then a function $f : X \to \mathbb{R}$ is said to satisfy the maximum principle if $f \leq 0$ on X provided $f \leq 0$ on the set $\operatorname{ext} X$ of extreme points of X. The fact that continuous convex functions satisfy the maximum principle follows immediately from the Krein-Milman theorem. The maximum principle for semicontinuous affine functions and affine functions of the first Baire class is a well-known part of the classical Choquet theory, see e. g. [**41**, Theorem 3.16, Theorem 3.85 and Corollary 4.23]. In a recent paper [**22**] it is shown that the maximum principle is valid for a larger class of affine functions, that is, the class of fragmented affine functions.

We recall that a function $f : X \to \mathbb{F}$, where $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$, is said to have the *point of continuity property* if $f|_F$ has a point of continuity for any $F \subset X$ nonempty closed, and the function f is called *fragmented* if for any $\epsilon > 0$ and nonempty closed set $F \subset X$ there exists a relatively open nonempty set $U \subset F$ such that $\operatorname{diam} f(U) < \epsilon$.

By [**41**, Theorem A.121], a function $f\colon X \to \mathbb{F}$ on a locally compact space X is fragmented if and only if it has the point of continuity property. Any semicontinuous function $f\colon X \to \mathbb{R}$ is fragmented (see [**41**, Proposition A.122]) and, if X is a completely metrizable, then the function f is fragmented if and only if f is a classical Baire-one function (see [**41**, Theorem A.127]).

The main result of the first paper of the thesis extends the result of [**22**] by working with convex fragmented functions instead of affine fragmented functions. It shows that the maximum principle is true not only for fragmented convex functions, but also for bounded monotone limits of sequences of fragmented convex functions, see [**49**, Theorem 2.6].

The maximum principles can also be considered for the case of abstract $\mathcal{H}$-convex functions defined with respect to a function cone $\mathcal{H} \subseteq \mathcal{C}(K)$, where K is a compact space. The maximum principle for semicontinuous or Baire $\mathcal{H}$-convex functions is well-known, see [**41**, Section 3.9]. The second paper of the thesis extends the result [**49**, Theorem 2.6] of the first paper by showing that the maximum principle holds for bounded monotone limits of sequences of fragmented $\mathcal{H}$-convex functions, see [**52**, Theorem 4.2]. The generalization of the result from the case of compact convex sets to this more abstract setting was not at all straightforward, as the main geometric idea (see [**49**, Lemma 2.1]) could not be easily used here. To be able to use similar geometric ideas, we needed to transfer this abstract setting to the area of the so-called ordered compact convex sets.

It is natural to ask to which extent is it possible to generalize the previous results. Intuitively, one might get the feeling that any convex function on a compact convex set should satisfy the maximum principle, but this is far from being true. In fact, there is quite a simple example of an affine function of the second Baire class on a metrizable compact convex set that does not satisfy the maximum principle, it is even a nonzero function that vanishes on the set of extreme points, see [**41**, Proposition 2.63]. Thus in this sense, it might seem that the known results are very close to being optimal: for affine functions of the first Baire class the maximum principle holds, for affine functions of the second Baire class it does not.

However, it is not that simple, as the question gets much more interesting when we restrict our attention to the so-called strongly affine functions. We recall that a function f on a compact convex set X is strongly affine if for each Radon probability measure μ on X, f is μ-measurable, and

$$f(r(\mu)) = \int_X f d\mu,$$

where $r(\mu)$ stands for the barycenter of the measure μ. Every strongly affine function is affine, but the converse is not true. In particular, the function from the example [**41**, Proposition 2.63] is not strongly affine. It turns out, however, that even strongly affine functions need not satisfy the maximum principle. The counterexample is much more involved, and the function constructed in this counterexample does not appear to be Borel (see [**55**, Theorem 5] or [**41**, Theorem 12.65]). Thus it is still an interesting open problem whether Borel strongly affine functions on a compact convex set X satisfy the maximum principle. If X is metrizable, then the answer is yes, which follows from the fact that in this case for any $x \in X$ there exists a Radon probability measure μ with $r(\mu) = x$ and such that μ is carried by the extreme points of X (see e. g. [**2**, Corolary I.4.9]). Thus it may be true that the known results are still quite far from being optimal, since, of course, there are plenty of Borel functions that are not fragmented.

The remaining papers contained in the thesis partly show an application of the above theory. In these papers, we obtained results in the spirit of the Banach-Stone theorem for spaces of continuous functions. In four of these research papers, the maximum principle for fragmented affine functions plays a crucial role. We first collect some of the known results in this area. Due to the large number of results we do not attempt to make this survey complete.

To start with, the well-known Banach-Stone theorem asserts that, given two (Hausdorff) compact spaces K_1, K_2, they are homeomorphic if and only if the spaces $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are isometric, that is, the underlying topological structure of a compact space K is completely determined by the Banach space structure of $\mathcal{C}(K)$. Holsztynski [**36**] proved the following theorem, which can be viewed as a one-sided version of the Banach-Stone theorem: Let K_1 and K_2 be compact Hausdorff spaces. Suppose that there exists a linear isometry from $\mathcal{C}(K_1)$ into $\mathcal{C}(K_2)$. Then there is a closed subset L of K_2 and a continuous function of L onto K_1. Since then, there has been a vast amount of results in the spirit of these two results. As it turns out, in most of the results the considered compact spaces and spaces of continuous functions can be replaced by locally compact spaces and spaces of continuous functions vanishing at infinity, respectively. Here we focus on results that could be described as isomorphic versions of the Banach-Stone theorem.

Roughly speaking, we can divide results in this area into the following two categories:

(i) Results, where the existence of an isomorphism T from $\mathcal{C}(K_1)$ onto $\mathcal{C}(K_2)$ with the number $\|T\| \, \|T^{-1}\|$ being small ensures that the compact Hausdorff spaces K_1 and K_2 are homeomorphic, and similar results for isomorphisms, that are not necessarily surjective.

(ii) Results, where the existence of an isomorphism T from $\mathcal{C}(K_1)$ onto $\mathcal{C}(K_2)$ ensures that one of the compact Hausdorff spaces K_1 and K_2 possesses a given property if and only if the other one does, and similar results for isomorphisms, that are not necessarily surjective.

The first result in the first group is the following improvement of the Banach-Stone theorem that was given independently by Amir [**3**] and Cambern [**6**]. They showed that compact spaces K_1 and K_2 are homeomorphic if there exists an isomorphism $T\colon \mathcal{C}(K_1,\mathbb{F}) \to \mathcal{C}(K_2,\mathbb{F})$ with $\|T\| \cdot \|T^{-1}\| < 2$. Cohen [**21**] and Drewnowski [**23**] gave alternative proofs of this result. Jarosz in [**37**] proved an extension of the theorem of Holsztynski by working with into isomorphims $T : \mathcal{C}_0(K_1) \to \mathcal{C}_0(K_2)$ with $\|T\| \, \|T^{-1}\| < 2$ instead of into isometries.

These results can be extended in various ways. To start with, one can consider spaces of vector-valued functions instead of scalar-valued functions. The first vector-valued version of the isomorphic Banach-Stone theorem is due to Cambern [**7**], who proved that if E is a finite-dimensional Hilbert space and $\mathcal{C}_0(K_1,E)$ is isomorphic to $\mathcal{C}_0(K_2,E)$ by an isomorphism T satisfying $\|T\| \cdot \|T^{-1}\| < \sqrt{2}$, then the locally compact spaces K_1 and K_2 are homeomorphic.

Later in [**8**], Cambern proved the first result in the spirit of isomorphic vector-valued Banach-Stone theorem for infinite-dimensional Banach spaces. He showed that if K_1 and K_2 are compact spaces, E is a uniformly convex Banach space and $T : \mathcal{C}(K_1,E) \to \mathcal{C}(K_2,E)$ is an isomorphism satisfying $\|T\|\cdot\|T^{-1}\| < (1-\delta(1))^{-1}$, then K_1 and K_2 are homeomorphic (here $\delta : [0,2] \to [0,1]$ denotes the modulus of convexity of E).

Since then, there have been improvements in this area proved e.g. in [**5**], [**4**] and [**38**].

Many of those results were recently unified and strengthened in [**19**], where it was showed that if E is a real or complex reflexive Banach space with $\lambda(E) > 1$, then for all locally compact spaces K_1, K_2, the existence of an isomorphism $T : \mathcal{C}_0(K_1,E) \to \mathcal{C}_0(K_2,E)$ with $\|T\| \cdot \|T^{-1}\| < \lambda(E)$ implies that the spaces K_1, K_2 are homeomorphic. Here

$$\lambda(E) = \inf\{\max\{\|e_1 + \lambda e_2\| : \lambda \in \mathbb{F}, |\lambda| = 1\} : e_1, e_2 \in S_E\}$$

is a parameter introduced by Jarosz in [**38**].

It is easy to check that $\lambda(\mathbb{F}) = 2$, thus this result recovers the theorem of Amir and Cambern. The authors of [**19**] also showed that the constant $\lambda(E) = 2^{\frac{1}{p}}$ is the best possible for $E = l_p$, where $2 \le p < \infty$. In the paper [**31**] the authors give a vector-valued extension of the Holsztynski theorem based on the above constant $\lambda(E)$.

In [**30**] it was shown how the constant $\lambda(E)$ can be improved, if one moreover assumes that E is a real Banach lattice and $T : \mathcal{C}_0(K_1,E) \to \mathcal{C}_0(K_2,E)$ is a Banach lattice isomorphism. The constant $\lambda(E)$ may be then replaced by

$$\lambda^+(E) = \inf\{\max\{\|e_1 + e_2\|, \|e_1 - e_2\|\}, e_1, e_2 \in S_E, e_1, e_2 \ge 0\}.$$

It is easily seen that $\lambda(E) \le \lambda^+(E)$ for each Banach lattice E, and in [**30**] it is shown that for $E = \ell_p$, where $1 \le p < 2$, the inequality is strict. Moreover, the constant $\lambda^+(E)$ is optimal for $E = \ell_p$, where $p \in [1,\infty)$. In the paper [**32**] the authors investigate positive isomorphisms between $\mathcal{C}(K,E)$ spaces that are not necessarily surjective.

Another way to extend the Amir-Cambern theorem is directed to replacing the number 2 by a larger number. After proving the result, Amir conjectured that the number 2 appearing in the Amir-Cambern theorem may be replaced by 3. Cohen [**20**] showed that this is not true in general by providing a counterexample. However, in [**34**], Gordon proved that it is true in the class of countable compact spaces: If K_1, K_2 are nonhomeomorphic countable compact spaces and $T : \mathcal{C}(K_1) \to \mathcal{C}(K_2)$ is an isomorphism, then $\|T\| \, \|T^{-1}\| \geq 3$.

The result of Gordon was extended in [**13**, Theorem 1.5], where the authors show that if E is a Banach space having non-trivial cotype, and such that for every $n \in \mathbb{N}$, E^n contains no subspace isomorphic to E^{n+1}, then countable compact spaces K_1 and K_2 are homeomorphic provided there exists an isomorphism $T : \mathcal{C}(K_1, E) \to \mathcal{C}(K_2, E)$ with $\|T\| \, \|T^{-1}\| < 3$. It is clear that every finite-dimensional Banach space satisfies the above condition, and the authors also show in [**13**, Remark 4.1] that there exist many infinite-dimensional Banach spaces that satisfy it.

Another way of generalization of the Amir-Cambern theorem is obtained by working with nonlinear mappings instead of linear isomorphisms. A mapping $T : \mathcal{C}_0(K_1, \mathbb{R}) \to \mathcal{C}_0(K_2, \mathbb{R})$ is called coarse (M, L)-quasi-isometry if for each $f, g \in \mathcal{C}_0(K_1, \mathbb{R})$ it holds that

$$\frac{1}{M} \|f - g\| - L \leq \|T(f) - T(g)\| \leq M \|f - g\| + L.$$

E.M. Galego and A.L. Porto da Silva in [**25**] proved the following extension of the Amir-Cambern theorem. If T is a function from $\mathcal{C}_0(K_1, \mathbb{R})$ to $\mathcal{C}_0(K_2, \mathbb{R})$, $T(0) = 0$, and both T and T^{-1} are bijective coarse $(M, 1)$-quasi-isometries with $M < \sqrt{2}$, then K_1 and K_2 are homeomorphic. Vector-valued extension of this result was given in [**26**]. In the paper [**27**] the authors prove a more general version of this result. As an application they obtain results also for the spaces $\mathcal{C}_0^{(1)}(K)$ of continuously differentiable functions on locally compact subspaces of the real line without isolated points. The paper [**28**] deals with nonlinear mappings that are not bijective.

Now, we turn our attention to results from the second category. Thus we are interested in the question of what properties of locally compact spaces are preserved by isomorphisms of the respective spaces of continuous functions. The first result in this area, also referred to as the weak Banach-Stone theorem, is due to Cengiz [**17**], who showed that locally compact Hausdorff spaces K_1 and K_2 have the same cardinality provided that the spaces $\mathcal{C}_0(K_1, \mathbb{F})$ and $\mathcal{C}_0(K_2, \mathbb{F})$ are isomorphic.

In the area of weak vector-valued Banach-Stone type theorems, Candido and Galego in [**16**] showed that if K_1, K_2 are locally compact Hausdorff spaces and E is a Banach space having nontrivial cotype, such that either E^* has the Radon-Nikodym property or E is separable, then either both K_1 and K_2 are finite or K_1 and K_2 have the same cardinality provided that the spaces $\mathcal{C}_0(K_1, E)$ and $\mathcal{C}_0(K_2, E)$ are isomorphic.

This result was improved by Galego and Rincón-Villamizar in [**29**], who showed that the same conclusion holds for Banach spaces not containing an isomorphic copy of c_0. The way to this improvement was using a classical characterization of Banach spaces not containing an isomorphic copy of c_0, see [**43**, Theorem 6.7], and a result of Plebanek, see [**45**, Theorem 3.3], which made it possible to remove the assumptions of separability and the Radon-Nikodym property.

The next property of locally compact spaces that is closely connected with isomorphisms between the respective spaces of continuous functions is the scattered structure. Since it is known that a compact space K is scattered if and only if $\mathcal{C}(K)$ is Asplund (see e. g. [**24**, Theorem 1.1.3]), and it is well-known that the class of Asplund Banach spaces is closed under isomorphisms, it follows that if

$\mathcal{C}(K_1)$ is isomorphic to $\mathcal{C}(K_2)$ then K_1 is scattered if and only if K_2 is scattered. This fact was improved in [**10**, Theorem 1.4], where the author shows that if E is a Banach space not containing an isomorphic copy of c_0, K_2 is a scattered locally compact space and $\mathcal{C}_0(K_1)$ embeds isomorphically into $\mathcal{C}_0(K_2, E)$, then K_1 is also scattered.

Moreover, there have been proven estimates of the Banach-Mazur distance of $\mathcal{C}(K)$ spaces from $\mathcal{C}_0(\Gamma, E)$ spaces, where Γ is a discrete set, and from $\mathcal{C}(F)$, where F is a compact space of height 2 (in particular for $F = [0, \omega]$), based on the height of the compact space K. It was proved in [**15**, Theorem 1.2] that if K is a compact space with the n-th derivative $K^{(n)}$ nonempty for some $n \in \mathbb{N}$, F is a compact space with $F^{(2)} = \emptyset$ and there exists an isomorphism $T : \mathcal{C}(K) \to \mathcal{C}(F)$, then $\|T\| \|T^{-1}\| \geq 2n - 1$. Moreover, if $\left|K^{(n)}\right| > \left|F^{(1)}\right|$, then $\|T\| \|T^{-1}\| \geq 2n + 1$. In [**11**, Theorem 1.1] it has been showed that if Γ is an infinite discrete space, E is a Banach space not containing an isomorphic copy of c_0 and $T : \mathcal{C}(K) \to \mathcal{C}_0(\Gamma, E)$ is an into isomorphism, then for each $n \in \mathbb{N}$, if $K^{(n)} \neq \emptyset$, then $\|T\| \|T^{-1}\| \geq 2n+1$. Similar results for isomorphisms with range in $\mathcal{C}_0(\Gamma, E)$ spaces were proven before in [**12**] and [**14**].

The papers [**42**] and [**45**] provide partial answers to the question of whether the class of Corson compact spaces is preserved by isomorphisms of the respective spaces of continuous functions.

In the paper [**44**] the author shows that positive embeddings of $\mathcal{C}(K)$ spaces produce upper-semicontinuous set-functions with finite values between the underlying compact spaces. As a consequence the author proves that if there exists a positive embedding of $\mathcal{C}(K_1)$ into $\mathcal{C}(K_2)$, then some topological properties of the compact space K_2, as countable tightness or Fréchetness, are inherited by K_1.

Starting in [**18**], and continuing in [**40**], [**22**], [**47**] and [**50**], the theorem of Amir and Cambern was extended to the context of subspaces. The papers [**18**], [**40**], [**22**], [**47**] deal with the spaces of continuous affine functions on compact convex sets. The paper [**18**] may be viewed as the first generalization of the Amir-Cambern theorem to the context of subspaces. The next papers [**40**] and [**22**] successively improved the result of [**18**] by removing redundant assumptions. The progress to final result for spaces of real continuous affine functions, contained in [**22**], was possible due to the maximum principle for fragmented affine functions. Now we come to the contribution of the papers contained in the thesis to this theory. The paper [**47**], contained in the thesis, extends the result of [**22**] to the case of complex continuous affine functions. The next paper of the thesis [**50**] futher extends the above results for affine functions on compact convex sets to general subspaces of $\mathcal{C}_0(K, \mathbb{F})$ spaces. We recall that if $\mathcal{H}$ is a subspace of $\mathcal{C}_0(K, \mathbb{F})$ then the Choquet boundary $\operatorname{Ch}_{\mathcal{H}} K$ of K with respect to $\mathcal{H}$ is defined as the set of points $x \in K$ such that the evaluation functional defined for $h \in \mathcal{H}$ as $i(x) : h \mapsto h(x)$ is an extreme point of the compact convex set $B_{\mathcal{H}^*}$ endowed with the w^* topology. If $\mathcal{H}$ is the space of affine continuous functions on a compact convex set X, then $\operatorname{Ch}_{\mathcal{H}} K = \operatorname{ext} X$, and if $\mathcal{H} = \mathcal{C}_0(K, \mathbb{F})$, then $\operatorname{Ch}_{\mathcal{H}} K = K$. The final result in the spirit of the Amir-Cambern theorem for subspaces of scalar functions (see [**50**, Theorem 1.1]), reads as follows. For $i = 1, 2$, let $\mathcal{H}_i \subseteq \mathcal{C}_0(K_i, \mathbb{F})$ be closed subspaces such that all points in their Choquet boundaries are weak peak points. If there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \|T^{-1}\| < 2$, then their Choquet boundaries $\operatorname{Ch}_{\mathcal{H}_i} K_i$ are homeomorphic (we recall that $x \in K_i$ is a *weak peak point* (with respect to $\mathcal{H}_i$) if for a given $\varepsilon \in (0, 1)$ and a neighborhood U of x there exists a function $h \in B_{\mathcal{H}_i}$ such that $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$

on $\mathrm{Ch}_{\mathcal{H}_i} K_i \setminus U$). It turns out that this result is in a sense optimal since the assumption of weak peak points cannot be omitted (see [**35**]), and the bound 2 is optimal even for $\mathcal{C}(K)$ spaces (see [**34**]). The theorem [**50**, Theorem 1.2] gives a one-sided version of [**50**, Theorem 1.1], and thus provides an extension of the theorem of Holsztynski to this context. The paper [**50**] also contains a weak Banach-Stone theorem for subspaces: If for $i = 1, 2$, $\mathcal{H}_i \subseteq \mathcal{C}_0(K_i, \mathbb{F})$ are closed subspaces such that all points in their Choquet boundaries are weak peak points and there exists an isomorphism $T\colon \mathcal{H}_1 \to \mathcal{H}_2$, then their Choquet boundaries $\mathrm{Ch}_{\mathcal{H}_i} K_i$ have the same cardinality (see [**50**, Theorem 1.3]). The remaining papers of the thesis deal with subspaces of vector-valued functions. The paper [**48**] gives vector-valued extensions of [**50**, Theorem 1.1] and [**50**, Theorem 1.3] in the spirit of papers [**19**] and [**16**], and the paper [**51**] provides a generalization of those results for Banach lattice-valued functions, inspired by [**30**]. Let us also note that isomorphic vector-valued Banach-Stone type theorems for subspaces were treated before by Al-Halees and Fleming in [**1**], but they were restricted to subspaces of vector-valued continuous functions, that are closed with respect to multiplication by scalar functions.

As mentioned above, in all papers [**47**], [**50**], [**48**] and [**51**], the maximum principle for fragmented affine functions is a key ingredient.

The last paper of the thesis contains an extension of the results of papers [**33**] and [**13**] to the context of subspaces. It shows that for some class of subspaces, the constant 2 appearing in the Amir-Cambern theorem may be replaced by 3, see [**46**, Theorem 1.2]. Also, it contains an extension of some of the results from [**10**] and [**15**]. We recall that in those papers, the authors obtained estimates of the Banach-Mazur distance of $\mathcal{C}(K)$, where K is a scattered compact space, from c_0 and $\mathcal{C}(F)$, where F is a scattered compact space of height 2. In [**46**, Theorem 1.3], we proved a more general result for subspaces of continuous functions. As a consequence of this result, we obtained the estimate of the Banach-Mazur distance between spaces $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$, where K_1 and K_2 are scattered compact spaces of finite height, see [**46**, Corollary 1.5]. From this result it also follows that if $\mathcal{C}(K_1)$ embeds isomorphically into $\mathcal{C}(K_2)$ and height of K_2 is finite, then also height of K_1 is finite (see again [**46**, Corollary 1.5]).

The techniques of proofs of the above results for subspaces $\mathcal{H}$ of $\mathcal{C}(K, E)$ spaces and their Choquet boundaries $\mathrm{Ch}_{\mathcal{H}} K$ can often imitate the techniques used for $\mathcal{C}(K, E)$ spaces, however, there are some extra difficulties present, apart from the fact that there is in general much more theory known for the case of $\mathcal{C}(K, E)$ spaces. We list the main difficulties that occurred in the process of trying to prove the results contained in the papers of the thesis.

- While K is compact, the Choquet boundary $\mathrm{Ch}_{\mathcal{H}} K$ is in general far from being closed in K, it need not be even Borel.
- The dual space of $\mathcal{H}$ is more complicated than the dual space of $\mathcal{C}(K, E)$ (we recall that $\mathcal{C}(K, E)^*$ is isometric to the space $\mathcal{M}(K, E^*)$ of E^*-valued regular Radon measures by the Singer's theorem, see [**54**, p. 192]). Elements of the dual space of $\mathcal{H}$ can be extended to measures with preservation of norm, which is a key tool in this theory (see [**48**, Lemma 2.2]), but this extension is in general not unique.
- The second dual space of $\mathcal{H}$ is more complicated than the second dual space of $\mathcal{C}(K, E)$ (we recall that the second dual space of $\mathcal{C}(K, \mathbb{F})$ is isometric to $\mathcal{C}(Z, \mathbb{F})$, where Z is a compact Hausdorff space depending on K, see [**39**], and this fact can be extended to vector-valued functions,

see [**9**]). Also, in the papers [**47**], [**50**], [**48**] and [**51**], an analogue of the concept of characteristic function of a point in the second dual space of $\mathcal{H}$ needed to be found, see e.g. [**50**, Lemma 2.6].

- Unlike $\mathcal{C}(K)$, its subspaces are generally not Banach lattices nor Banach algebras. This played an important role in the paper [**46**], since for the proofs of analogous results in the case of $\mathcal{C}(K)$ spaces, both lattice operations (see e.g [**13**, Claim 1 of Theorem 2.1]) and pointwise multiplication of functions (see [**13**, Proposition 3.1]) were used.
- Urysohn functions are not present in the context of subspaces. Those have to be replaced by the functions appearing in the definition of weak peak points. These functions, however, approximate only points of K (instead of subsets of K), and only up to some given $\varepsilon > 0$. Moreover, system of these functions is not monotone, if considered as a net.
- Meanwhile operators from $\mathcal{C}(K, E_1)$ spaces to E_2, where E_1 and E_2 are Banach spaces, may be represented by Borel measures on K with values in $L(E_1, E_2^{**})$, the space of bounded linear operators from E_1 to E_2^{**}, such a representation is not available for operators from $\mathcal{H}_1 \subset \mathcal{C}(K, E_1)$ to E_2, which caused technical problems in the paper [**51**], see [**51**, Remark 5.1].
- The Choquet theory of subspaces of vector-valued continuous functions is not very well established, as many classical concepts and facts of the scalar Choquet theory have not been studied extensively in the vector-valued case. In fact, in papers [**48**], [**51**] and [**46**], dealing with vector-valued functions, both the definitions of Choquet boundary and weak peak points for subspaces of vector-valued functions are new (a notion of a Choquet boundary of a vector-valued subspace has been known before, but it is different from the one that we use, see e.g. [**53**, page 1154]). Actually, a slightly different definition of a weak peak point is used in each of the papers [**48**], [**51**] and [**46**], as for vector-valued spaces, there are various reasonable ways to define a notion of a weak peak point that agrees with the definition for scalar subspaces. Those differences, however, do not seem to be significant, and all of them coincide for a large number of subspaces, in particular, for spaces of continuous vector-valued affine functions on compact convex sets.

Working in the context of subspaces, on the other hand, also has its advantages. Apart from the evident fact that the results are overall more general, it also allows to work efficiently at the same with functions defined on compact spaces as well as on locally compact spaces, since subspaces of $\mathcal{C}_0(K, E)$, where K is a locally compact space and E is a Banach space, may be naturally viewed as subspaces of $\mathcal{C}(J, E)$, where J is the one-point compactification of K, see [**48**, Lemma 2.10]. This is most apparent in the paper [**46**], where, for example, Theorem 1.3 covers quite naturally results for c_0 and $\mathcal{C}([0, \omega])$, that were proven separately before. Also, the results for subspaces make more apparent the exact properties of $C(K, E)$ spaces that are necessary for the results to hold.

Finally, from the survey of the results for $\mathcal{C}(K, E)$ spaces above, it is evident that there are plenty of results that are not known for subspaces. For some of those results, it is unclear whether some reasonable generalization to the context of subspaces might hold. There are many other results, however, where all we know is that the proofs that were used for $\mathcal{C}(K, E)$ spaces do not work for subspaces. For example, in the results dealing with quasi-isometries ([**25**], [**26**], [**28**] and [**27**]), the compactness of the underlying spaces is so crucial that is not clear how to try to prove analogous results for spaces, that do not have closed Choquet boundaries. As another example we can give the results of Plebanek for positive embeddings [**44**]. Here the results rely too heavily on the description of the dual space of $\mathcal{C}(K)$, making it impossible to imitate the proof in the context of subspaces. Still, it is of course possible that these results hold in some way also for subspaces, just another way of proof needs to be found.

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Attachments

A.1 On fragmented convex functions

A.2 Maximum principle for abstract convex functions

A.3 Isomorphisms of spaces of affine continuous complex functions

A.4 Small-bound isomorphisms of function spaces

A.5 Isomorphisms of subspaces of vector-valued continuous functions

A.6 An Amir-Cambern theorem for subspaces of Banach lattice-valued continuous functions

A.7 On the Banach-Mazur distance between continuous function spaces with scattered boundaries