UNIVERSIDAD DE MURCIA | Facultad de Matemáticas

REPORT ON THE HABILITATION THESIS 'CONSTRUCTIONS AND DECONSTRUCTIONS OF LOCALLY WELL-BEHAVED COMPLEX OBJECTS'
by Jan Saroch

All throughout this report we will denote by Mod $-R$ the category of right modules over an associative unital ring R. Essentially, the whole thesis consists of five published papers, listed at the beginning of the introduction. We use the term 'paper n', where n is any number from 1 to 5, to refer to them.

This thesis tackles several open problems on rings and modules that have been going around for a long time and which, according to the interest showed on them and the time passed without being answered, are of great relevance to different areas of Mathematics, including Ring and Module Theory, Representation Theory, (absolute and Gorenstein) Homological Algebra and Algebraic Geometry. In almost all of them, the author has been successful in either answering them or getting a strong advance. We will expose those problems in the course of the description of the contents of the thesis. I have checked the report made by the program Tunitin, that has confirmed that there is no indication at all of scientific misconduct in the preparation of this thesis.

In paper 1 the author answers a question proposed by Maurice Auslander in a conference in 1975. Auslander asked whether the codomain of a right almost split morphism in Mod$-R$ is a finitely presented module. The affirmative answer is given in Theorem 1.4.4. The main ingredient of the proof is a combinatorial object that Saroch introduces and that he names 'tree module'. But, even before introducing test modules and in the course of the preparation for the proof of the last mentioned theorem, the author is able to answer a question proposed by Bergman. Concretely, Theorem 1.3.2 shows that if R is a countable ring such that R^ω is a flat Mittag-Leffler right R-module, then a module M is cotorsion if, and only if, the map $\mathrm{Hom}(\iota, M)$, induced by the inclusion $\iota : R^{(\omega)} \to R^\omega$, is surjective.

Papers 2 and 3 are devoted to approximations and Mittag-Leffler conditions. The class of flat Mittag-Leffler modules were introduced by Grothendieck and played an important role in the proof of the locality of the notion of vector bundle. The class of flat Mittag-Leffler modules was known to be neither deconstructible nor precovering over rings with some extra conditions. It has an open question for one decade whether those extra conditions were necessary. The main goal of paper 2 was to get rid of those extra conditions. The first main result of that paper, Theorem 2.3.3, shows that if R is non right perfect then the class $\mathcal{FM}$ of flat Mittag-Leffler R-modules is not precovering (and not deconstructible). The main tool for its proof was Lemma 2.3.2 of that paper, where, for a class $\mathcal{F}$ of modules, the author develops a test, using Baer modules over $\mathcal{F}$, to discard the precovering condition of the class $\mathcal{L}$ of locally $\mathcal{F}$-free R-modules. This tool is further used to prove a general (i.e. non-hereditary) version of the Countable Telescope Conjecture. Namely, the second main result

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of the paper, Theorem 2.6.1, states that if $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\text{Mod} - R$, where $\mathcal{B}$ is closed under taking direct limits, then $\mathcal{B}$ is definable and $\mathfrak{C}$ is generated by the countably presented modules in $\mathcal{A}$.

The tools developed in paper 2 were heavily used in paper 3, joint with L. Angeleri-Hugel and J. Trlifaj, where several nice applications are found. The paper generally considers the case of a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in $\text{Mod} - R$, where $\mathcal{B}$ is closed under taking direct limits. Cotorsion pairs of finite type and, in particular, tilting cotorsion pairs are the prototypes. The authors prove in Theorems 3.3.6, 3.4.1 and 3.5.2 that the following conditions are all equivalent:

(a) $\mathcal{A}$ is also closed under taking direct limits;

(b) $\mathfrak{C}$ is cogenerated by a (discrete) pure-injective module;

(c) $\mathcal{A}$ is closed under taking pure epimorphic images (and pure submodules);

(d) All Bass modules over $\mathcal{A}^{\leq\omega}$ are contained in $\mathcal{A}$ (equivalently, the class $\mathcal{A}^{\leq\omega}$ of countably presented modules in $\mathcal{A}$ is closed under taking countable direct limits);

(e) The class $\mathcal{L}$ of $\mathcal{A}^{\leq\omega}$-free modules is deconstructible;

(f) $\mathcal{A}$ is a covering class;

(g) $\text{Ker}(\mathfrak{C}) := \mathcal{A} \cap \mathcal{B}$ is closed under taking countable direct limits;

(h) $\mathcal{L}$ consists precisely of the pure-epimorphic images of modules in $\mathcal{A}$.

The authors further prove in Lemma 3.5.4 that a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, with $\mathcal{B}$ closed under taking direct limits, has the property that $\text{Ker}(\mathfrak{C}) = \text{Add}(K)$, for some module K. Once they know this, they are able to prove in Corollary 3.5.5 that the conditions (a)-(f) above are further equivalent to:

(i) Any module in $\text{Ker}(\mathfrak{C})$ has a semiregular endomorphism ring;

(j) The module K has a perfect decomposition.

Paper 3 ends with a nice connection of the results with the Pure Semisimplicity Conjecture (PSSC). This conjecture has been open for more than sixty years. Stated on the left, it says that any left pure-semisimple ring should be of finite representation type or, equivalently, it should be also pure-semisimple on the right. The surprising result in connection with PSSC is that the verification of the conjecture for hereditary rings can be expressed in terms of properties of the cotorsion pairs in $\text{Mod} - R$. Summing up Theorem 3.6.1 and Proposition 3.6.3 one gets that, for a left pure-semisimple hereditary ring R, the following conditions are equivalent:

(a) R satisfies PSSC;

(b) Every cotorsion pair in $\text{Mod} - R$ is tilting;

(c) Every cotorsion pair in Mod$R$ is of finite type;

(d) The cotorsion pairs in Mod $-$ R form a set;

(e) Each 1-tilting right R-module is Σ-pure-split;

(f) There are only finitely many left (resp. right) tilting modules, up to equivalence.

In paper 4, joint with Jan Stovicek, the authors address the following two questions for a class $\mathcal{B}$ in Mod $-$ R:

- Is there a cardinal $\kappa = \kappa(\mathcal{B})$ such that if $\mathrm{Ext}^1_R(M, \mathcal{B}) = 0$, then $M = \varinjlim M_i$ is a direct limit of κ-presented modules M_i such $\mathrm{Ext}^1_R(M_i, \mathcal{B}) = 0$ for all $i \in I$?

- Conversely, if $M = \varinjlim M_i$, where $\mathrm{Ext}^1_R(M_i, \mathcal{B}) = 0$ for all $i \in I$, when is it true that $\mathrm{Ext}^1_R(M, \mathcal{B}) = 0$?.

The general study of these two questions allows the authors to apply them to two seemingly distant classes of modules. Namely, the Σ-cotorsion modules on one side and, on the other, the Gorenstein flat and Gorenstein injective modules, respectively. The authors show in Theorem 4.3.3 that if C is a Σ-cotorsion module then each module in the smallest definable subcategory of Mod $-$ R that contains C is also Σ-cotorsion. On the side of Gorensten Homological Algebra, there have been two open questions that have deserved a lot of attention in the last two decades. The first one asks whether the class of Gorenstein flat modules is covering and, dually, if the class of Gorenstein injective modules is enveloping. The other one asks whether any Gorenstein flat module is Gorenstein projective. In this paper the authors answer the first question, both in the direct and dual version, and get advance in the second. The crucial point is the introduction of a class of modules, that they name 'projectively coresolved Gorenstein flat', that lies strictly between the classes of Gorenstein flat and Gorenstein projective modules. Then they show in Theorem 4.4.9 that the class $\mathcal{PGF}$ of projectively coresolved Gorenstein flat modules is the left constituent of a complete hereditary cotorsion pair whose right part $\mathcal{PGF}^{\perp}$ is thick, i.e. if any two of the members of a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ is in $\mathcal{PGF}^{\perp}$ then so is the the other one. With this result at hand, they are able to characterize Gorenstein flat modules in Theorem 4.4.11 as particular epimorphic (resp. monomorphic) images of projectively coresolved Gorenstein flat modules with flat kernel (resp. cokernel). From this they derive the desired consequence. Concretely, in Corollary 4.4.12 they show that the class $\mathcal{GF}$ of Gorenstein flat modules is the left consitutent of a complete hereditary cotorsion pair whose kernel is the class of flat cotorsion modules, from which it easily follows that $\mathcal{GF}$ is a covering class. The also prove the 'dual' result in Theorem 4.5.6, showing that the class $\mathcal{GI}$ of Gorenstein injective modules is the right part of a hereditary perfect cotorsion pair, and so, in particular, $\mathcal{GI}$ is an enveloping class.

In the last paper 5, joint with J. Trlifaj, the authors show that, under the addition of some extra axiom to ZFC, it is possible to characterize when a cotorsion pair in Mod $-$ R is generated or cogenerated by a set of modules. More generally, under those extra axioms, given such a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, they characterize when there is a monomorphism $\iota : X \to Y$ (resp. an epimorphism $p : X \to Y$) such that $\mathcal{B}$ (resp. $\mathcal{A}$) consists of the modules M such that $\mathrm{Hom}_R(\iota, M)$ (resp. $\mathrm{Hom}_R(M, p)$) is a surjective map.

I find the results of this thesis really outstanding. The impact of them is visible in several fields of research in Algebra. Moreover, the techniques used in the proof are very involved and include heavy machinery of Module and Ring Theory, Logic, Set Theory and Homological Algebra. In my opinion, the thesis is of a level much higher than the average that one finds in most European Universities. For these reasons, I strongly support that it be publicly defended.

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