# INTEGRATED EXPONENTIALS OF BROWNIAN MOTION AND RELATED PROCESSES WITH APPLICATIONS TO ASIAN OPTION VALUATION

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# Contents

# Introduction

This habilitation thesis summarizes contributions of its author in the area of valuation of Asian options. As the work spans now a period of two completed decades, we can put all the existing results into a broader perspective. Asian options are contracts whose payoff depends on the average of the price of a specific asset. As the average is one of the basic statistics of the price (the other being the price itself at a given moment, also known as a spot, or the maximum of the price), its study attracts constant attention in both practice and academia. The simplest distribution of an asset price is modeled by geometric Brownian motion, which is obtained by exponentiation of Brownian motion $W(t)$. More specifically, the asset price in the geometric Brownian motion is modeled by a process

$$S(t) = S(0)\exp(\sigma W(t) - \tfrac{1}{2}\sigma^2 t).$$

The average price is then represented by an integral

$$\frac{1}{T}\int_0^T S(t)dt = \frac{1}{T}S(0)\int_0^T \exp(\sigma W(t) - \tfrac{1}{2}\sigma^2 t)dt.$$

The main problem is that from the mathematical point of view, the average of the price is – perhaps surprisingly – too complicated to admit a tractable analytical formula even in the simplest price models, such as in the geometric Brownian motion model.

Historically, the problem of analytical characterization of the average of the geometric Brownian motion was studied from several different perspectives. The two most important papers that shaped the field are Geman and Yor [31] and Rogers and Shi [71]. These two papers are also most cited works in this direction. The paper by Geman and Yor found a Laplace transform representation of the average price, but the exact transformation is in the form of an integral, and its inversion comes with a substantial computational challenge. In addition, it applies only in the situation of continuous averaging, but the applications in practice require discrete averaging. The approach of Rogers and Shi is to find a representation of the average price in terms of a partial differential equation. This approach does not lead to an analytical representation of the distribution of the average, but the partial differential equation characterization allows for a numerical approach with arbitrary precision. The PDE approach was already known from Ingersoll [45], but his formulation was in terms of a partial differential equation with 2 spatial variables. Rogers and Shi were able to reduce the PDE to one spatial variable by choosing a stock as a reference asset, making it more numerically tractable. However, they used a running average as their explanatory variable, which in today's perspective seems like a suboptimal choice for the problem. The running average cannot be obtained by a self-financing trading strategy, which means that its stochastic evolution is not a martingale. Non-martingale evolutions in turn produce some extra terms in the resulting PDE, making it more complicated and less numerically stable. The PDE approach can also be applied to the discretely sampled averages as shown in Andreasen [1], but the discrete averaging applied to the running average introduces discontinuities at each sampling point. This approach requires that the PDE has to be pasted at these points, making this approach somewhat complicated.

It should be mentioned that papers focused on Monte-Carlo simulation methods dominated most of the literature on Asian options from 1990's. It has been known that a specific choice of a control variable – the geometric average – was a well-suited choice since the geometric average of a geometric Brownian motion admits a closed form solution. However, the Monte-Carlo methods are in general ill-suited for evaluation of the arithmetic average price since the average requires to compute all values at the sampling points, making such methods computationally inefficient. The simulation methods were later replaced by methods based on numerical solutions of the partial differential equations, an approach that is promoted by the author of this work. We do not list these papers in this part of the text as the references to such papers can be found in the individual publications presented in this thesis. We would like to mention two results, namely Dufresne [24] Milevsky and Posner [63]. These works looked at the distribution of the average price in a perpetual setup (when $T \to \infty$). Interestingly, they found that the distribution of the perpetual average is inverse gamma, which is itself regarded as a remarkable observation.

The author's main contribution is in a proper martingale formulation of the problem, which leads to the simplest possible form of the resulting partial differential equation that also easily applies to a discretely sampled average. The martingale formulation allows for natural extensions of the price evolution, in particular, it can be extended to models with stochastic volatility or models with jump evolution of the price. In addition, one can simplify Geman and Yor's formula for Laplace transform for the continuously sampled option and find a representation which does not involve an integral by using Whittaker functions. The author later showed that the price of the Asian option admits analytical representation analogous to the Nobel Prize winning Black-Scholes formula (Black and Scholes [8]). The representation of the hedging formula also becomes available. Lastly, the approach can be applied to other types of price averaging, in particular to the harmonic average. In total, we present six papers in this direction. These papers attracted a number of citations that rank these results somewhat favorably and place them among the top publications in this area. Only two of them have another coauthor (Steven Shreve, the Ph.D. adviser of the author, and Mingxin Xu, the author's schoolmate), which indicates that most of these ideas are indeed entirely original contributions. The existence of such a number of solo papers is somewhat unusual even for the author himself as most of his other publications are results of teamwork.

The thesis has the following structure. The first part reviews the theory of martingale pricing with fundamental results that are used in our study of Asian option pricing. Our explanatory part consists of three parts:

**Martingale pricing theory,** where we introduce the basic principles of mathematical finance,

**Diffusion models,** where we illustrate these principles on models driven by Brownian motion, and

**Asian options,** where we focus on explanation of the basic principles behind Asian option pricing.

We explain the basic finance principles, in particular we review no-arbitrage principle (no agent in the market can produce a risk-free profit). According to the First Fundamental Theorem of Asset Pricing, no arbitrage is equivalent to the fact that prices are martingales. This central result dates back to Harrison and Kreps [36] and Harrison and Pliska [37], but we use a generalized version using an arbitrary reference asset that is needed for our purposes. While various choices of a reference asset have been used in some particular situations, for instance in a work of Geman et al. [32], such a general exposition is original. The author published an entire monograph Vecer [81] "Stochastic Finance: A Numeraire Approach" in this direction. The explanatory part of the habilitation thesis is partly based on some parts that appear in this monograph. The idea of using a general reference asset in pricing is motivated by the fact that the valuation of the Asian option is simpler if we use stock as a reference asset in contrast to using money as a reference asset. This approach proved to be very fruitful as it simplifies many pricing problems that have been historically valued in more complicated fashion.

Once we have all the necessary theoretical foundation, we apply these techniques to pricing the Asian options. We also put the published papers on this topic in a broader perspective, so a contribution of each paper for understanding the problem is clearly explained. The first part thus serves as a brief introduction to a broader area of mathematical finance related to this problem so that it allows a non-specialist to understand and follow the core ideas presented in the papers in the second part of the thesis.

The second part includes the published papers of the author in this direction of study. Specifically, it includes the following six papers:

1. **Shreve, S., J. Vecer (2000): "Options on a Traded Account: Vacation Calls, Vacation Puts and Passport Options",** ***Finance and Stochastics*****, Vol. 4, No. 3, 255-274.** [77]

   This paper studies pricing options on a traded account. A traded account is a portfolio created by a client who trades in several underlying assets – typically just two, such as money and a stock. The client is restricted by contractual trading constraints, such as on a maximal and a minimal position in the stock, but he is free to change the positions at any time with any frequency. One novelty of the paper is to study different constraints on the trading position in the stock. At the end of a monitoring period, he can keep all the resulting profits of a such trading and he is forgiven any losses. In this sense, this represents an insurance on an actively managed portfolio. In practice, this contracts did not become mainstream financial products for several reasons. Such contracts are relatively complicated, leading to a decreased demand for them. Some clients may regard such insurance as relatively expensive. Moreover, the formulation of this problem is not symmetric with respect to the assets involved in the contract, which makes it unnatural in markets with equal treatment of the assets (such as in the foreign exchange markets).

From the mathematical point of view, this is an interesting application in the field of stochastic optimal control. As the client is free to use any contractually allowed strategy, the seller of this product must be ready to face the worst case scenario in terms of the implied costs. It turns out that the most costly strategy maximizes the volatility. The strategy can be found both from the probabilistic arguments or by using analytical techniques by solving the resulting Hamilton-Jacobi-Bellman equation. This technique allowed us to find the analytical formulas for the price of such products.

This paper made an observation that turned out to be critical in the subsequent research, namely that the price of the actively traded portfolio represented in terms of a stock is Markovian in one dimension, while the price of the same portfolio represented in terms of money is Markovian in two dimensions. The reason is the asymmetry of the contractual constraints, the constraints on the stock position are deterministic, but the constraints on the money position are implied by the current value of the portfolio, and thus random. As a consequence, this contract has the stock as a preferential reference asset, which leads to a simpler formulation of the pricing problem. The resulting partial differential equation has only one spatial variable.

2. **Vecer, J. (2001): "A new PDE approach for pricing arithmetic average Asian options",** ***Journal of Computational Finance*****, Vol. 4, No. 4, 105-113.** [79]

This paper extends the work on the options on a traded account. It makes a new observation that the portfolios that depend on an average price can be replicated by an active trading strategy, thus representing a particular form of an option on a traded account. This approach covers both discrete and continuous averaging. Interestingly, the strategy replicating the average is deterministic, and thus the pricing problem is somewhat simpler than the previously studied problem of an option on a traded account that allowed for a choice of a strategy for the holder of the option. The observation that a self-financing strategy can replicate the average price is indeed critical. It means that the price evolution of a portfolio that replicates the average is a martingale and thus according to the theory, its price must be a martingale under the probability measures corresponding to a reference asset. This approach finds a proper choice of an explanatory variable.

As it was previously known from our previous paper, the choice of the reference asset is asymmetric, the one-dimensional formulation corresponds to the stock chosen as a reference asset. Even though the paper is from the mathematical point of view relatively simple, it did provide a computationally simple method to determine the price of the Asian options. It has been receiving a constant citation response.

It is interesting to note that while the problem for the options on a traded account with that allowed the client to change the positions randomly led to an analytical solution, a seemingly simpler problem of the Asian option that has a deterministic position in a stock has no simple analytical representation. The problem is that the deterministic position is a function of time $t$ so the spatial term $u_{zz}$ in the pricing PDE is multiplied by a coefficient that also depends on time. This fact makes the problem complicated from the analytical point of view. Even for the simplest choice of parameters, there is no simple analytical formula. We know it as a fact since in the later work, the Laplace transform is expressed analytically, but the inversion cannot be expressed in a simple form.

3. **Vecer, J. (2002): "Unified Asian Pricing", *Risk*, Vol. 15, No. 6, 113-116.** [80]

   This paper is an extension of the "A new PDE approach for pricing arithmetic average Asian options" paper. A simple and numerically stable 2-term partial differential equation characterizing the price of any type of arithmetically averaged Asian option is given. The PDE is improved, it is formulated in a proper martingale fashion that absorbs all discounting into the explanatory variables and thus it removes the $u_z$ term that appeared previously. The approach includes both continuously and discretely sampled options, and it is easily extended to handle continuous or discrete dividend yields. In contrast to the previous methods, this approach does not require to implement jump conditions for sampling or dividend days. Numerical examples are given.

4. **Vecer, J., M. Xu (2004): "Pricing Asian Options in a Semimartingale Model", *Quantitative Finance*, Vol. 4, No. 2, 170-175.** [84]

   This paper shows that the previously presented approach for pricing Asian options also applies to all semimartingale evolutions of the underlying price as the results in the previous papers were limited to the geometric Brownian motion models. In particular, it applies to models with jumps. The paper derived the corresponding partial integro-differential equation. In the meantime, it was shown by Fouque and Han [27] that the author's approach can be used in the stochastic volatility setup.

5. **Vecer, J. (2014): "Black-Scholes Representation for Asian Options", *Mathematical Finance*, Vol. 24, No. 3, 598-626.** [83]

   The author revisits this topic after 10 years. In the meantime, he published a monograph Vecer [81] (Stochastic Finance: A Numeraire Approach), which puts the previous research to a new perspective. The paper is mathematically quite heavy both in terms of the theory and the actual computations involved. The main contribution is in finding a novel representation for the Asian option price in terms of probabilities under two different martingale measures associated with a stock $S$ and an abstract asset $A$ (an "average asset") that pays off a weighted average of the stock price number of units of a dollar at time $T$. They are model independent formulas and they are analogous to the Black-Scholes formula for the plain vanilla options; they are expressed in terms of probabilities under the corresponding martingale measures that the Asian option will end up in the money. Computation of these probabilities is relevant for hedging. In contrast to the plain vanilla options, the probabilities for the Asian options do not admit a simple closed form solution. However, the paper shows that it is possible to obtain the numerical values in the geometric Brownian motion model efficiently, either by solving a partial differential equation numerically or by computing the Laplace transform. The Laplace transform formula is given in terms of Whittaker functions and does not involve any integration in contrast to the previous result of Geman and Yor [31]. Models with stochastic volatility or pure jump models can also be priced within the Black-Scholes framework for the Asian options.

6. **Vecer, J. (2014): "Asian Options on the Harmonic Average", *Quantitative Finance*, Vol. 14, No. 8, 1315-1322.** [82]

   This paper extends the pricing techniques to contracts written on the harmonic average. As the arithmetic average dominates the harmonic average, the contracts covering the increase of the average price are cheaper in the situation of the harmonic average. For this feature, the contracts on the harmonic average enjoy some popularity on the foreign exchange market. If $X$ denotes the foreign currency and $Y$ denotes the domestic currency, the payoff of the contract is a function of a price of an asset $H$ which is defined as

$$H(T) = \left[\int_0^T [X_Y(t)]^{-1}\eta(t)dt\right]^{-1} Y(T) = \left[\frac{1}{\int_0^T Y_X(t)\eta(t)dt}\right] Y(T).$$

   The harmonic average resembles a quanto option: the price $Y_X(t)$ is monitored with respect to the foreign currency $X$, but the payoff is settled in the domestic currency $Y$. Although the pricing problem appears to be rather complex, it can be ultimately simplified to a partial differential equation in one spatial variable after a numeraire change and using the time reversal argument.

# Chapter 1

# Martingale Pricing Theory

This chapter provides a summary of contemporary techniques used in mathematical finance in relation to the author's work in this direction. Today, the entire mathematical theory of pricing is based on a principle that no agent can produce a risk-free profit by trading in the available assets. The possibility to produce a risk-free profit is called an arbitrage and the assumption of the pricing theory is that there is no arbitrage. While this condition may seem obvious, it was not the first principle to be used. The traditional economic theory has been based on the fact that agents on the market have supply and demand functions, the number of assets they are willing to sell and buy that is a function of the market price. The price where these two functions intersect is called an equilibrium. While the approach based on supply and demand functions may explain the market behavior in the traditional markets such as individual stocks, it does not explain the derivatives markets that are dependent on some primary asset. The no-arbitrage principle sets restrictions on the prices of derivatives and provides a typically narrow interval of possible values where the price could be.

The key result of the pricing theory is the First Fundamental Theorem of Asset Pricing by Harrison and Kreps [36] and Harrison and Pliska [37] which states that no-arbitrage assumption is equivalent to the martingale evolution of the prices. Martingales are processes that keep a constant conditional expectation, so the intuition of this result is that the prices should not exhibit a predictable drift. In particular, this means when the prices are continuous, they cannot have a derivative, otherwise the agents in the market would be able to lock a risk-free profit from observing the drift. As a consequence, the drift component must be zero and the entire evolution of the price is due to a diffusion process (an integrated Brownian motion).

Some of the basic concepts of finance are widely understood in broad terms; however this chapter will introduce them from a novel perspective of prices being treated relative to a reference asset. We first show the difference between an asset and the price of an asset. The price of an asset is always expressed in terms of another reference asset. The reference asset is also called a **numeraire.** The numeraire asset should never become worthless so that the price with respect to this asset is well defined. The relationship between prices of an asset expressed with respect to two different reference assets is known as a **change of numeraire.** The concept of price appears in different markets under different names, so it may not be obvious that it is just a particular instance of a more general concept. For instance, an exchange rate is in fact a price representing a pairwise relationship of two currencies. An even less obvious example of a price is a forward London Interbank Offer Rate (LIBOR). By adopting a precise definition of price, we are able to treat various markets (equities, foreign exchange, fixed income) in one single unified framework, which simplifies our analysis.

The second section introduces the concept of arbitrage – the possibility of making a risk free profit. We study models of markets where no agent allows an arbitrage opportunity. One can create an arbitrage opportunity just by holding a single asset such as a banknote. This is known as a time value of money. Thus the concept of no arbitrage splits assets into two groups: no-arbitrage assets – the assets that do not allow any arbitrage opportunities; and arbitrage assets – the assets that do allow arbitrage opportunities. In theory, the market should have only no-arbitrage assets. Financial contracts are typically no-arbitrage assets; they become arbitrage assets only when their holder takes some suboptimal action (such as not exercising the American put option at the optimal exercise time). On the other hand, real markets include arbitrage assets such as currencies.

Currencies, in terms of banknotes, are losing an interest rate when compared to the corresponding bond or money market account. Since the loss of the currency value is typically small, money still serves as a primary reference asset in the economy. However, in order to avoid this loss of value in pricing contingent claims, one should use discounted prices rather than dollar prices of the assets. Discounted prices correspond to either a bond or a money market account as a reference asset. Stocks paying dividends are arbitrage assets when the dividends are taken out, but an asset representing the equity plus the dividends is a no-arbitrage asset. We find a simple relationship between the dividend paying stock and the portfolio of the stock and the dividends.

In the section that follows, we introduce the concept of a portfolio. A portfolio is a combination of several assets, and it is important to realize that it has no numerical value. In fact, one should not confuse the concept of a portfolio (viewed as an asset) with the price of a portfolio (number that represents a pairwise relationship of two assets). It should be noted that a portfolio may be staying physically the same, but the price of this portfolio with respect to some reference asset may be changing. We also introduce the concept of trading. Self-financing trading is exchanging assets that have the same price at a given moment. As a consequence, portfolios may be evolving in time by following a self-financing trading strategy.

When no arbitrage exists in the markets, all prices are martingales with respect to the probability measure that comes with the specific no-arbitrage reference asset. Martingales are processes whose best estimator of the future value is its present value. Mathematically, a process $\mathcal{M}$ that satisfies $\mathbb{E}_s[\mathcal{M}(t)] = \mathcal{M}(s), s \leq t$, is a martingale, where $\mathbb{E}_s[.]$ denotes conditional expectation. The reader should refer to the Appendix for more details about martingales and conditional expectation. The result that prices are martingales under the probability measure that is related to the reference asset is known as the First Fundamental Theorem of Asset Pricing. In particular, every no-arbitrage asset has its own pricing **martingale measure.** Other no-arbitrage assets have different martingale measures. The martingale measure associated with the money market account is known as a **risk-neutral measure.** The martingale measures associated with bonds are known as **T-forward measures.** Stocks have martingale measures known as a **stock measure.** Arbitrage assets, such as currencies, do not have their own martingale measures. In particular, there is no dollar martingale measure.

Many authors do not regard currencies as true arbitrage assets because this arbitrage opportunity is one sided for the issuer of the currency. It is also easy to confuse money (in terms of banknotes) with the money market account. Banknotes deposited in a bank start to earn the interest rate and become a part of the money market account. When borrowing money, the debt is not a currency, but rather the corresponding money market account. The debt earns the interest to the lender, and thus it behaves like the money market account. However, arbitrage pricing theory applies only to no-arbitrage assets, such as the money market account, bonds, or stocks. It does not apply to money in terms of banknotes. No-arbitrage assets have their own martingale measure, while arbitrage assets do not.

An important consequence of the First Fundamental Theorem of Asset Pricing is that the prices are martingales with respect to a probability measure associated with a particular reference asset. Martingales in continuous time models are under some assumptions just combinations of continuous martingales, and purely discontinuous martingales. Moreover, continuous martingales are stochastic integrals with respect to Brownian motion. This limits possible evolutions of the price to this class of stochastic processes since other types of evolutions allow for an existence of arbitrage.

Another related question to the concept of no arbitrage is a possibility of replicating a given financial contract by trading in the underlying primary assets. The martingale measure from the First Fundamental Theorem of Asset Pricing may not necessarily be unique; each reference asset may have infinitely many of such measures. However, each martingale measure under one reference asset has a corresponding martingale measure under a different reference asset that agrees on the prices of the financial contracts. The two measures are linked by a Radon–Nikodým derivative. In particular, when there is a unique martingale measure under one reference asset, the martingale measures that correspond to other reference assets are also unique due to the one-to-one correspondence of the martingale measures.

In the case when the martingale measure is unique, all financial contracts can be perfectly replicated. This result is known as the Second Fundamental Theorem of Asset Pricing. The market is complete essentially in situations when the number of different noise factors does not exceed the number of assets minus one. Thus models with two assets are complete when there is only one noise factor, which is, for instance, the case in the binomial model, in the diffusion model driven by one Brownian motion, or in the jump model with a single jump size. When the market is complete, the financial contracts are in principle redundant since they can be replicated by trading in the underlying primary assets. The replication of the financial contracts is also known as **hedging.**

## 1.1 Price

This section defines price as a pairwise relationship of two assets.

**Price is a number representing how many units of an asset $Y$ are required to obtain a unit of an asset $X$.**

We denote this price at time $t$ by

$$X_Y(t).$$

Here an asset $Y$ serves as a reference asset. The reference asset is known as a **numeraire. Price is always a pairwise relationship of two assets.**

For practical purposes the role of a reference asset is typically played by **money**, a choice of the reference asset $Y$ being a dollar \$. However, the choice of the reference asset is in principle arbitrary as long as the reference asset is not worthless. The reader should note that some financial assets may become worthless at a certain stage (such as options expired out of the money), and such contracts would be a poor choice of the reference asset. There are also some desirable properties that the reference asset should satisfy: it should be sufficiently durable, and there should exist enough identical copies of the asset. From this perspective, consumer goods (such as cars, electronic products, most food products) may be used as a reference asset, but this choice would not be appropriate since the asset itself has time value; it is deteriorating in time.

In practice, a small loss of the value of the reference asset is acceptable. Currencies in particular lose value in time by allowing an arbitrage opportunity with respect to the money market account, and they still play a role of a primary reference asset in the economy. However, when the loss of the value becomes large, for instance in a period of hyperinflation, such currency may no longer be accepted as a reference asset. The property of having sufficient identical copies of the asset ensures that the individuals in the economy can easily acquire the reference asset. The reference asset should be sufficiently liquid. For instance some art works (paintings, sculptures, buildings) have a significant value, but they cannot be easily bought or sold and thus using them as a reference asset would not be a good choice.

Typical choices of a reference asset used in practice are currencies (denoted by \$, €, £, ¥, etc.), bonds (denoted by $B^T$), a money market (denoted by $M$), or stocks and stock indices (denoted by $S$). A **bond** $B^T$ is an asset that delivers one dollar at time $T$. The **money market** $M$ is an asset that is created by the following procedure. The initial amount equal to one dollar is invested at time $t = 0$ in the bond with the shortest available maturity (ideally in the next infinitesimal instant), and this position is rolled over to the bond with the next shortest maturity once the first bond expires. The resulting asset, the money market $M$, is a result of an active trading strategy involving a number of these bonds. In principle, there is a counter party risk involved in delivering a unit of a currency at some future time. The counter party may fail to deliver the agreed amount at the specified time. The following text assumes situations when there is no such risk present, as in the case when the delivery of the asset is guaranteed by the government.

The reference asset itself does not need to be a traded asset. As we will see in the chapter on pricing exotic options, some natural reference assets that are useful for pricing complex financial contracts do not exist in real markets. For instance, one can use an asset that represents the running maximum of the price $\max_{0 \leq s \leq t} X_Y(s)$ for pricing lookback options, or one can use an asset that represents the average price for pricing Asian options. A price of a financial contract that is expressed in terms of an asset which is not traded can be easily converted to a price expressed in terms of a traded asset. Thus for practical purposes it does not matter if the reference asset exists or not in real markets.

Let us introduce the following notation. **By $X(t)$ we mean a unit of an asset $X$ at time $t$, not its price in terms of a different asset.** In principle, an asset $X$ that has no time value stays the same at all times (think of an ounce of gold), so there is really no need to index it with time. However, by adding the time coordinate we express that a particular asset is used at that time for trading, pricing, hedging, or for settling some contract. When there is no ambiguity, we will simply drop the time index, and write only $X$ to stress that the asset in fact stays the same.

Recall that price is a pairwise relationship of two assets denoted by $X_Y(t)$ – the number of units of an asset $Y$ required to obtain one unit of an asset $X$. The asset $Y$ is known as a reference asset, or as a numeraire. We can write that

$$1 \text{ unit of } X = X_Y(t) \text{ units of } Y,$$

or simply

$$X = X_Y(t) \cdot Y. \tag{1.1}$$

Assets $X$ and $Y$ on their own do not have any numerical value (such as an ounce of gold), and the above equality does not mean that the assets on the left hand side and on the right hand side of the equation are physically the same. Note that we cannot divide by $Y$ in the above equation since $Y$ is an asset.

The relation "=" when used for assets as in Equation (1.1) is an **equivalence relation. We will write $X(t) = Y(t)$ in the sense of assets when $X_Y(t) = 1$ in the sense of numbers.** Clearly, the relation "=" for assets is

- reflexive: $X(t) = X(t)$,
- symmetric: $X(t) = Y(t)$ implies $Y(t) = X(t)$,
- transitive: $X(t) = Y(t)$ and $Y(t) = Z(t)$ imply $X(t) = Z(t)$.

The assets are also ordered according to their prices. We can write $X(t) \geq Y(t)$ in terms of assets when $X_Y(t) \geq 1$ in terms of numbers. It should be noted that two assets $X$ and $Y$ with an equal price at time $t_1$ (meaning $X_Y(t_1) = 1$) may differ in price at some other time $t_2$ (meaning $X_Y(t_2) \neq 1$). If two assets $X$ and $Y$ have the same price at time $t$, they can be exchanged for each other at that time. This procedure is known as a **self-financing trade.**

It may not be clear as to why we should adopt notation $X_Y(t)$ for the price, instead of using just a single letter for it, say $S(t)$, which is typically used for the price of a stock in terms of dollars. The following examples illustrate that the concept of price appears in different markets, such as in equity markets, in the foreign exchange markets, or in fixed income markets. By using our notation, we are able to treat these prices in one single framework, rather than studying them separately.

**Example 1.1 (Examples of the price)**

- *The* **dollar price** *of an asset $S$, $S_\$(t)$, where the role of the asset $X$ is played by the stock $S$, and the role of the reference asset $Y$ is played by the dollar \$. Most of the current literature writes simply $S(t)$ for the dollar price $S_\$(t)$ of this asset, but we want to avoid in our text confusing the asset $S$ itself with the price of the asset $S_\$(t)$.*
- *The price of a stock $S$ in terms of the money market $M$, $S_M(t)$, where the asset $X$ is a stock $S$, and the asset $Y$ is a money market $M$ with $M(0) = \$(0)$. The price $S_M(t)$ is known as a* **discounted price** *of an asset $S$.*
- *The price of a stock $S$ in terms of a zero coupon bond $B^T$ with maturity $T$, $S_{B^T}(t)$, where the asset $X$ is a stock $S$, and the asset $Y$ is a bond $B^T$. This is also a form of a* **discounted price** *which is more appropriate than $S_M$ for pricing derivative contracts that depend on $S$ and \$. Note that we have $S_{B^T}(T) = S_\$(T)$.*
- *The* **exchange rate**, *$€_\$(t)$, where $X$ is the foreign currency (€), and $Y$ is the domestic currency \$. The choice of domestic and foreign currency is relative, and thus $\$_€(t)$ is also an exchange rate.*
- **Forward London Interbank Offered Rate**, *or forward LIBOR for short,*

$$\left[B^T - B^{T+\delta}\right]_{\delta B^{T+\delta}}(t),$$

*where the role of the asset $X$ is played by a portfolio of two bonds $[B^T - B^{T+\delta}]$, and the reference asset $Y$ is $\delta \cdot B^{T+\delta}$.*

We will discuss these examples of price in more detail after introducing the concepts of inverse price, and change of numeraire. Since the assets $X$ and $Y$ considered in the above are arbitrary, it also makes perfect sense to consider the inverse relationship when $X$ is chosen as a reference asset. For instance, one may think about $X$ and $Y$ as two currencies. When $X = €$, and $Y = \$$, we have both the exchange rate $€_{\$}(t)$ – the number of dollars required to obtain a unit of a euro, and the exchange rate $\$_{€}(t)$ – the number of euros required to obtain a unit of a dollar. Thus we can also write

$$1 \text{ unit of } Y = Y_X(t) \text{ units of } X,$$

or simply

$$Y = Y_X(t) \cdot X. \tag{1.2}$$

The price $Y_X(t)$ is the **inverse price** to $X_Y(t)$. Let us show the relationship between $Y_X(t)$ and $X_Y(t)$. Suppose that an agent starts with a unit of an asset $Y$. He can change it for $Y_X(t)$ units of an asset $X$. This amount can be split in two parts: $Y_X(t) - X_Y(t)^{-1}$ and $X_Y(t)^{-1}$ units of an asset $X$. The part of $X_Y(t)^{-1}$ units of an asset $X$ can be exchanged back for a unit $Y$, which follows from the relationship

$$X = X_Y(t) \cdot Y,$$

which is equivalent to

$$Y = X_Y(t)^{-1} \cdot X.$$

We can rewrite the above trading procedure using the following identities

$$\begin{aligned} Y &= Y_X(t) \cdot X \\ &= (Y_X(t) - X_Y(t)^{-1}) \cdot X + X_Y(t)^{-1} \cdot X \\ &= (Y_X(t) - X_Y(t)^{-1}) \cdot X + Y. \end{aligned}$$

Thus the net result of this exchange is $Y_X(t) - X_Y(t)^{-1}$ units of an asset $X$, which must be zero in order not to allow a risk-free profit. Therefore the prices $X_Y(t)$ and $Y_X(t)$ are related by the following relationship

$$\boxed{Y_X(t) = \frac{1}{X_Y(t)}.} \tag{1.3}$$

This relationship is valid when $0 < X_Y(t) < \infty$, which is the case that neither the asset $X$ nor the asset $Y$ is worthless. In this case, $X_Y(t)$ and its inverse price $Y_X(t)$ have the same information.

In general, it should not matter which reference asset is chosen, one should observe similar price evolutions. We will use this as a key principle for pricing derivative contracts studied in this book. One can look at it as a **theory of relativity in finance**: how one views prices depends on one's choice of the reference asset.

Given an asset $X$ and two reference assets $Y$ and $Z$, we can write the price of $X$ with respect to the reference asset $Y$ using

$$X = X_Y(t) \cdot Y. \tag{1.4}$$

Similarly, we can write

$$X = X_Z(t) \cdot Z \tag{1.5}$$

when we use $Z$ as a reference asset. Thus we have

$$\boxed{X = X_Y(t) \cdot Y = X_Z(t) \cdot Z,} \tag{1.6}$$

which is known as a **change of numeraire formula.** The above relationship is written in terms of assets. We can rewrite the above relationship in terms of the price as

$$\boxed{X_Y(t) = X_Z(t) \cdot Z_Y(t).} \tag{1.7}$$

This relationship is valid for assets $X$, $Y$, and $Z$ that are not worthless.

**Example 1.2 (Foreign Exchange Market)**
*Let us illustrate the concepts of the inverse price and the change of numeraire on the foreign exchange market. Prices in the real markets satisfy the relationship*

$$Y_X(t) = X_Y(t)^{-1}$$

*at all times (up to the rounding errors). For instance, on January 8th, 2010, at 8:00PM EST, the exchange rates between € and \$ were:*

$$€_\$ = 1.4415 \qquad \$_€ = 0.6937.$$

*We can easily check that*

$$\$_€^{-1} = \frac{1}{0.6937} = 1.441545...$$

*Thus the inverse exchange rate* $\$_€^{-1}$ *matches the first four digits of the exchange rate* $€_\$$*. The exact match is typically not possible since these exchange rates are quoted in four decimal digits. However, the arbitrage is still not possible due to the difference of the prices offered and asked. An agent who wants to acquire a unit of an asset should be ready to pay more than an agent who wants to sell a unit of the same asset.*

*More specifically, the market exchange works in the following way: Agents who want to buy a particular asset place their orders on the market exchange, and wait until they find corresponding counter parties that are willing to match their orders. The orders compete according to the price that is quoted; a higher quote has a higher priority of being executed. The highest quote is known as the* **best bid.** *Similarly, agents who want to sell a particular asset place their orders on the market exchange. A smaller price asked for a unit of a given asset has a higher priority. The smallest price asked is known as the* **best ask.** *Clearly, the best ask is larger than the best bid. The smallest difference between two possible quoted prices on the exchange is known as a* **tick.** *In the case of euro/dollar exchange rates, the tick is equal to 0.0001. The difference between the best bid and the best ask is known as a* **bid-ask spread.** *Bid-ask spreads may be larger than a tick. More liquid assets have smaller* *bid-ask spreads, the difference between the buying and the selling price being smaller.*

*From the perspective of having both $X_Y(t)$ and $Y_X(t)$ as prices, there is no absolute direction of up and down in the market. Each trade has two sides, a seller and a buyer. If the market moves in one direction, it is either to the benefit of the seller and at the expense of the buyer, or vice versa. This is another way of saying that when one of the prices $X_Y(t)$ or $Y_X(t)$ goes up, the inverse price must go down.*

*Exchange rates also serve as an example of the change of numeraire formula. Table 1.1 shows the exchange rate table for four major currencies: dollars, euros, pounds, and yen as seen on January 8th, 2010 at 8:00PM EST. For instance the entry (\$, €) gives the price $\$_{€} = 0.6937$, etc.*

Table 1.1: Exchange Rate Table.

| | \$ | € | £ | ¥ |
|---|---|---|---|---|
| \$ | 1 | 0.6937 | 0.6238 | 92.6300 |
| € | 1.4415 | 1 | 0.8991 | 133.5260 |
| £ | 1.6032 | 1.1122 | 1 | 148.5040 |
| ¥ | 0.0108 | 0.0075 | 0.0067 | 1 |

*From the change of numeraire formula, we should also have among other similar relationships*

$$\$_{€} = \$_{£} \cdot £_{€}. \tag{1.8}$$

*In fact,*

$$\$_{£} \cdot £_{€} = 0.6238 \times 1.1122 = 0.693790...$$

*This matches the original $\$_{€}$ rate in four decimal digits if we neglect the rounding error in the fourth digit. This match is close enough not to allow for any arbitrage opportunities due to the market imperfections such as the bid-ask spread, or transaction costs.*

**Remark 1.3** *The change of numeraire formula (1.7) applies to all assets, with or without time value. Note that Equation (1.8) is an example of the change of numeraire formula for assets with time value.*

**Example 1.4 (Forward London Interbank Offered Rate)**
*The Forward London Interbank Offered Rate, or* **LIBOR** *for short, is defined as a simple interest rate that corresponds to borrowing money over the time interval $T$ and $T + \delta$ as seen at time $t \leq T$. We denote forward LIBOR by $L(t, T)$. When $t = T$, $L(T, T)$ is known as a spot LIBOR since it corresponds to borrowing money at the present time $T$.*

*Suppose that one dollar is borrowed at time $T$, and assume that $L(t, T)$ is the simple interest rate for the period between $T$ and $T + \delta$. Then the agent should* *return* $1 + \delta L(t, T)$ *dollars at time* $T + \delta$. *Thus* $L(t, T)$ *can be defined by the following relationship:*

$$(1 + \delta L(t, T)) \cdot B^{T+\delta}(t) = B^T(t). \tag{1.9}$$

*The right hand side of the above relationship indicates that one dollar will be delivered at time* $T$. *The left hand side indicates that* $(1 + \delta L(t, T))$ *dollars will be returned at time* $T + \delta$. *Therefore*

$$\delta L(t, T) \cdot B^{T+\delta}(t) = B^T(t) - B^{T+\delta}(t). \tag{1.10}$$

*We can rewrite this relationship in the following form:*

$$\boxed{L(t, T) = \left[B^T - B^{T+\delta}\right]_{\delta \cdot B^{T+\delta}}(t),} \tag{1.11}$$

*showing that forward LIBOR* $L(t, T)$ *is in fact a price, where the asset* $X$ *is a portfolio* $[B^T - B^{T+\delta}]$ *(long the* $B^T$ *bond, and short the* $B^{T+\delta}$ *bond), and the reference asset* $Y$ *is* $\delta$ *units of the bond* $B^{T+\delta}$.

*If we wanted to compute* $X_Y(t)$ *for two general assets, we can do so from the dollar prices of the assets* $X$ *and* $Y$*:*

$$X_Y(t) = X_\$(t) \cdot \$_Y(t) = \frac{X_\$(t)}{Y_\$(t)}, \tag{1.12}$$

*where we substitute* $Z$ *for* $\$$ *in the change of numeraire formula. Using Equation (1.12), we can determine forward LIBOR from dollar prices of bonds by using*

$$\begin{aligned} L(t, T) &= \left[B^T - B^{T+\delta}\right]_{\delta B^{T+\delta}}(t) \\ &= \left[B^T - B^{T+\delta}\right]_\$(t) \cdot \$_{\delta B^{T+\delta}}(t) = \frac{B^T_\$(t) - B^{T+\delta}_\$(t)}{\delta B^{T+\delta}_\$(t)}. \end{aligned} \tag{1.13}$$

*Here we have used the change of numeraire formula, and linearity of the prices:*

$$[aX + bY]_Z(t) = aX_Z(t) + bY_Z(t). \tag{1.14}$$

Foreign exchange markets, or fixed income markets that trade on LIBORs, are in fact much larger than the equity markets in terms of the volume traded, and thus the main focus of financial markets is on prices that are not expressed exclusively in dollar terms. It is also not an obvious observation that exchange rates and forward LIBORs are in fact prices. Calling them the exchange rates or forward LIBORs is slightly misleading, and the literature tends to study the asset prices, foreign exchange rates, and forward LIBORs separately. In our approach, they are just special cases of a more general concept of price.

Price is always a pairwise relationship of two assets, and we will use this notation throughout this book to indicate the reference asset. This distinction will help us study derivative contracts later on in the text that are written on more than one underlying asset. The second (or the third asset when applicable in the case of exotic options) asset also serves as a viable reference asset for pricing a given derivative contract. This notation is especially helpful when studying quantos and other exotic options, which represent financial contracts that are written on three underlying assets. The reader should also note here that every contract is settled in units of particular assets (dollars, stocks, bonds) rather than in the price itself – the price indicates only how many units of a particular asset are needed.

## 1.2 Arbitrage

This section discusses another fundamental concept of finance, namely arbitrage.

**Arbitrage is an opportunity to make a risk free profit in the market.**

It is important to distinguish an arbitrage opportunity from a profitable trading strategy. Arbitrage means that there is at least one agent that can make money for sure, while a profitable trading strategy simply works on average, meaning that some scenarios may lead to a loss.

An arbitrage opportunity means that one can create a guaranteed profit starting from a portfolio with a zero initial price. It is easy to see that if a portfolio has a zero price with respect to one asset, it has a zero price with respect to any reference asset. A typical example of an arbitrage opportunity is the ability to purchase an asset at a given price and then sell the same asset immediately or some later time for a higher price. The guarantee of a higher price is necessary to make it an arbitrage opportunity, assuring that the portfolio always ends up with more assets than when it started. Such arbitrage trades can happen when a purchase price in one market is less than the selling price in a different market.

**Example 1.5** *Assume that at time* $t = 0$*, the price of an asset* $X$ *with respect to an asset* $Y$ *is* $X_Y(0) = K$*. Suppose that at a fixed time* $T \geq 0$*, the price will be exactly* $X_Y(T) = J$ *with* $J > K$*. In such a case one can construct a portfolio, starting at time* $t = 0$ *with* $P^0(0) = 0$*, exchange it for the portfolio* $P^1(0) = X - K \cdot Y$ *that has a zero price (long one unit of* $X$ *and short* $K$ *units of* $Y$*), and end up with a portfolio* $P^1(T) = X - K \cdot Y$ *at time* $T$*. This portfolio can be exchanged by selling a unit of an asset* $X$ *for* $J$ *units of an asset* $Y$ *for a portfolio with the same price* $P^2(T) = (J - K) \cdot Y > 0$*. This is clearly an arbitrage opportunity.*

A slightly less obvious arbitrage opportunity is a free lottery ticket. Although in most cases a typical lottery ticket does not win any prize, one is certain not to lose any money and still have a possibility of winning something. That qualifies as an arbitrage opportunity.

**Example 1.6** *Assume that there is a free lottery ticket* $L$ *whose price in terms of the dollar \$ is zero:* $L_\$(0) = 0$*. We have seen in the previous example that* *having dollars in a portfolio provides an arbitrage opportunity, but let us assume for the purpose of this example that dollars keep their value with respect to bonds in order to illustrate a different kind of arbitrage. The lottery ticket either expires worthless, or it wins $N$ dollars at time $T$. One can construct the portfolio starting from zero $P^0(0) = 0$, acquiring one zero price lottery ticket, thus creating a portfolio $P^1(0) = L(0)$. This portfolio will convert to $P^1(T) = N \cdot \mathbb{I}(\omega = Win) \cdot \$$, where $\mathbb{I}(\omega = Win)$ is the indicator function of the win. We have that $P^1(T) \geq 0$ for sure, with the possibility of $P^1(T) > 0$. This also constitutes an arbitrage opportunity.*

Another example of an arbitrage opportunity is when the price $X_Y(t)$ of an asset $X$ in terms of an asset $Y$ does not correspond to the price $Y_X(t)$ of an asset $Y$ in terms of an asset $X$.

**Example 1.7 (Arbitrage opportunity when $X_Y(t) \neq Y_X(t)^{-1}$.)**
*If the relationship*

$$X_Y(t) = \frac{1}{Y_X(t)}$$

*does not hold, it is possible to realize a risk-free profit. Assume for instance*

$$\frac{1}{X_Y(t)} < Y_X(t).$$

*In this case, we can start with a unit of an asset $Y$, and exchange it for $Y_X(t)$ units of an asset $X$. We can split this position in two parts: $Y_X(t) - X_Y(t)^{-1}$ and $X_Y(t)^{-1}$ units of an asset $X$. The second part, $X_Y(t)^{-1}$ units of an asset $X$, can be exchanged back for a unit of an asset $Y$. This follows from*

$$X = X_Y(t) \cdot Y,$$

*which is equivalent to*

$$Y = X_Y(t)^{-1} \cdot X.$$

*Therefore one can generate a certain profit of $Y_X(t) - X_Y(t)^{-1} > 0$ units of an asset $X$.*

**Example 1.8** *Assume that $X_Y(t) = 3$, and $Y_X(t) = \frac{1}{2}$. How can one realize a risk free profit? First check that $Y_X(t) = \frac{1}{2} \neq X_Y(t)^{-1}(t) = \frac{1}{3}$. Therefore the prices allow for an arbitrage opportunity. Following the method described in the previous example, we can start with borrowing one unit of $Y$. Using $Y_X(t) = \frac{1}{2}$, we can immediately exchange the unit of $Y$ for $\frac{1}{2}$ units of $X$. We can split $\frac{1}{2}$ units of $X$ in two parts, consisting of $\frac{1}{6}$ and $\frac{1}{3}$ units of $X$. The first part $\frac{1}{6}$ units of $X$ is a net profit from this transaction; the second part can be used for an acquisition and return of a borrowed unit $Y$ using the price relationship $X_Y(t) = 3$.*

Formally, an arbitrage opportunity is defined by:

**If one starts with a zero initial portfolio $P(0) = 0$, follows a self-financing strategy, and ends up with $P(T) \geq 0$ with probability 1, and has a possible outcome of $P(T) > 0$ with positive probability at any given**

**time $T$, then an arbitrage opportunity is available in the market.**

Note that the definition of an arbitrage opportunity does not depend on the choice of the reference asset $Y$. If $P_Y = 0$ or $P_Y > 0$ for the reference asset $Y$, then $P_U = 0$ or $P_U > 0$ for any other reference asset $U$.

## 1.3 Time Value of Assets, Arbitrage and No-Arbitrage Assets

As stated in the previous section, an asset can either stay the same over time or change over time. In the first case, we say that the asset has **no time value.** Examples of assets that do not change over time include precious metals, a contract to deliver a particular asset in some fixed future time, or a stock that reinvests dividends. One should not confuse the concept of an asset with no time value with the concept of the price of an asset with no time value. For instance an ounce of gold is an asset with no time value, and it does not change over time, but the price of this asset with respect to a dollar may be changing over time.

When the asset is changing over time, we say that the asset has a **time value.** Assets with time value may deteriorate over the passage of time or not. Examples of time value assets that deteriorate over time include currencies, stocks that pay out dividends, and most consumer goods. However, some assets may change over time and not deteriorate, for instance portfolios that actively exchange assets with no time value.

One certainly does not create an arbitrage opportunity by holding an asset that has no time value. On the other hand, assets that have time value may or may not create arbitrage opportunities. It depends if the asset with time value deteriorates (or appreciates) in time or not. If one creates an arbitrage opportunity by holding a given asset, we will call this asset an **arbitrage asset.** If an arbitrage opportunity is not possible by holding a given asset, we call this asset a **no-arbitrage asset.** There is a simple method to determine whether a given asset $X$ is an arbitrage or a no-arbitrage asset. Let $V$ be a contract to deliver a unit of the asset $X$ at some future time $T$. We can write

$$V(T) = X(T).$$

When $V(t) = X(t)$ at all times $t \leq T$, the asset $X$ is a no-arbitrage asset. When $V(t) \neq X(t)$ for some $t \leq T$, the asset $X$ is an arbitrage asset.

The identity $V(t) = X(t)$ means that $V$, the contract to deliver a unit of an asset $X$, is identical to the asset $X$ itself. The only way to deliver a no-arbitrage asset is to hold it at all times up to time $T$. For instance the contract to deliver a stock costs the stock itself, a contract to deliver an ounce of gold costs the ounce of gold (neglecting a possible cost of carry which is close to zero for financial assets). Some hedge funds try to realize arbitrage opportunities even in these primary assets, so it may be hard to tell which asset is a no-arbitrage asset without observing the corresponding contract to deliver. A contract to deliver usually does not exist for a no-arbitrage asset since it coincides with the asset itself, and thus it is completely redundant. However, the nonexistence of the contract to deliver can happen for two reasons: the underlying asset is a no-arbitrage asset, or there is no market for the contract to deliver. This makes it harder to determine whether the asset is a no-arbitrage asset.

Rational investors do not allow any arbitrage opportunities, and thus their portfolios hold only no-arbitrage assets, or arbitrage assets that provide one sided advantage for the investor. If the market has only rational investors, there would be no arbitrage assets at all. For a given asset $X$, **the contract $V$ to deliver an asset $X$ is always a no-arbitrage asset,** even when the asset $X$ to be delivered is an arbitrage asset. This is easily seen from the following argument. Let $U$ be a contract to deliver the asset $V$ at time $T$, or in other words, $U(T) = V(T)$. From the identity $V(T) = X(T)$, we also have $U(T) = X(T)$. Thus $U$ is also a contract to deliver $X$ at time $T$, and therefore $U$ is identical to $V$. This proves that $V$, a contract to deliver an asset $X$ at time $T$, is a no-arbitrage asset. In particular, bonds are no-arbitrage assets.

On the other hand, assets with $V(t) \neq X(t)$ for some $t < T$ are arbitrage assets. We have either $V(t) < X(t)$, or $V(t) > X(t)$. When $V(t) < X(t)$, it is possible to deliver the asset $X$ at time $T$ at a cheaper price than just holding the asset $X$ itself. The exact procedure to lock the arbitrage opportunity for an arbitrage asset is described in Example 1.9 which follows. When $V(t) < X(t)$, one should buy a contract to deliver $V$ and sell a corresponding number of units of an asset $X$.

Arbitrage assets do exist in real markets, mostly representing assets with deteriorating time value (food, consumer goods, banknotes). However, these assets are not typically included in financial portfolios as holding them would create arbitrage opportunities that are not favorable for the holders of such assets. But the arbitrage assets still may appear in the payoffs of financial contracts, such as a contract to deliver a unit of the asset in a fixed future time. We have already seen that a contract to deliver any asset is always a no-arbitrage asset. Such derivative contracts facilitate trading of assets with deteriorating time value. While the underlying asset creates arbitrage opportunities, the contract to deliver does not, and as such it may be included in a financial portfolio that does not deteriorate over time.

Examples of arbitrage assets that appear in such payoffs include certain food products (orange juice, coffee, pork bellies), currencies, or stocks that pay dividends. A stock together with the corresponding dividend payments is a no-arbitrage asset. However, a stock when taken separately without the dividends is an arbitrage asset. Taking away the dividends is an obvious arbitrage opportunity. Another example of an arbitrage asset is an asset that corresponds to a maximum price of an asset $X$ with respect to a reference asset $Y$ defined as $[\max_{0 \le s \le t} X_Y(s)] \cdot Y(t)$. This asset appears in the payoff of a lookback option, and although it does not exist in the real markets, it can still be used as a reference asset for pricing lookback options.

Arbitrage assets do change over some periods of time; in particular we have

$$\$(t) > \$(t+1), \tag{1.15}$$

which means that a dollar today is worth more than a dollar tomorrow. Inequality (1.15) is known as the **time value of money.**

**Example 1.9 (Arbitrage opportunity created by an arbitrage asset)**
*Let $V$ be a contract that delivers a unit of an asset $X$ at time $T$, or in other words,*

$$V(T) = X(T).$$

*This equality is written in the sense of two assets, the contract to deliver $V$ has the same price as an asset $X$ at time $T$. In terms of prices, we can write*

$$V_X(T) = 1,$$

*which means that the price of the contract to deliver $V$ with respect to the reference asset $X$ is one at time $T$. When $V(0) < X(0)$, we can realize a risk free profit by buying a unit of an asset $V$, and sell $V_X(0) < 1$ units of an asset $X$, thus creating a zero price portfolio*

$$P(0) = 1 \cdot V - V_X(0) \cdot X.$$

*Clearly, $P_X(0) = 1 \cdot V_X(0) - V_X(0) = 0$. This portfolio is kept until time $T$, when it becomes*

$$\begin{aligned} P(T) &= 1 \cdot V - V_X(0) \cdot X \\ &= (1 - V_X(0)) \cdot X > 0. \end{aligned}$$

*Thus one can get a portfolio with a guaranteed positive price starting from a portfolio with a zero price.*

The most typical examples of arbitrage assets are currencies. Let $X$ be a dollar \$. A contract to deliver a dollar at time $T$ is known as a **bond,** and it is denoted by $B^T$. The dollar price of the bond is typically less than one ($B^T_\$(0) < 1$), making a dollar an arbitrage asset. In order to lock the risk free profit, one would have to buy a bond $B^T$, and sell $B^T_\$(0)$ units of a dollar. This means one would have to borrow money to get a short position in dollars, which leads us to the following important remark.

**Remark 1.10 (Borrowing money)**
**When one borrows money in terms of a dollar \$, the resulting asset that is owed is not money but rather a money market account $M$, an interest bearing account.** *The asset that is borrowed is different from the asset that is owed. In contrast, if one borrows a stock $S$ (in terms of short-selling on the stock exchange), the debt is still the same stock $S$. The exchange may charge a fee for that, but the asset that is borrowed is the same as the asset that is owed.*

Even governments have to pay interest when borrowing money. The only exception when interest is not paid is when governments issue banknotes. Governments typically have a limited intention to print more banknotes in order to finance their debts, and thus exploration of this arbitrage opportunity is not significant.

## 1.4 Money Market, Bonds, and Discounting

The fact that currencies have time value means that prices in terms of a dollar may not be consistent in time. This is known as time value of money: A dollar today is worth more than a dollar tomorrow. Thus when one expresses prices of an asset $S$ in terms of a dollar, these prices will have an upward drift component that corresponds to the loss of value of the reference asset.

In order to remove the effect of the depreciation of the reference asset, one can express the price of the asset $S$ in terms of no-arbitrage proxy assets to a dollar, such as a money market $M$, or a bond $B^T$. Prices $S_M(t)$ and $S_{B^T}(t)$ are known as **discounted prices** of the asset $S$.

Recall that the money market $M$ is an asset created by the following procedure. The initial amount equal to one dollar is invested at time $t = 0$ in the bond with the shortest available maturity (ideally in the next infinitesimal instant), and this position is rolled over to the bond with the next shortest maturity once the first bond expires. The resulting no-arbitrage asset, the **money market** $M$, is a result of an active trading strategy involving a number of these no-arbitrage bonds. The dollar price of the money market is given by

$$\boxed{M_\$(t) = \exp\left(\int_0^t r(u)du\right),} \tag{1.16}$$

where $r(t)$ is a parameter known as the **interest rate**. In practice, the money market asset is replicated as a portfolio of different bonds by banks or investment funds.

Equation (1.16) can be written in a differential form as

$$dM_\$(t) = r(t)M_\$(t)dt. \tag{1.17}$$

The interest rate $r(t)$ can be viewed as a rate of deterioration of an arbitrage asset \$ with respect to a no-arbitrage asset $M$, the money market account. Since the parameter $r(t)$ is related only to the shortest available bond, in this case $B^t$, a bond that matures immediately at time $t$, a simple analog of Equation (1.17) for a bond $B^T$ is not available. Only if we take a simplifying assumption that the interest rate $r(t)$ is **deterministic**, can we also write

$$B^T_\$(t) = \exp\left(-\int_t^T r(u)du\right). \tag{1.18}$$

The reason is that there is only one way to deliver one dollar at time $T$ by investing in the money market account $M$. If one starts with $\exp\left(-\int_t^T r(u)du\right)$ units of a dollar at time $t$ and invests it in the money market account $M$, it will be worth

$$\exp\left(-\int_t^T r(u)du\right) \cdot \exp\left(\int_t^T r(u)du\right) = 1$$

unit of a dollar at time $T$. Therefore the price of the bond $B^T$ at time $t$ must be given by Equation (1.18); otherwise we would have an arbitrage opportunity. In this case, the price of the bond $B^T$ and the price of the money market $M$ are related by the formula

$$B^T(t) = \exp\left(-\int_0^T r(u)du\right) \cdot M(t). \tag{1.19}$$

Thus the money market $M$ is just a constant multiple of the bond $B^T$.

In the case of a deterministic interest rate $r(t)$, we can also write

$$dB_\$^T(t) = r(t)B_\$^T(t)dt, \tag{1.20}$$

which is similar to Equation (1.17). Moreover, when the interest rate is constant, the above relationships lead to

$$M(t) = e^{rt} \cdot \$(t), \tag{1.21}$$

$$B^T(t) = e^{-r(T-t)} \cdot \$(t), \tag{1.22}$$

and

$$B^T(t) = e^{-rT} \cdot M(t). \tag{1.23}$$

The relationship between the money market $M$ and the bond $B^T$ is no longer trivial when the interest rate $r(t)$ is stochastic. In this case, the price of the money market starts at a deterministic value $M_\$(0) = 1$, but at later time $t$, $M_\$(t)$ will be stochastic in general. On the other hand, the price of the bond $B_\$^T(t)$ is random in general for times $t < T$ before the expiration of the bond, but it becomes one at time $T$ ($B_\$^T(T) = 1$), which is a deterministic value. We study the evolution of bond prices in detail in the chapter on term structure models.

As seen earlier, we can regard both $S_M(t)$ and $S_{B^T}(t)$ as discounted prices of an asset $S$. When we express the price of $S$ with respect to the money market $M$ using the change of numeraire formula for assets $X = S$, $Y = M$, and $Z = \$$, we get

$$S_M(T) = S_\$(T) \cdot \$_M(T) = \exp\left(-\int_0^T r(u)du\right) \cdot S_\$(T) \le S_\$(T), \tag{1.24}$$

with

$$S_M(0) = S_\$(0) \cdot \$_M(0) = S_\$(0). \tag{1.25}$$

Similarly, when we express the price of $S$ with respect to the bond $B^T$ using the change of numeraire formula for assets $X = S$, $Y = M$, and $Z = \$$, we get

$$S_{B^T}(T) = S_\$(T) \cdot \$_{B^T}(T) = S_\$(T), \tag{1.26}$$

with

$$S_{B^T}(0) = S_\$(0) \cdot \$_{B^T}(0) = \frac{S_\$(0)}{B_\$^T(0)} \ge S_\$(0). \tag{1.27}$$

The two types of discounting are also related by

$$S_{B^T}(t) = S_M(t) \cdot M_{B^T}(t). \tag{1.28}$$

In particular, when the interest rate $r$ is constant, the relation between $S_{B^T}$ and $S_M$ is simply

$$S_{B^T}(t) = e^{rT} \cdot S_M(t). \tag{1.29}$$

The important difference between $S_M$ and $S_{B^T}$ is that the price of $S_M$ agrees with the price $S_\$$ at time $t = 0$, while the price of $S_{B^T}$ agrees with the price $S_\$$ at time $T$. The reference point for discounting with the money market $M$ is at time $t = 0$, while the reference point for discounting with the bond $B^T$ is at time $T$. Since typical European-type derivative contracts explained in the next chapter pay off $f(S_\$(T))$ for some function $f$, discounting with respect to the bond $B^T$ makes more sense as $S_{B^T}(T) = S_\$(T)$.

Bonds usually deliver units of a currency at multiple times until their maturity. However, without loss of generality we consider only bonds with a single delivery time $T$. A bond $B^T$ that pays one dollar at time $T$ is also known as a zero coupon bond. A bond with multiple delivery times is just a combination of several zero coupon bonds. A zero coupon bond is also a possible choice of a no-arbitrage reference asset.

## 1.5 Portfolio

This section addresses the following questions: What is a portfolio? What is the price of a portfolio? What is a self-financing trading strategy?

**A portfolio is a sum of one's assets**

$$\boxed{P(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot X^i,} \tag{1.30}$$

**where $\Delta^i(t)$ represents how many units of an asset $X^i$ are held at time $t$.**

When $\Delta^i(t) > 0$, we say that the portfolio has a **long position** in the asset $X^i$. When $\Delta^i(t) < 0$, we say that the portfolio has a **short position** in the asset $X^i$. When $\Delta^i(t) = 0$, we say that the portfolio has a **neutral position** in the asset $X^i$.

Note that a portfolio is not a number. A car, a house, paintings, and jewelery are assets that do not take numerical values. Thus a portfolio is a distinct concept from the **price of a portfolio**, the number of units of the reference asset that is required to acquire the entire portfolio. As mentioned earlier, price is relative to the chosen reference asset. If we fix $Y = X^0$ to be the reference asset, the price of a portfolio with respect to the reference asset (numeraire) $Y$ is given by

$$\boxed{P_Y(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot X^i_Y(t).} \tag{1.31}$$

In other words, $P_Y(t)$ is the number of units of the asset $Y$ that one would obtain, should one exchange all assets in one's portfolio for an asset $Y$ at time $t$.

The individual portfolio position $\Delta^i(t)$ has to be known at time $t$; it cannot be set in retrospect after observing prices in the future. It is similar to betting in a casino – one first places the stake before observing the outcome of a given game. Mathematically, each $\Delta^i(t)$ has to be a **predictable process,** which means that the portfolio position is set before the market observes the price move. Predictable processes are generated by the processes that have left continuous paths.

A portfolio, $P(t)$, together with prices $X^i_Y(t)$ determine the price of a portfolio $P_Y(t)$. On the other hand, different portfolios may have the same price at a given time $t$. We assume that one can exchange one's portfolio for any other portfolio that has an equal price at time $t$. We also assume that all assets in the portfolio are no-arbitrage assets. This procedure of exchanging no-arbitrage assets with equal price is known as a **self-financing trading strategy.** Trading portfolios with equal prices means that no asset is either added or withdrawn from the portfolio without being properly exchanged with a combination of assets of an equal price. Holding only no-arbitrage assets ensures that the resulting portfolio is also a no-arbitrage asset. If the prices of two portfolios are the same with respect to one asset $Y$, the prices are also the same with respect to any other asset $Z$. This is easily seen from the change of numeraire formula

$$P_Z(t) = P_Y(t) \cdot Y_Z(t).$$

Since exchanging portfolios with equal price can be done in principle at any given time $t$, one can have continuously rebalanced portfolios as a result.

Let us give an example of self-financing trading.

**Example 1.11 (Self-financing trading)** *The portfolio*

$$P^1(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot X^i$$

*can be exchanged for the portfolio*

$$P^2(t) = \left[\sum_{i=0}^{N} \Delta^i(t) \cdot X^i_Y(t)\right] \cdot Y$$

*since the two have the same price. This is easily seen from*

$$P^1_Y(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot X^i_Y(t),$$

*and*

$$P^2_Y(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot X^i_Y(t).$$

*Therefore we have*

$$P_Y^1(t) = P_Y^2(t).$$

*However, the two portfolios are physically different. The first portfolio* $P^1(t)$ *has* $\Delta^i(t)$ *units of an asset* $X^i$*, for* $i = 1, \ldots, N$*, while the second portfolio* $P^2(t)$ *has* $\sum_{i=0}^{N} \Delta^i(t) \cdot X_Y^i(t)$ *units of an asset* $Y$*, and zero positions in the remaining assets. But since they have the same price, they can be exchanged for each other at time* $t$*.*

**Remark 1.12** *Note that self-financing trading may come with some limitations. For instance in the economy consisting of just two assets* $X$ *and* $Y$*, portfolios of the form*

$$P = \Delta^X(t) \cdot X + (P_Y(t) - \Delta^X(t) X_Y(t)) \cdot Y$$

*have the same price* $P_Y(t)$ *with respect to the reference asset* $Y$*, where* $\Delta^X(t)$ *is an arbitrary number. But in reality, one usually cannot take arbitrarily large or arbitrarily small (negative) positions in the underlying assets. These positions are usually bounded. For instance, sometimes it may not be possible to take a short position in a particular asset. The bounds on the portfolio position may depend on a given situation, and they may even be different for different agents (think about credit lines). Therefore it is not clear how to define acceptable portfolio positions in order to reflect the reality of the market. There can be also a physical limit on the number of assets that can be held: some assets are nondivisible, and thus one can have only an integer number of them in a given portfolio.*

*Another limit is that the price of the portfolio may be required to stay above a certain minimal threshold; otherwise a bankruptcy occurs. An adapted portfolio process* $\Delta^i(t)_{i=0}^N$ *that guarantees* $P_Y(t) \geq L$ *for some lower bound* $L$ *for all* $t$ *is called* **admissible.**

*The last concern we mention is continuous trading. The traders in the real markets are allowed to change their portfolio positions rather frequently, but only finitely many times in a given time interval. However, mathematical models in continuous time assume that the portfolio positions can be changed continuously. Such an approach gives realistic results, but one should be careful not to construct portfolios that require an infinite number of trades that are not the result of a limit of discrete trading.*

*We will not be specific in this text about these limitations since this is not a prime focus of the book, but the reader should be aware of them.*

## 1.6 Evolution of a Self-Financing Portfolio

Let us discuss how the portfolio can evolve in time, using a self-financing trading strategy. **We also assume that all assets are no-arbitrage assets; otherwise the portfolio itself is an arbitrage asset.** Consider first the discrete time case.

Let the portfolio at time $k$ be given by

$$P(k) = \sum_{i=0}^{N} \Delta^i(k) \cdot X^i.$$

At time $k+1$, the portfolio will have the same positions $\Delta^i(k)$ in each asset $X^i$:

$$P(k+1) = \sum_{i=0}^{N} \Delta^i(k) \cdot X^i,$$

but since $X^i$ stays the same over time for each $i = 0, 1, \ldots, N$, the portfolios $P(k)$ and $P(k+1)$ are the same, only taken at two different time periods.

While the portfolio remains unchanged, its price with respect to a reference asset may be changing. When we write the difference of the prices of the portfolio taken at two consecutive times $k$ and $k+1$, we get

$$\boxed{P_Y(k+1) - P_Y(k) = \sum_{i=0}^{N} \Delta^i(k) \cdot \left[X_Y^i(k+1) - X_Y^i(k)\right].} \tag{1.32}$$

Note that we can omit the changes in the reference asset $Y = X^0$ since

$$Y_Y(k+1) - Y_Y(k) = 1 - 1 = 0.$$

For example, one ounce of gold in the portfolio will still be one ounce of gold in the portfolio in the next time interval, and its price will stay unchanged if the reference asset is chosen to be gold. Similarly, a particular asset will remain the same in the portfolio, but its price with respect to gold may fluctuate in time.

Equation (1.32) says that the change of the price of the portfolio is explained only by the changes of the prices of individual assets in the portfolio. On the other hand, possible changes in the asset positions $\Delta^i(k)$ from time $k$ to $k+1$ do not enter this equation. At time $k+1$, the holder of the portfolio is free to exchange his present portfolio for a portfolio that has the same price. If we denote the old portfolio that was inherited from time $k$ by $P^{old}(k+1) = P(k) = \sum_{i=0}^{N} \Delta^i(k) \cdot X^i$, and the newly exchanged portfolio at time $k+1$ by $P^{new}(k+1) = \sum_{i=0}^{N} \Delta^i(k+1) \cdot X^i$, we have

$$P_Y^{old}(k+1) = P_Y^{new}(k+1).$$

The holder of the portfolio can change his position in the underlying assets $X^i$ from $\Delta^i(k)$ to $\Delta^i(k+1)$ given that the two portfolios under consideration have the same price. It means that

$$\sum_{i=0}^{N} \Delta^i(k) \cdot X_Y^i(k+1) = \sum_{i=0}^{N} \Delta^i(k+1) \cdot X_Y^i(k+1),$$

or in other words,

$$\boxed{\sum_{i=0}^{N} \left[\Delta^i(k+1) - \Delta^i(k)\right] \cdot X_Y^i(k+1) = 0.} \tag{1.33}$$

This is the condition a discretely rebalanced portfolio must satisfy in order to be self-financing. The above identity can be also expressed as

$$\sum_{i=0}^{N} \Big[ (\Delta^i(k+1) - \Delta^i(k)) \cdot \left[X_Y^i(k+1) - X_Y^i(k)\right] \\ + (\Delta^i(k+1) - \Delta^i(k)) \cdot X_Y^i(k) \Big] = 0. \tag{1.34}$$

When we consider continuous time models, the above identities will take the following forms. For the evolution of the price of the portfolio, we have

$$\boxed{dP_Y(t) = \sum_{i=0}^{N} \Delta^i(t) \cdot dX_Y^i(t),} \tag{1.35}$$

a continuous analog of Equation (1.32). Similarly, the identity corresponding to Equation (1.34) is

$$\boxed{\sum_{i=0}^{N} \left[(d\Delta^i(t)) \cdot dX_Y^i(t) + (d\Delta^i(t)) \cdot X_Y^i(t)\right] = 0.} \tag{1.36}$$

Indeed, if we applied Ito's formula for the evolution of the price of the portfolio, we would get

$$\begin{aligned} dP_Y(t) &= d\left(\sum_{i=0}^{N} \Delta^i(t) \cdot X_Y^i(t)\right) \\ &= \sum_{i=0}^{N} \left[\Delta^i(t) \cdot dX_Y^i(t) + (d\Delta^i(t)) \cdot dX_Y^i(t) + (d\Delta^i(t)) \cdot X_Y^i(t)\right], \end{aligned}$$

But since the last two terms of the above identity sum to zero from (1.36), we have Equation (1.35).

**Example 1.13** *Consider a portfolio $P$ that holds $\Delta^X(t) = \left[1 - \frac{t}{T}\right]$ units of an asset $X$, and $\Delta^Y(t) = \left[\frac{1}{T}\int_0^t X_Y(s)ds\right]$ units of an asset $Y$ at time $t$, where $t \in [0, T]$. In other words,*

$$P(t) = \left[1 - \tfrac{t}{T}\right] \cdot X + \left[\tfrac{1}{T}\int_0^t X_Y(s)ds\right] \cdot Y. \tag{1.37}$$

*We can show that this is a self-financing portfolio. The condition of self-financing trading (1.36) reads as*

$$(d\Delta^Y(t)) \cdot dY_Y(t) + (d\Delta^Y(t)) \cdot Y_Y(t) \\ + (d\Delta^X(t)) \cdot dX_Y(t) + (d\Delta^X(t)) \cdot X_Y(t) = 0,$$

*where we substituted $X^0 = Y$, and $X^1 = X$. Since $Y_Y(t) = 1$, the above relationship simplifies to*

$$\boxed{(d\Delta^Y(t)) + (d\Delta^X(t)) \cdot dX_Y(t) + (d\Delta^X(t)) \cdot X_Y(t) = 0.} \tag{1.38}$$

*Note that*

$$\begin{aligned} d\Delta^Y(t) &= \tfrac{1}{T} X_Y(t) dt, \\ d\Delta^X(t) &= -\tfrac{1}{T} dt, \end{aligned}$$

*and thus*

$$\begin{aligned} &d\Delta^Y(t) + d\Delta^X(t) \cdot dX_Y(t) + d\Delta^X(t) \cdot X_Y(t) = \\ &\qquad = \tfrac{1}{T} X_Y(t) dt + (-\tfrac{1}{T} dt) \cdot dX_Y(t) + (-\tfrac{1}{T} dt) X_Y(t) = 0. \end{aligned}$$

*Therefore we have the self-financing evolution of the prices of the portfolio from (1.35). When we choose $Y$ to be the reference asset, we have*

$$dP_Y(t) = \Delta^X(t) dX_Y(t) = \left[1 - \tfrac{t}{T}\right] dX_Y(t),$$

*when we choose $X$ to be the reference asset, we have*

$$dP_X(t) = \Delta^Y(t) dY_X(t) = \left[\tfrac{1}{T} \int_0^t X_Y(s) ds\right] dY_X(t).$$

*Note that the portfolio $P(t)$ starts with $P(0) = X(0)$ and ends with $P(T) = \left[\frac{1}{T} \int_0^T X_Y(s) ds\right] \cdot Y(T)$. Therefore the above described self-financing strategy delivers $\left[\frac{1}{T} \int_0^T X_Y(s) ds\right]$ units of $Y$ at time $T$. The number $\left[\frac{1}{T} \int_0^T X_Y(s) ds\right]$ represents the average price of the asset $X$ in terms of the reference asset $Y$.*

The trading strategy described in Example 1.13 does not depend on the evolution of the underlying price $X_Y(t)$. Also, $d\Delta^X(t)$ and $d\Delta^Y(t)$ have only a $dt$ term, so $\Delta^X(t)$ and $\Delta^Y(t)$ are smooth. Because of that, the $(d\Delta^X(t)) \cdot dX_Y(t)$ cross term is zero. However, the positions $\Delta^X(t)$ and $\Delta^Y(t)$ in the underlying assets can be even diffusions, such as in the following example. In that case, the $(d\Delta^X(t)) \cdot dX_Y(t)$ cross term may not disappear.

**Example 1.14** *Assume that an asset price follows geometric Brownian motion*

$$dX_Y(t) = \sigma X_Y(t) dW^Y(t),$$

*where $X$ and $Y$ are two no-arbitrage assets. Consider a portfolio $P(t)$ which is given by*

$$P(t) = [N(d_+)] \cdot X + [-KN(d_-)] \cdot Y,$$

*where*

$$N(x) = \int_{-\infty}^{x} \tfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy,$$

*and*

$$d_{\pm} = \frac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\frac{X_Y(t)}{K}\right) \pm \frac{1}{2}\sigma\sqrt{T-t}.$$

*The portfolio $P$ holds $\Delta^X(t) = N(d_+)$ units of an asset $X$, and $\Delta^Y(t) = -KN(d_-)$ units of an asset $Y$. It turns out that this portfolio is indeed self-financing. The self-financing condition is given by*

$$\begin{aligned} d\Delta^Y(t) + d\Delta^X(t) \cdot dX_Y(t) + d\Delta^X(t) \cdot X_Y(t) = \\ = -KdN(d_-) + dN(d_+) \cdot dX_Y(t) + dN(d_+) \cdot X_Y(t). \end{aligned}$$

*It is not trivial to show that*

$$-KdN(d_-) + dN(d_+) \cdot dX_Y(t) + dN(d_+) \cdot X_Y(t) = 0,$$

*but it is true. Thus we have*

$$dP_Y(t) = N(d_+)\, dX_Y(t),$$

*and*

$$dP_X(t) = -KN(d_-)\, dY_X(t).$$

*The portfolio $P(t)$ in this example is in fact a hedging portfolio for a European option with a payoff $(X(T) - K \cdot Y(T))^+$ in a geometric Brownian motion model.*

## 1.7 Fundamental Theorems of Asset Pricing

The general assumption in finance is that the market does not contain arbitrage. If an arbitrage opportunity appears, the market usually corrects itself in a short time period. On the other hand, profitable trading strategies may exist for long periods. Some profitable trading strategies may even come with a risk of a catastrophic loss.

Obviously, the entire theory depends upon the fact that the assets in the portfolio are no-arbitrage assets to start with; otherwise the portfolio is not arbitrage free. The central result of finance theory is the First Fundamental Theorem of Asset Pricing:

**Theorem 1.15 (First Fundamental Theorem of Asset Pricing)** *If there exists a probability measure $\mathbb{P}^Y$ such that the price processes $X_Y(t)$ are $\mathbb{P}^Y$-martingales, where $X$ is an arbitrary no-arbitrage asset, and $Y$ is an arbitrary no-arbitrage asset with a positive price, then there is no arbitrage in the market.*

Proof: Let $Y$ be a fixed reference asset. If there is an arbitrage opportunity, one can start with a zero price portfolio $P_Y(0) = 0$ and obtain a portfolio $P_Y(T)$ in the form $P_Y(T) = \xi(\omega)$, where $\xi(\omega)$ is a non-negative random variable with $\mathbb{P}^Y(\xi(\omega) > 0) > 0$. In this case, $P_Y(T)$ cannot be a martingale since $\mathbb{E}^Y[P_Y(T)] > 0 = P_Y(0)$.

**Remark 1.16 (Market interpretation of $\mathbb{P}^Y$)** *The probability measure $\mathbb{P}^Y$ associated with a no-arbitrage reference asset $Y$ has the following market interpretation. Let $A$ be an event in $\mathcal{F}_T$, which can be viewed as a set of market scenarios $\omega$ that satisfy a condition $\omega \in A$. As an example of such an event, consider $A = \{\omega \in \Omega : X_Y(T,\omega) \geq K\}$. This is a set of scenarios where the market price of $X$ with respect to the reference asset $Y$ exceeds a fixed constant $K$ at time $T$. Each set $A$ from the information set $\mathcal{F}_T$ has some objective probability $\mathbb{P}(A)$. The probability measure $\mathbb{P}$ is known as the* **real probability measure.** *However, the real probability measure does not play any role in the First Fundamental Theorem of Asset Pricing, and thus its role in pricing financial contracts is limited.*

*What is relevant to pricing financial contracts is the probability measure $\mathbb{P}^Y$. Imagine that there is a security $V$ that pays off one unit of the asset $Y$ at time $T$ when the scenario $\omega$ is in $A$; otherwise it pays nothing. In mathematical notation,*

$$V(T) = \mathbb{I}_A(\omega) \cdot Y(T),$$

*where $\mathbb{I}$ denotes an indicator function. We can also rewrite the above equation in terms of the prices as*

$$V_Y(T) = \mathbb{I}_A(\omega).$$

*The contract $V$ is known as a* **digital option**. *If we want to find the price of this contract at time $t = 0$, we can use the fact that $V_Y(t)$ is a martingale under the probability measure $\mathbb{P}^Y$. Therefore*

$$V_Y(0) = \mathbb{E}^Y[V_Y(T)] = \mathbb{E}^Y[\mathbb{I}_A(\omega)] = \mathbb{P}^Y(A).$$

*In terms of the assets, we have*

$$V(0) = \mathbb{P}^Y(A) \cdot Y(0).$$

*In other words, $\mathbb{P}^Y(A)$ is the initial market price of the contract $V$ in terms of the asset $Y$. Clearly, delivering a unit of $Y$ at time $T$ for a set of scenarios in $A$ should cost at most a unit of $Y$ at time $t = 0$. So $\mathbb{P}^Y(A)$ indicates what fraction of $Y$ is required to start with in order to deliver the digital option at time $T$. The probability $\mathbb{P}^Y$ does not indicate directly how likely is the event $A$ to occur, but rather how costly it is with respect to the asset $Y$.*

*When the number of possible scenarios in $\Omega$ is finite, we can consider events with a single scenario only, meaning $A = \{\omega\}$. The price corresponding to the Arrow–Debreu security for this event, $\mathbb{P}^Y(\omega)$, is known as an* **Arrow–Debreu state price.** *This concept generalizes to a countable number of states. When the number of states is not countable, representing a continuous random variable, the Arrow–Debreu state price can be interpreted as a density:*

$$\mathbb{P}^Y(A) = \int_{\omega \in A} d\mathbb{P}^Y(\omega).$$

*In this situation, $d\mathbb{P}^Y(\omega)$ is known as an* **Arrow–Debreu state price density.**

*Note that the probability measure $\mathbb{P}^X$ that is associated with a different no-arbitrage reference asset $X$ is in general different from $\mathbb{P}^Y$. The corresponding digital option $U$ would pay off one unit of an asset $X$ when a scenario $\omega$ is in $A$; otherwise it would pay nothing. In other words,*

$$U(T) = \mathbb{I}_A(\omega) \cdot X(T).$$

*This contract differs from $V$ only in the underlying asset. The initial price of $U$ is given by*

$$U(0) = \mathbb{P}^X(A) \cdot X(0).$$

*In general, the fraction $\mathbb{P}^Y(A)$ of the asset $Y$ needed for the security $V$ and the fraction $\mathbb{P}^X(A)$ of the asset $X$ needed for the security $U$ will differ.*

*Consider for instance a geometric Brownian motion model for an asset price*

$$X_Y(t) = X_Y(0) \cdot \exp\left(\sigma W^Y(t) - \tfrac{1}{2}\sigma^2 t\right).$$

*Let $A$ be the set of scenarios where the terminal price $X_Y(T)$ of the asset ends up below the initial price of the asset $X_Y(0)$, or in other words,*

$$A = \{\omega \in \Omega : X_Y(T,\omega) \le X_Y(0)\}.$$

*Let $V$ be the corresponding digital option that delivers a unit of the asset $Y$ at time $T$ when the asset price $X_Y(T)$ ends up below $X_Y(0)$, and let $U$ be the corresponding digital option that delivers a unit of the asset $X$ at time $T$. It turns out that $\mathbb{P}^Y(A) > \frac{1}{2}$, but $\mathbb{P}^X(A) < \frac{1}{2}$. Take as an example $X$ to be a stock market, and $Y$ to be a money market. The set of scenarios in $A$ represents outcomes when the market makes a downturn with respect to the reference asset $Y$. Should one deliver a unit of $Y$ on the downturn, this happens to cost more than half a unit of $Y$ to start with. But that is not surprising; when the market takes a downturn, the reference asset $Y$, such as the money market in this case, becomes more expensive to deliver. The reason is that $Y$ has appreciated with respect to $X$, and thus it takes more than one half units of $Y$ to cover the payoff of the corresponding digital option. On the other hand, it costs less than half a unit of $X$ to deliver a unit of $X$ on the market downturn. This is also not surprising, since on the downturn, the asset $X$ becomes less valuable, and cheaper to deliver.*

**Remark 1.17** *The inverse statement in the First Fundamental Theorem of Asset Pricing that a no-arbitrage condition implies existence of a martingale measure $\mathbb{P}^Y$ is also true, at least in typical mathematical models. This means no arbitrage implies that prices are martingales with respect to the corresponding probability measure. A proper mathematical statement of this theorem requires a careful definition of an admissible trading strategy. The interested reader should refer to academic literature on this topic.* **For practical purposes, it is enough that we start with a martingale evolution of the price.** *Furthermore, martingales in continuous time are just combinations of diffusions and jumps, so no other processes (such as a* *fractional Brownian motion for Hurst index* $\neq \frac{1}{2}$*) can be considered for a no-arbitrage description of the prices.*

*We now consider how to determine the probability measure* $\mathbb{P}^Y$*. One should start with describing the set of possible outcomes* $\Omega$ *that represent the individual scenarios of a price evolution. One can consider discrete time and discrete space models, which are known as tree models (binomial or trinomial tree). Continuous time models can have either continuous paths, which lead to diffusion models, or they can have jumps, which lead to purely discontinuous models. Note that requiring prices to be martingales limits possible types of the price evolution.*

*A general martingale in continuous time can be written as a sum of a martingale with continuous paths and a purely discontinuous martingale:*

$$\mathcal{M}(t) = \mathcal{M}^c(t) + \mathcal{M}^d(t). \tag{1.39}$$

*A martingale* $\mathcal{M}^d(t)$ *is called purely discontinuous if its product with any continuous martingale remains a martingale. For instance, a compensated Poisson process* $N(t) - \lambda t$ *is a purely discontinuous martingale. Note that a purely discontinuous martingale may have continuous paths. Continuous martingales adapted to a filtration* $\mathcal{F}_t^W$ *generated by a Brownian motion* $W$ *are in fact diffusions; they can be represented as stochastic integrals with respect to Brownian motion. Thus*

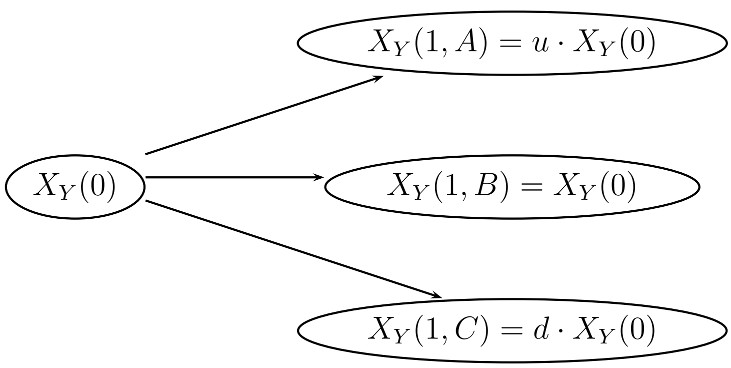
$$\mathcal{M}^c(t) = \mathcal{M}^c(0) + \int_0^t \phi(s) dW(s), \tag{1.40}$$

*where* $\phi(t)$ *is adapted to* $\mathcal{F}_t^W$*. This result is known as the Martingale Representation Theorem.*

The following example lists some possible martingale evolutions of the price.

**Example 1.18 (Martingale evolution of the price)**

**Trinomial Model** *The price* $X_Y(0)$ *is assumed to take three possible values in the next time instant: event* $A$ *– go up to* $u \cdot X_Y(0)$ *(*$u > 1$*), event* $B$ *– stay the same, or event* $C$ *– go down to* $d \cdot X_Y(0)$ *(*$d < 1$*).*



*When the probabilities of the events $A$, $B$ and $C$ are given by*

$$\mathbb{P}^{Y,\xi}(A) = \frac{1-d}{u-d} \cdot \xi, \quad \mathbb{P}^{Y,\xi}(B) = 1 - \xi, \quad \mathbb{P}^{Y,\xi}(C) = \frac{u-1}{u-d} \cdot \xi. \tag{1.41}$$

*where $\xi \in [0,1]$, the price process $X_Y(n)$ is a martingale. Note that each $\xi$ defines a different probability measure, so in this case there exist infinitely many martingale measures $\mathbb{P}^{Y,\xi}$. One can check that*

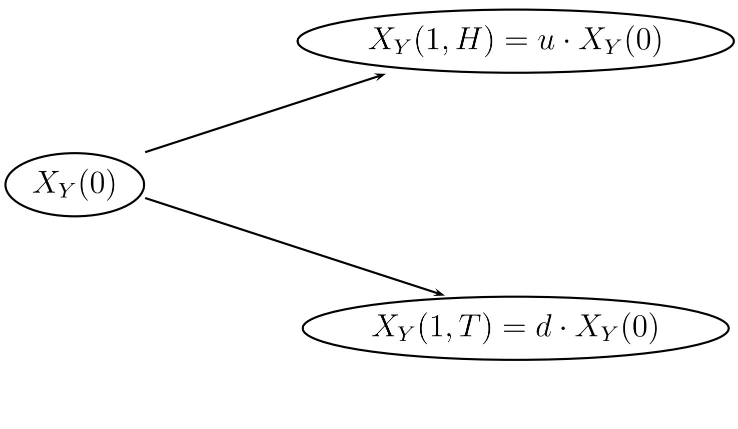
$$\begin{aligned}
\mathbb{E}^{Y,\xi} X_Y(1) &= X_Y(1,A) \cdot \mathbb{P}^{Y,\xi}(A) + X_Y(1,B) \cdot \mathbb{P}^{Y,\xi}(B) \\
&\qquad + X_Y(1,C) \cdot \mathbb{P}^{Y,\xi}(C) \\
&= u \cdot X_Y(0) \cdot \frac{1-d}{u-d} \cdot \xi + X_Y(0) \cdot (1-\xi) \\
&\qquad + d \cdot X_Y(0) \cdot \frac{u-1}{u-d} \cdot \xi \\
&= X_Y(0).
\end{aligned}$$

*It means that the prices of Arrow–Debreu securities may not be uniquely defined, meaning that there exists a range of the prices when there is no arbitrage present. Consider for instance an Arrow–Debreu security that pays off one unit of $Y$ when the scenario $A$ happens. The initial price of this security is*

$$\mathbb{P}^{Y,\xi}(A) = \frac{1-d}{u-d} \cdot \xi,$$

*which can be any number in the interval $[0, \frac{1-d}{u-d}]$, depending on the value of the parameter $\xi$. The market can quote any price in that interval, and there would be no arbitrage opportunity. The question is which martingale measure should one use when there is more than one in order to determine the prices of financial securities? The answer is that it is the market that chooses the martingale measure. For instance, if the market quotes the price of the above mentioned Arrow–Debreu security, it already determines the value of the parameter $\xi$, thus effectively choosing only one martingale measure.*

**Binomial Model** *A binomial model is a special case of a trinomial model with $\xi = 1$. The price either goes up to $u \cdot X_Y(0)$ ($u > 1$), or goes down to $d \cdot X_Y(0)$ ($0 < d < 1$).*



*We have a martingale evolution of the price when*

$$\mathbb{P}^Y(H) = \frac{1-d}{u-d}, \quad \mathbb{P}^Y(T) = \frac{u-1}{u-d}. \tag{1.42}$$

*Note that the martingale measure here is unique.*

**Geometric Brownian Motion** *Geometric Brownian motion is a process that satisfies the following stochastic differential equation:*

$$dX_Y(t) = \sigma X_Y(t) dW^Y(t). \tag{1.43}$$

*The parameter $\sigma$ is known as the volatility. The above stochastic differential equation admits a closed form solution*

$$X_Y(t) = X_Y(0) \cdot \exp\left(\sigma W^Y(t) - \tfrac{1}{2}\sigma^2 t\right), \tag{1.44}$$

*which is a martingale. The market noise process, namely Brownian motion $W^Y(t)$, comes with the reference asset $Y$ and determines the martingale measure $\mathbb{P}^Y$.*

**Geometric Poisson Process** *Geometric Poisson process satisfies the following stochastic differential equation:*

$$dX_Y(t) = (e^\gamma - 1) \cdot X_Y(t-) d(N(t) - \lambda^Y t). \tag{1.45}$$

*The price with the above dynamics is given by*

$$X_Y(t) = X_Y(0) \exp\left(\gamma \cdot N(t) - (e^\gamma - 1)\lambda^Y t\right). \tag{1.46}$$

*This is also a martingale process. In contrast to a geometric Brownian motion model, the market noise process $N(t)$ that represents Poisson jumps does not come with a particular asset. However, different assets come with different martingale measures, which is captured by the intensity of jumps $\lambda^Y$ that comes with a particular reference asset $Y$.*

Note that when there is more than one asset with a positive price available, any of them can be used as a reference asset. Consider a situation when both $X$ and $Y$ are no-arbitrage assets with a positive price, and let $V$ be an arbitrary no-arbitrage asset. Then we have that $V_Y(t)$ is a $\mathbb{P}^Y$ martingale, but also $V_X(t)$ is a $\mathbb{P}^X$ martingale. The relationship between martingale measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ is explained in detail in the following text. It turns out that an important assumption is that both prices $X_Y(t)$ and $Y_X(t)$ stay positive, which is a reasonable assumption for primary reference assets that are represented by currencies, stocks, or precious metals. It is possible that even such basic assets may become worthless, in which case the worthless asset cannot be used as a numeraire. For instance when $X = 0$, we still have a well-defined price $X_Y(t) = 0$, but $Y_X(t)$ is not well defined. Note that derivative contracts can have in principle any price, they may even take negative values, but in this case they cannot be used as reference assets.

An example of a situation when we have two assets with a positive price is a foreign exchange market, where $X$ stands for a domestic bond and $Y$ stands for a foreign bond. Bonds serve as no-arbitrage proxies to the respective currencies. However, an asset is domestic relative to a location, and thus $Y$ is a domestic asset and $X$ is a foreign asset for somebody else. Therefore it makes sense to consider the price of the foreign asset in terms of the domestic asset $X_Y(t)$, and vice versa, the price of the domestic asset in terms of the foreign asset $Y_X(t)$.

When the underlying asset is a bond $B^T$ with maturity $T$, the corresponding $\mathbb{P}^T$ measure is known as a **T-forward measure.** The term **risk-neutral measure** is used when the underlying asset is the money market account $M$. We will denote the risk-neutral measure by $\mathbb{P}^M$. The risk-neutral measure and T-forward measure coincide when the interest rate evolution is deterministic. The reader should note that the natural choice for the pricing measure for contracts that are settled in money is the T-forward measure which works also in situations of random interest rates. The risk-neutral measure can be used for pricing such contracts only when the interest rate is deterministic. There is no martingale measure $\mathbb{P}^{\$}$ that would correspond to a dollar as a reference asset since the dollar is an arbitrage asset. Other no-arbitrage reference assets have their own martingale measure. When the underlying reference asset is a stock $S$, the corresponding $\mathbb{P}^S$ measure is known as a **stock measure.**

The price of an arbitrary no-arbitrage asset $V$ can be computed from the First Fundamental Theorem of Asset Pricing, which gives us a stochastic representation of the prices. The theorem states that the prices are martingales under a proper probability measure, and thus their expected value does not change with time. We have the following relationship:

$$V_Y(t) = \mathbb{E}_t^Y \left[ V_Y(T) \right], \tag{1.47}$$

where $V$ and $Y$ are two no-arbitrage assets. The symbol $\mathbb{E}_t[.]$ denotes conditional expectation. Rewriting the above relationship in terms of assets, we get

$$\boxed{V = \mathbb{E}_t^Y \left[ V_Y(T) \right] \cdot Y.} \tag{1.48}$$

This literally means that $V$ is worth $\mathbb{E}_t^Y \left[ V_Y(T) \right]$ units of $Y$ at time $t$. Note that $\mathbb{E}_t^Y \left[ V_Y(T) \right]$ is an $\mathcal{F}_t$ measurable random variable that represents the price $V_Y(t)$. Computing this conditional expectation is a key aspect of pricing financial contracts. The computation can be done in the following ways: finding a closed form solution for a particular contract; using Monte Carlo simulation to estimate the expected value; or by using differential methods to compute the price as explained later in the text.

**Remark 1.19 (Computing dollar prices)** *The First Fundamental Theorem of Asset Pricing does not apply when a dollar is used as a reference asset since it is an arbitrage asset. The dollar prices have to be computed from the change of numeraire formula. Consider a contingent claim $V$ with a payoff at a fixed maturity $T$. The* *claim will pay $V_\$(T)$ units of a dollar \$ at time $T$. We can use any no-arbitrage asset $Y$ to compute the price of $V$ using the formula*

$$V_Y(t) = \mathbb{E}_t^Y[V_Y(T)].$$

*From the change of numeraire formula, we can compute the dollar price of the contract by*

$$V_\$(t) = V_Y(t) \cdot Y_\$(t).$$

*A natural no-arbitrage asset to use is the bond $B^T$ that matures at time $T$. In this case we can write*

$$V_{B^T}(t) = \mathbb{E}_t^T[V_{B^T}(T)].$$

*Converting to dollar prices by the change of numeraire formula and using the fact that $B_\$^T(T) = 1$, we can also write*

$$V_\$(t) = V_{B^T}(t) \cdot B_\$^T(t) = \mathbb{E}_t^T[V_\$(T) \cdot \$_{B^T}(T)] \cdot B_\$^T(t) = \mathbb{E}_t^T[V_\$(T)] \cdot B_\$^T(t).$$

*Thus we have*

$$\boxed{V_\$(t) = B_\$^T(t) \cdot \mathbb{E}_t^T[V_\$(T)].} \tag{1.49}$$

*Equation (1.49) is of central importance in the current literature on derivative pricing. The advantage is that one can immediately obtain the dollar value of a given contingent claim by using the corresponding T-forward measure. Note that the interest rate $r(t)$ does not enter the formula. It appears only indirectly in the price of the bond $B_\$^T(t)$ if we assumed some dependence of this price on the interest rate. However, such a step is not needed as we can get the value of $B_\$^T(t)$ directly from the price quoted on the market.*

*Another possible choice of a no-arbitrage proxy asset to a dollar is the money market $M$. We can write*

$$V_M(t) = \mathbb{E}_t^M[V_M(T)].$$

*Converting to dollar prices, we get*

$$V_\$(t) = V_M(t) \cdot M_\$(t) = \mathbb{E}_t^M[V_\$(T) \cdot \$_M(T)] \cdot M_\$(t).$$

*We have already seen in Equation (1.16) that $M_\$(t)$ is given by $M_\$(t) = \exp\left(\int_0^t r(s)ds\right)$, and thus the above formula simplifies to*

$$V_\$(t) = \mathbb{E}_t^M\left[\exp\left(-\int_t^T r(s)ds\right) \cdot V_\$(T)\right]. \tag{1.50}$$

*Equation (1.50) says that "the price of a contingent claim $V$ is the expected value of its discounted payoff under the risk-neutral measure." Some authors use this equation as a starting point of pricing financial contracts, but this method can be safely used only in the case of a deterministic interest rate $r$. When the interest rate process $r(t)$ is stochastic, which is a typical case in real markets, the random variables $\exp\left(-\int_t^T r(s)ds\right)$ and $V_\$(T)$ that show up in the expectation in (1.50) could be correlated, and the problem of pricing a contingent claim $V$ would have to* *address the joint distribution of* $\exp\left(-\int_t^T r(s)ds\right)$ *and* $V_\$(T)$. *This may not be a trivial task, especially when* $V$ *itself is an interest rate product.*

*When the interest rate is deterministic, the discount factor* $\exp\left(-\int_t^T r(s)ds\right)$ *is also deterministic and thus independent of the payoff* $V_\$(T)$. *Thus it can be factored out from the expectation, and we have*

$$V_\$(t) = \exp\left(-\int_t^T r(s)ds\right) \cdot E_t^M[V_\$(T)].$$

*However, in the case of a deterministic interest rate we also have* $B_\$^T(t) = \exp\left(-\int_t^T r(s)ds\right)$, *and we can rewrite Equation (1.49) as*

$$V_\$(t) = \exp\left(-\int_t^T r(s)ds\right) \cdot E_t^T[V_\$(T)].$$

*This shows that*

$$E_t^M[V_\$(T)] = E_t^T[V_\$(T)]$$

*for an arbitrary claim* $V$, *and thus the* $T$*-forward measure* $\mathbb{P}^T$ *and the risk-neutral measure* $\mathbb{P}^M$ *are the same, but only when the interest rate is deterministic.*

*When the contingent claim* $V$ *depends on a stock* $S$, *we can also choose* $S$ *as the reference asset. Converting the price to dollar values, we obtain*

$$V = \mathbb{E}_t^S\left[V_S(T)\right] \cdot S = \mathbb{E}_t^S\left[V_S(T)\right] \cdot S_\$(t) \cdot \$. \tag{1.51}$$

Due to the symmetry of the First Fundamental Theorem of Asset Pricing, a similar formula is valid for the choice of a different reference asset $X$:

$$V_X(t) = \mathbb{E}_t^X\left[V_X(T)\right], \tag{1.52}$$

or in other words,

$$\boxed{V = \mathbb{E}_t^X\left[V_X(T)\right] \cdot X.} \tag{1.53}$$

This means that $V$ is worth $\mathbb{E}_t^X\left[V_X(T)\right]$ units of $X$ at time $t$.

Let us illustrate the concept of $X$ being a reference asset on a trinomial model. The cases of a binomial model, a geometric Brownian motion model, and a geometric Poisson model are discussed in detail in the corresponding chapters.

**Example 1.20 (Trinomial model with $X$ as a reference asset)**
*Recall that the trinomial model assumes the following evolution of the price process. The price can take three different values in one time step. When* $Y$ *was chosen as a reference asset, the price can go up to* $X_Y(1, A) = u \cdot X_Y(0)$ *for* $u > 1$ *(event* $A$*), it can stay the same* $X_Y(1, B) = X_Y(0)$ *(event* $B$*), or it can go down to* $X_Y(1, C) = d \cdot X_Y(0)$ *for* $0 < d < 1$ *(event* $C$*). Let us take* $X$ *as a reference asset, and let us study the inverse price process* $Y_X$. *On event* $A$, *the price* $Y_X(1)$ *is equal to* $Y_X(1, A) = \frac{1}{u} \cdot Y_X(0)$. *This follows from the relationship between the price* $X_Y$

*and its inverse price $Y_X$: $Y_X(t) = X_Y(t)^{-1}$. When the price $X_Y$ goes up (such as in the case of event $A$), the inverse price $Y_X$ goes down, and vice versa. On event $B$, the price $Y_X$ stays the same: $Y_X(1,B) = Y_X(0)$. On event $C$, the price $Y_X$ goes up to $Y_X(1,C) = \frac{1}{d} \cdot Y_X(0)$.*

$$
\begin{tikzcd}
 & Y_X(1,A) = \frac{1}{u} \cdot Y_X(0) \\
Y_X(0) \arrow[ur] \arrow[r] \arrow[dr] & Y_X(1,B) = Y_X(0) \\
 & Y_X(1,C) = \frac{1}{d} \cdot Y_X(0)
\end{tikzcd}
$$

*When the probabilities of the events $A$, $B$ and $C$ are given by*

$$\mathbb{P}^{X,\xi}(A) = u \cdot \frac{1-d}{u-d} \cdot \xi, \quad \mathbb{P}^{X,\xi}(B) = 1 - \xi, \quad \mathbb{P}^{X,\xi}(C) = d \cdot \frac{u-1}{u-d} \cdot \xi. \tag{1.54}$$

*where $\xi \in [0,1]$, the price process $Y_X(n)$ is a martingale. As in the case of $Y$ being a reference asset, we get infinitely many martingale measures $\mathbb{P}^{X,\xi}$, one for each choice of the parameter $\xi$. One can check that*

$$
\begin{aligned}
\mathbb{E}^{X,\xi} Y_X(1) &= Y_X(1,A) \cdot \mathbb{P}^{X,\xi}(A) + Y_X(1,B) \cdot \mathbb{P}^{X,\xi}(B) + Y_X(1,C) \cdot \mathbb{P}^{X,\xi}(C) \\
&= \tfrac{1}{u} \cdot Y_X(0) \cdot u \cdot \frac{1-d}{u-d} \cdot \xi + Y_X(0) \cdot (1 - \xi) \\
&\qquad\qquad + \tfrac{1}{d} \cdot Y_X(0) \cdot d \cdot \frac{u-1}{u-d} \cdot \xi \\
&= Y_X(0).
\end{aligned}
$$

*The probability measure $\mathbb{P}^X$ corresponds to Arrow–Debreu securities that use the asset $X$ as the underlying asset. For instance, a security $U$ that pays off one unit of an asset $X$ when the event $A$ happens has the initial price*

$$\mathbb{P}^{X,\xi}(A) = u \cdot \frac{1-d}{u-d} \cdot \xi.$$

*The price of $U$ is also not uniquely defined, it can be any number in the interval $[0, u \cdot \frac{1-d}{u-d}]$.*

We have two possible representations of the price of a contract $V$: it is either $\mathbb{E}_t^Y [V_Y(T)]$ units of an asset $Y$, or $\mathbb{E}_t^X [V_X(T)]$ units of an asset $X$. This leads to the following variant of the **change of numeraire formula**

$$\boxed{V = \mathbb{E}_t^Y [V_Y(T)] \cdot Y = \mathbb{E}_t^X [V_X(T)] \cdot X.} \tag{1.55}$$

The reference asset appears in three places in the pricing formula: $X$ – the reference asset; $\mathbb{E}_t^X$ – the conditional expectation that is associated with the reference asset; and $X$ – the discount factor in the payoff function. Thus if one wants to price a contract under a different numeraire $Y$, one just needs to replace the formula with $Y$ at these three locations.

Note that the probability measure $\mathbb{P}^Y$ in the change of numeraire formula (1.55) may not be unique, and the price $V_Y(t) = \mathbb{E}_t^Y\left[V_Y(T)\right]$ of a general contingent claim $Y$ may depend on a particular choice of $\mathbb{P}^Y$. We have seen this situation in the trinomial model. Similarly, the probability measure $\mathbb{P}^X$ may not be unique, and the price $V_X(t) = \mathbb{E}_t^X\left[V_X(T)\right]$ may depend on a particular choice of $\mathbb{P}^X$. However, to one particular probability measure $\mathbb{P}^Y$ corresponds one particular probability measure $\mathbb{P}^X$ that agrees on the prices in the sense of the change of numeraire formula (1.55). It turns out that the two measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ are linked by a Radon–Nikodým derivative as we will show in the next section.

**Example 1.21 (One-to-one correspondence of the probability measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ in the trinomial model)**

*Let us show a one-to-one correspondence of the probability measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ in the trinomial model. Let $V$ be an arbitrary contingent claim. Since the price $V_Y$ is a martingale with respect to $\mathbb{P}^Y$, we can write*

$$V_Y(0) = \mathbb{E}^Y[V_Y(1)].$$

*This expectation depends on a particular choice of the probability measure $\mathbb{P}^{Y,\xi}$. When we fix a parameter $\xi$, we get*

$$V_Y(0) = \mathbb{E}^{Y,\xi}[V_Y(1)].$$

*Note that a different choice of $\xi$ may lead to a different value of $V_Y(0)$. Expanding the expectation, we can also write*

$$V_Y(0) = V_Y(1,A)\cdot\mathbb{P}^{Y,\xi}(A) + V_Y(1,B)\cdot\mathbb{P}^{Y,\xi}(B) + V_Y(1,C)\cdot\mathbb{P}^{Y,\xi}(C).$$

*Using the change of numeraire formula $V_Y = V_X \cdot X_Y$, the above equality can be rewritten as*

$$\begin{aligned}
&V_X(0)\cdot X_Y(0)\\
&\qquad = V_X(1,A)\cdot X_Y(1,A)\cdot\mathbb{P}^{Y,\xi}(A) + V_X(1,B)\cdot X_Y(1,B)\cdot\mathbb{P}^{Y,\xi}(B)\\
&\qquad\qquad + V_X(1,C)\cdot X_Y(1,C)\cdot\mathbb{P}^{Y,\xi}(C).
\end{aligned}$$

*After dividing by $X_Y(0)$, we can also write*

$$\begin{aligned}
&V_X(0)\\
&\qquad = V_X(1,A)\cdot\frac{X_Y(1,A)}{X_Y(0)}\cdot\mathbb{P}^{Y,\xi}(A) + V_X(1,B)\cdot\frac{X_Y(1,B)}{X_Y(0)}\cdot\mathbb{P}^{Y,\xi}(B)\\
&\qquad\qquad + V_X(1,C)\cdot\frac{X_Y(1,C)}{X_Y(0)}\cdot\mathbb{P}^{Y,\xi}(C). \qquad (1.56)
\end{aligned}$$

*But $V_X$ is a martingale under some probability measure $\mathbb{P}^X$, and thus we have*

$$V_X(0) = \mathbb{E}^X[V_X(1)],$$

*or*

$$V_X(0) = V_X(1,A) \cdot \mathbb{P}^X(A) + V_X(1,B) \cdot \mathbb{P}^X(B) + V_X(1,C) \cdot \mathbb{P}^X(C) \tag{1.57}$$

*after expanding the expectation. The prices in (1.56) and (1.57) should agree, so we must have*

$$\mathbb{P}^X(\omega) = \frac{X_Y(1,\omega)}{X_Y(0)} \cdot \mathbb{P}^{Y,\xi}(\omega). \tag{1.58}$$

*Thus for a particular choice of the martingale measure $\mathbb{P}^{Y,\xi}$ there is a single corresponding measure $\mathbb{P}^X$ given by (1.58) that gives the same prices of contingent claims $V$. Since*

$$\frac{X_Y(1,A)}{X_Y(0)} = u, \qquad \frac{X_Y(1,B)}{X_Y(0)} = 1, \qquad \frac{X_Y(1,C)}{X_Y(0)} = d,$$

*the measure $\mathbb{P}^X$ is given by*

$$\mathbb{P}^X(A) = u \cdot \mathbb{P}^{Y,\xi}(A), \qquad \mathbb{P}^X(B) = \mathbb{P}^{Y,\xi}(B), \qquad \mathbb{P}^X(C) = d \cdot \mathbb{P}^{Y,\xi}(C). \tag{1.59}$$

*It turns out that the probability measure $\mathbb{P}^X$ corresponds to the probability measure $\mathbb{P}^{X,\xi}$ that is given in (1.54). Therefore the price of a contingent claim $V$ would be the same if computed both under $\mathbb{P}^{Y,\xi}$ or under $\mathbb{P}^{X,\xi}$ for a fixed parameter $\xi$. The relationship between $\mathbb{P}^Y$ and $\mathbb{P}^X$ in a general model is given by the so-called Radon–Nikodým derivative, and it is studied in the next section.*

**Remark 1.22 (All martingale measures $\mathbb{P}^Y$ agree on the price of a contract to deliver)**

*A given model of a price evolution may come with infinitely many martingale measures $\mathbb{P}^Y$, and the price of a general contingent claim may depend on the choice of the probability measure $\mathbb{P}^Y$. We have seen this situation in the trinomial model. However, all martingale measures $\mathbb{P}^Y$ agree on a price of a contract to deliver a no-arbitrage asset $Y$. Let us denote this contract by $V$, with $V(T) = Y(T)$ at the delivery time $T$. Since $V_Y(t)$ is a martingale, the initial price $V_Y(0)$ is given by*

$$V_Y(0) = \mathbb{E}^Y[V_Y(T)] = \mathbb{E}^Y[Y_Y(T)] = \mathbb{E}^Y[1] = 1,$$

*and thus $V(0) = Y(0)$. This result is independent on the choice of the probability measure $\mathbb{P}^Y$. From the change of numeraire formula (1.55), we would get the same price of the contract $V$ by using any probability measure $\mathbb{P}^X$. Similarly, all martingale measures $\mathbb{P}^X$ and $\mathbb{P}^Y$ agree on the price of a contract to deliver a no-arbitrage asset $X$.*

In perfectly symmetric situations when the roles of $X$ and $Y$ can be exchanged, it makes sense to study models where $X_Y(t)$ and its inverse price $Y_X(t)$ admit similar evolutions. That would make the observer unable to identify the reference asset just by looking at the price process. More specifically, we can consider the situation when the distribution of the price $\frac{X_Y(T)}{X_Y(0)}$ under the probability measure $\mathbb{P}^Y$ has the same distribution as $\frac{Y_X(T)}{Y_X(0)}$ under the probability measure $\mathbb{P}^X$. When this is the case, we can also write

$$\mathcal{L}^Y\left(\frac{X_Y(T)}{X_Y(0)}\right) = \mathcal{L}^X\left(\frac{Y_X(T)}{Y_X(0)}\right), \tag{1.60}$$

meaning that the laws of the two distributions agree. We will call this principle the **exchangeability of the reference assets.** We show in the following text that it is possible to model the prices of assets in a way that the role of $X$ and $Y$ can be freely exchanged, for instance in the binomial model or in the diffusion model.

Another important question is: when is it possible to replicate a contingent claim $V$ whose payoff depends on underlying assets $X^i$ by trading in these assets? Or in other words, is there a portfolio $P$ of the form

$$P = \sum \Delta^i(t) \cdot X^i$$

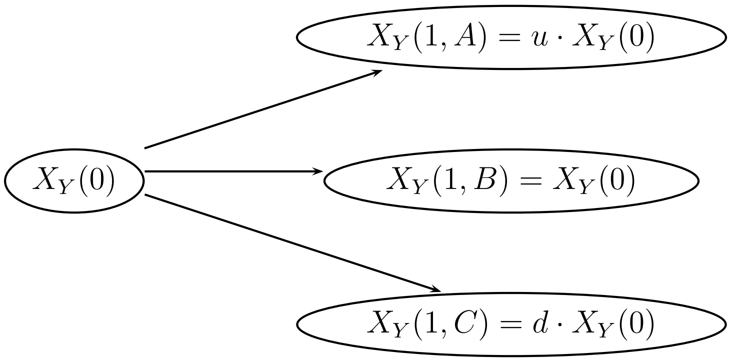
such that $P(t) = V(t)$? We call such a situation a **complete market.** A market is **incomplete** if it is not complete.

**Theorem 1.23 (Second Fundamental Theorem of Asset Pricing)**
*A market is complete if and only if the martingale measure $\mathbb{P}^Y$ is unique.*

**Rule of Thumb:** The market is typically complete in situations when the number of different noise factors does not exceed the number of assets minus one asset that serves as a numeraire.

**Example 1.24 (Complete models)**
*Consider a situation when there are just two assets $X$ and $Y$. The binomial model has one noise factor which can be thought of as a coin toss, and the market is complete. Similarly, the market is complete in a geometric Brownian motion model, where the only source of uncertainty is the Brownian motion. In the case when the asset price has stochastic volatility, there are two noise factors (the original Brownian motion, and stochastic volatility), and the market is incomplete. Jump models are complete only if the jump size takes one single value, such as in a geometric Poisson process which represents the only noise factor. Jump models with multiple jump sizes are incomplete.*

**Example 1.25 (Trinomial model)**
*We have already seen that a trinomial model, with $X_Y(1,A) = u \cdot X_Y(0)$, $X_Y(1,B) = X_Y(0)$, and $X_Y(1,C) = d \cdot X_Y(0)$, where $0 < d < 1 < u$, does not have a unique probability measure $\mathbb{P}^Y$.*




*The price process is a martingale when the probability $\mathbb{P}^Y$ is given by*

$$\mathbb{P}^{Y,\xi}(A) = \frac{1-d}{u-d} \cdot \xi, \quad \mathbb{P}^{Y,\xi}(B) = 1 - \xi, \quad \mathbb{P}^{Y,\xi}(C) = \frac{u-1}{u-d} \cdot \xi,$$

*where $\xi \in [0, 1]$. Here one can think of $\mathbb{P}^Y(A)$ and $\mathbb{P}^Y(B)$ as two sources of uncertainty (or noise factors). The probability of event $C$, $\mathbb{P}^Y(C)$ is already determined since $\mathbb{P}^Y(C) = 1 - \mathbb{P}^Y(A) - \mathbb{P}^Y(B)$. Let us show that this is indeed an incomplete market. Let $V$ be a contingent claim that pays off $V_Y(1)$ units of $Y$ at time $T = 1$. A hedging portfolio for this claim takes the form*

$$P(0) = \Delta^X(0) \cdot X + \Delta^Y(0) \cdot Y. \tag{1.61}$$

*If $P$ replicates a contract $V$, we should have $P(1) = V(1)$ for all outcomes $A$, $B$, and $C$. Note that the portfolio $P$ remains unchanged from time $t = 0$ to time $t = 1$, and thus we also have*

$$P(1) = \Delta^X(0) \cdot X + \Delta^Y(0) \cdot Y.$$

*The identity $P(1) = V(1)$ can also be written in terms of the prices as $P_Y(1) = V_Y(1)$. Thus we have three equations, one for each outcome:*

$$\begin{aligned} V_Y(1, A) &= \Delta^X(0) \cdot X_Y(1, A) + \Delta^Y(0), \\ V_Y(1, B) &= \Delta^X(0) \cdot X_Y(1, B) + \Delta^Y(0), \\ V_Y(1, C) &= \Delta^X(0) \cdot X_Y(1, C) + \Delta^Y(0). \end{aligned}$$

*However, we have only two unknowns, $\Delta^X(0)$, and $\Delta^Y(0)$ and there is no way to match all three different values of $V_Y(1)$ in general. Since $P(1) = V(1)$ cannot be satisfied in general, this model is incomplete.*

*One way to overcome the incompleteness of the model is to consider more underlying assets that may exist in the real markets, thus completing the model. Let us assume for instance that the market trades an Arrow–Debreu security $Z$ that pays one unit of an asset $Y$ when the outcome $A$ happens. The quote of the price $Z_Y(0)$ already determines the probability measure $\mathbb{P}^{Y,\xi}$ uniquely from the relationship*

$$Z_Y(0) = \mathbb{P}^{Y,\xi}(A) = \frac{1-d}{u-d} \cdot \xi,$$

*and thus*

$$\xi = Z_Y(0) \cdot \frac{u-d}{1-d}.$$

*The market becomes complete if we consider a portfolio in the form*

$$P(0) = \Delta^X(0) \cdot X + \Delta^Y(0) \cdot Y + \Delta^Z(0) \cdot Z.$$

*At time $t = 1$, the portfolio $P$ will remain unchanged. In order to match $P(1) = V(1)$ for a general claim $V$, we must have*

$$\begin{aligned}
V_Y(1, A) &= \Delta^X(0) \cdot X_Y(1, A) + \Delta^Y(0) + \Delta^Z(0), \\
V_Y(1, B) &= \Delta^X(0) \cdot X_Y(1, B) + \Delta^Y(0), \\
V_Y(1, C) &= \Delta^X(0) \cdot X_Y(1, C) + \Delta^Y(0),
\end{aligned}$$

*where we used the fact that $Z_Y(1, \omega) = \mathbb{I}_A(\omega)$. We can always find a solution for $\Delta^X(0)$, $\Delta^Y(0)$, and $\Delta^Z(0)$ that would match the payoff of the contingent claim $V$.*

*An alternative way to complete the market with other securities is to change the condition on the hedging portfolio $P$. Instead of requiring $P(1) = V(1)$ which corresponds to a perfect hedge, one may require $P(1) \geq V(1)$ which corresponds to a* **superhedge.** *A superhedging portfolio guarantees that the contractual payoff represented by a claim $V$ is always met, but in some scenarios the resulting portfolio $P$ may have a higher price than $V$. Unfortunately, it often happens the the superhedging portfolio $P$ has a substantially higher price than the actual claim $V$. Even the superhedging portfolio that has the smallest initial price $P_Y(0)$ may give unrealistically high prices. For this reason, superhedging is almost never used in practice.*

*A perfect hedge in the assets $X$ and $Y$ is only possible in two notable situations: either when $V_Y(1) = 1$, or when $V_Y(1) = X_Y(1)$. The first case represents a situation when $V(1) = Y(1)$, so the payoff is the asset $Y$ itself. In this case, $V$ becomes a contract to deliver an asset $Y$, and the corresponding hedge is $\Delta^X(0) = 0$ and $\Delta^Y(0) = 1$. All martingale measures $\mathbb{P}^{Y,\xi}$ do agree that the initial price of this contract is simply $V(0) = Y(0)$. The second case represents a situation when $V(1) = X(1)$, so the payoff is the asset $X$ itself. The contract $V$ becomes a contract to deliver the asset $X$, with the initial price $V(0) = X(0)$ that is independent of the choice of the martingale measure $\mathbb{P}^{Y,\xi}$ and the corresponding hedge is $\Delta^X(0) = 1$, $\Delta^Y(0) = 0$.*

## 1.8 Change of Measure via Radon–Nikodým Derivative

This section describes the relationship between measures implied by using a different numeraire. Suppose that $X$ is a no-arbitrage reference asset, $Y$ is another no-arbitrage reference asset, and $V$ is a contract to be priced. From the change of numeraire formula, we have

$$V = \mathbb{E}^Y \left[V_Y(T)\right] \cdot Y = \mathbb{E}^X \left[V_X(T)\right] \cdot X. \tag{1.62}$$

Recall that we may have in principle infinitely many different martingale measures $\mathbb{P}^Y$ and $\mathbb{P}^X$, but the change of numeraire formula links one probability measure $\mathbb{P}^Y$ with another probability measure $\mathbb{P}^X$ that agrees with $\mathbb{P}^Y$ on the same prices for an arbitrary claim $V$.

The two measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ can be also related through a scaling factor $\mathbb{Z}(T)$ in the following sense:

$$\mathbb{E}^Y[V_X(T)\cdot\mathbb{Z}(T)] = \mathbb{E}^X[V_X(T)]. \tag{1.63}$$

Rewriting this equation in integral form

$$\int_\Omega V_X(T,\omega)\mathbb{Z}(T,\omega)d\mathbb{P}^Y(\omega) = \int_\Omega V_X(T,\omega)d\mathbb{P}^X(\omega)$$

which is valid for any integrable random variable $V_X(T,\omega)$, we get the following representation of $\mathbb{Z}(T)$:

$$\mathbb{Z}(T) = \frac{d\mathbb{P}^X}{d\mathbb{P}^Y}. \tag{1.64}$$

In other words,

$$\mathbb{P}^X(A) = \int_A \mathbb{Z}(T,\omega)d\mathbb{P}^Y(\omega), \quad A\in\mathcal{F}. \tag{1.65}$$

Intuitively this represents how much one must increase or decrease the weight placed upon the probability of $\omega$ under the $\mathbb{P}^Y$ measure so that one gets the same answer as if one used the $\mathbb{P}^X$ measure to start with. The scaling factor $\mathbb{Z}$ is known as the **Radon–Nikodým derivative**. When the space of outcomes $\Omega$ is discrete, Equation (1.65) can be expressed as

$$\mathbb{P}^X(\omega) = \mathbb{Z}(T,\omega)\cdot\mathbb{P}^Y(\omega), \quad \omega\in\Omega. \tag{1.66}$$

We can also consider a reciprocal change of measure

$$\frac{1}{\mathbb{Z}(T)} = \frac{d\mathbb{P}^Y}{d\mathbb{P}^X}, \tag{1.67}$$

meaning that

$$\mathbb{E}^Y[V_Y(T)] = \mathbb{E}^X\left[\frac{V_Y(T)}{\mathbb{Z}(T)}\right]. \tag{1.68}$$

The Radon–Nikodým derivative has the following financial interpretation. We can write

$$\mathbb{E}^X\left[V_X(T)\right]\cdot X(0) = \mathbb{E}^Y\left[V_X(T)\cdot\mathbb{Z}(T)\right]\cdot X(0) = \mathbb{E}^Y\left[V_Y(T)\right]\cdot Y(0),$$

where the first equality results from changing measures, and the second equality comes from the change of numeraire formula. Since this relationship is valid for an arbitrary payoff $V$, we must have

$$[V_X(T)\cdot\mathbb{Z}(T)]\cdot X(0) = [V_Y(T)]\cdot Y(0),$$

or

$$\boxed{\mathbb{Z}(T) = \frac{d\mathbb{P}^X}{d\mathbb{P}^Y} = \frac{X_Y(T)}{X_Y(0)}.} \tag{1.69}$$

We used that

$$\frac{V_Y(T)}{V_X(T)} = X_Y(T),$$

which follows from the change of numeraire formula. Note that the Radon–Nikodým derivative for the reciprocal change of measure is given by

$$\boxed{\frac{1}{\mathbb{Z}(T)} = \frac{d\mathbb{P}^Y}{d\mathbb{P}^X} = \frac{Y_X(T)}{Y_X(0)},} \tag{1.70}$$

which preserves the symmetry between assets $X$ and $Y$.

**Remark 1.26 (Condition for equivalence of the martingale measures $\mathbb{P}^X$ and $\mathbb{P}^Y$)** *When both $\mathbb{Z}(T)$ and $\frac{1}{\mathbb{Z}(T)}$ stay positive, the two measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ agree on zero probability events in $\mathcal{F}_T$. When $\mathbb{P}^Y(A) = 0$ for $A \in \mathcal{F}_T$ we also have $\mathbb{P}^X(A) = 0$ and vice versa, $\mathbb{P}^X(A) = 0$ implies $\mathbb{P}^Y(A) = 0$. This follows from the relationships*

$$\mathbb{P}^X(A) = \int_A \mathbb{Z}(T,\omega)d\mathbb{P}^Y(\omega),$$

*and*

$$\mathbb{P}^Y(A) = \int_A \frac{1}{\mathbb{Z}(T,\omega)}d\mathbb{P}^X(\omega).$$

*When two probability measures agree on zero probability events in $\mathcal{F}_T$, they are* **equivalent.** *Thus the probability measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ are equivalent when both prices $X_Y(T)$ and $Y_X(T)$ stay positive.*

**Remark 1.27 (The risk-neutral measure $\mathbb{P}^M$ agrees with the T-forward measure $\mathbb{P}^T$ when the interest rate is deterministic)** *We have already seen that when the interest rate is deterministic, the risk-neutral measure $\mathbb{P}^M$ that comes with the money market account $M$ and the $T$-forward measure $\mathbb{P}^T$ that comes with the bond $B^T$ that matures at time $T$ give the same prices of contingent claims. This means that the two measures are the same. We can also check this result using the Radon–Nikodým derivative*

$$\mathbb{Z}(T) = \frac{d\mathbb{P}^M}{d\mathbb{P}^T} = \frac{M_{B^T}(T)}{M_{B^T}(0)} = \frac{M_\$(T)\cdot \$_{B^T}(T)}{M_\$(0)\cdot \$_{B^T}(0)} = \frac{\exp\left(\int_0^T r(t)dt\right)\cdot 1}{1\cdot \exp\left(\int_0^T r(t)dt\right)} = 1, \tag{1.71}$$

*which implies that*

$$\mathbb{P}^M(A) = \int_A \mathbb{Z}(T,\omega)d\mathbb{P}^T(\omega) = \int_A 1 d\mathbb{P}^T(\omega) = \mathbb{P}^T(A), \quad A \in \mathcal{F}. \tag{1.72}$$

*Therefore the two martingale measures $\mathbb{P}^T$ and $\mathbb{P}^M$ are the same when the interest rate is deterministic. When the interest rate is stochastic, the Radon–Nikodým derivative becomes*

$$\mathbb{Z}(T) = \frac{d\mathbb{P}^M}{d\mathbb{P}^T} = \frac{M_{B^T}(T)}{M_{B^T}(0)} = \frac{M_\$(T) \cdot \$_{B^T}(T)}{M_\$(0) \cdot \$_{B^T}(0)}$$
$$= \frac{\exp\left(\int_0^T r(t)dt\right) \cdot 1}{1 \cdot \$_{B^T}(0)} = \exp\left(\int_0^T r(t)dt\right) \cdot B_\$^T(0),$$

*which is no longer one, and the relationship between the risk-neutral measure $\mathbb{P}^M$ and the T-forward measure $\mathbb{P}^T$ is no longer trivial.*

**Remark 1.28 (Radon–Nikodým derivative for conditional expectations)** *The Radon–Nikodým derivative as described in the above text corresponds to changing the measure at time $t = 0$. However, we can generalize this concept to any time $t \leq T$. From the change of numeraire formula, we have*

$$\mathbb{E}_t^X[V_X(T)] \cdot X(t) = \mathbb{E}_t^Y[V_Y(T)] \cdot Y(t) = \mathbb{E}_t^Y[V_X(T) \cdot X_Y(T)] \cdot Y(t).$$

*This can be rewritten as*

$$\mathbb{E}_t^X[V_X(T)] = \mathbb{E}_t^Y\left[V_X(T) \cdot \frac{X_Y(T)}{X_Y(t)}\right] = \mathbb{E}_t^Y\left[V_X(T) \cdot \frac{\mathbb{Z}(T)}{\mathbb{Z}(t)}\right].$$

*Therefore we have*

$$\boxed{\mathbb{E}_t^X[V_X(T)] = \frac{1}{\mathbb{Z}(t)} \cdot \mathbb{E}_t^Y[V_X(T) \cdot \mathbb{Z}(T)]} \tag{1.73}$$

*in terms of the original Radon–Nikodým derivative $\mathbb{Z}$. This relationship is known as the* **Bayes formula.**

**Remark 1.29 (European call option)** *A* **European call option** *is a contract that pays off $(X(T) - K \cdot Y(T))^+$ at maturity time $T$, where $K$ is a constant defined by the contract and is known as the* **strike.** *Let us denote the European call option contract as $V$. We can assume that both assets $X$ and $Y$ are no-arbitrage assets. If not, we can consider corresponding no-arbitrage assets that deliver a unit of an asset $X$, or a unit of an asset $Y$ respectively, at time $T$. We can rewrite the option payoff as*

$$V(T) = (X(T) - K \cdot Y(T))^+ = \mathbb{I}\left(X_Y(T) \geq K\right) \cdot X - K \cdot \mathbb{I}\left(X_Y(T) \geq K\right) \cdot Y.$$

*The above expression suggests that a European option is simply a combination of two digital options, one that pays off a unit of an asset $X$ when $X_Y(T) \geq K$, and one that pays off $K$ units of an asset $Y$ on the same event when $X_Y(T) \geq K$. We have already seen in Remark 1.16 that the initial value of the digital option that pays off a unit of an asset $X$ when event $A$ happens is $\mathbb{P}^X(A)$ units of an asset $X$. Similarly, the initial value of the digital option that pays off a unit of an asset $Y$ when event*

*A happens is $\mathbb{P}^Y(A)$ units of an asset $Y$. If we consider $A$ to be event $X_Y(T) \geq K$, the value of the European call option at time $t$ is simply*

$$\boxed{V(t) = \mathbb{P}^X_t(X_Y(T) \geq K) \cdot X - K \cdot \mathbb{P}^Y_t(X_Y(T) \geq K) \cdot Y.} \tag{1.74}$$

*The above relationship is known as the* **Black–Scholes formula.** *Note that deriving the Black–Scholes formula in this form does not require any computation. The question how to determine the probabilities $\mathbb{P}^X_t(X_Y(T) \geq K)$ and $\mathbb{P}^Y_t(X_Y(T) \geq K)$ more explicitly for more specific martingale models of the price evolution is the subject of following chapters.*

*Note that the choice of the probability measure $\mathbb{P}^Y$ in situations when there is more than one such measure already determines the corresponding probability measure $\mathbb{P}^X$, and vice versa. The two probability measures must agree on the prices of all contingent claims, and thus they are related by the Radon–Nikodým derivative. This follows from*

$$\begin{aligned}
V &= \mathbb{E}^Y_t\left[(X - K \cdot Y)^+_Y(T)\right] \cdot Y \\
&= \mathbb{E}^Y_t\left[X_Y(T) \cdot \mathbb{I}(X_Y(T) \geq K)\right] \cdot Y - \mathbb{E}^Y_t\left[K \cdot Y_Y(T) \cdot \mathbb{I}(X_Y(T) \geq K)\right] \cdot Y \\
&= \mathbb{E}^X_t\left[X_X(T) \cdot \mathbb{I}(X_Y(t) \geq K)\right] \cdot X - K \cdot \mathbb{P}^Y_t(X_Y(T) \geq K) \cdot Y \\
&= \mathbb{P}^X_t(X_Y(T) \geq K) \cdot X - K \cdot \mathbb{P}^Y_t(X_Y(T) \geq K) \cdot Y,
\end{aligned}$$

*where we have used the change of numeraire formula*

$$\mathbb{E}^Y_t\left[X_Y(T) \cdot \mathbb{I}(X_Y(T) \geq K)\right] \cdot Y = \mathbb{E}^X_t\left[X_X(T) \cdot \mathbb{I}(X_Y(T) \geq K)\right] \cdot X.$$

*This shows that the probability measures $\mathbb{P}^X$ and $\mathbb{P}^Y$ are indeed linked by the Radon–Nikodým derivative.*

## 1.9 Leverage: Forwards and Futures

**Leverage** is one of the most important concepts of finance. It allows investors to magnify their positions in the underlying assets. Let us consider a situation when an investor believes that the price $X_Y$ of a specific asset $X$ will appreciate in the near future. A straightforward way how to realize the potential profit is to buy the asset $X$ now, and sell it at some subsequent time $T$. The result of this trading is summarized in Table 1.2. At time $t = 0$, the investor has one unit of an asset $X$ that costs him $X_Y(0)$ units of an asset $Y$. At time $t = T$, the asset $X$ is sold for $X_Y(T)$ units of an asset $Y$. Therefore at time $t = T$, the position in the asset $X$ is zero, and the position in the asset $Y$ is $X_Y(T) - X_Y(0)$. The net profit or loss of this trading is thus $X_Y(T) - X_Y(0)$ units of an asset $Y$. When $X_Y(T) - X_Y(0)$ is positive, this trade results in a net profit, when $X_Y(T) - X_Y(0)$ is negative, this trade results in a net loss.

There is an alternative way to realize this profit or loss by trading in contracts to deliver. Instead of buying the asset $X$ at time $t = 0$, one can buy a contract $U$ that

Table 1.2: Trading in the asset $X$.

| | Time $t = 0$ | Time $t = T$ |
|---|---|---|
| Asset $X$ | 1 | 0 |
| Asset $Y$ | $-X_Y(0)$ | $X_Y(T) - X_Y(0)$ |

delivers the asset $X$ at time $T$, and pay for it in terms of a contract $V$ that delivers the asset $Y$ at time $T$. Consider first the case that $X$ and $Y$ are both no-arbitrage assets. We have seen that a contract to deliver a no-arbitrage asset agrees with the asset itself at all times, so $U(t) = X(t)$, and $V(t) = Y(t)$. Thus it may not be obvious why this approach gives any advantage over the case when the investor trades in the primary assets $X$ and $Y$. Table 1.3 shows the positions in the assets $U$, $V$, $X$ and $Y$. Note that at time $t = 0$, the investor has zero positions in the assets $X$ and $Y$. The major advantage in this trade is that the investor does not need to have a short position in the reference asset $Y$. The choice of $Y$ is typically a money market account. In contrast to the previous case, the investors do not need to decrease their position in the money market by paying $X_Y(0)$ units of an asset $Y$ for a unit of an asset $X$.

Table 1.3: Trading in the contracts to deliver $U$ and $V$.

| | Time $t = 0$ | Delivery, $t = T$ | Sale of $X$, $t = T$ |
|---|---|---|---|
| Asset $U$ | 1 | 0 | 0 |
| Asset $V$ | $-X_Y(0)$ | 0 | 0 |
| Asset $X$ | 0 | 1 | 0 |
| Asset $Y$ | 0 | $-X_Y(0)$ | $X_Y(T) - X_Y(0)$ |

The contract $U$ delivers a unit of an asset $X$ at time $T$. Similarly, $X_Y(0)$ units of the contract $V$ delivers the corresponding number of units of an asset $Y$ to the counter party of this trade at time $T$. Furthermore, the holder of the asset $X$ may immediately sell it for $X_Y(T)$ units of an asset $Y$, resulting in the net profit or loss of $X_Y(T) - X_Y(0)$ units of an asset $Y$. This is the same as in the case when the asset $X$ was bought at time $t = 0$, and sold at time $T$.

Developing this idea even further, one can introduce a contract that pays off one unit of an asset $X$ for $K$ units of an asset $Y$ at time $T$:

$$F(T) = X(T) - K \cdot Y(T). \tag{1.75}$$

The contract $F$ is known as a **forward.** When $X$ and $Y$ are no-arbitrage assets, the price of the forward contract is given by

$$\begin{aligned} F_Y(t) = \mathbb{E}_t^Y[X_Y(T) - K \cdot Y_Y(T)] \\ = \mathbb{E}_t^Y[X_Y(T) - K \cdot Y_Y(T)] = X_Y(t) - K. \end{aligned} \tag{1.76}$$

Thus we have

$$F(t) = X(t) - K \cdot Y(t)$$

at all times $t \leq T$. More generally, the forward can be written as

$$F(t) = U(t) - K \cdot V(t),$$

where $U$ is a contract that delivers a unit of an asset $X$, and $V$ is a contract that delivers a unit of an asset $Y$. This relationship is valid in both cases when assets $X$ and $Y$ are arbitrage or no-arbitrage assets.

**The forward price** $\mathrm{For}(t,T)$ is the value of $K$ that makes the forward contract $F$ have zero price at time $t$. It is obvious that

$$\mathrm{For}(t,T) = X_Y(t) \tag{1.77}$$

when $X$ and $Y$ are no-arbitrage assets. Table 1.4 shows that one receives $X_Y(T) - X_Y(0)$ units of an asset $Y$ at time $T$ by buying a forward contract $F$. The forward contract itself has a zero price at time $t = 0$, and entering this contract does not require any change of positions in the assets $X$ and $Y$. Since the price of the forward contract $F$ is zero, one can potentially enter an unlimited number of forward contracts at a given time. Although the forward contract should formally deliver a unit of the asset $X$, it is still typically settled entirely in the asset $Y$. Thus the number of the forward contracts may exceed the total supply of the asset $X$. This is indeed the case for many typical assets. For instance there are many more contracts to deliver gold or oil than is physically available. However, these contracts are typically settled in money; the asset itself is delivered only in rare cases.

Table 1.4: Trading in the forward contract $F$.

| | Time $t = 0$ | Time $t = T$ |
|---|---|---|
| Asset $F$ | 1 | 0 |
| Asset $X$ | 0 | 0 |
| Asset $Y$ | 0 | $X_Y(T) - X_Y(0)$ |

Obviously, entering a huge number of forward contracts comes with a significant risk of a bankruptcy. The contractual payoff $X_Y(T) - X_Y(0)$ at time $T$ can be both positive or negative, and having a substantial number of such contracts may lead to a significant gain, or to a significant loss. In order to prevent the situation that one of the contractual parties fails to meet its obligations, one can split the payoff $X_Y(T) - X_Y(0)$ into a series of daily payments that reflect the change of the price of the forward contract.

Splitting the payoff into a series of payments is done in the following way. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the times of the payments. One can think about them as days if the payments come on a daily basis. At time $t_0 = 0$, one enters a forward contract $F^{t_0} = X - X_Y(t_0) \cdot Y$ that has a zero price. At time $t_1$, the price of $F^{t_0}$ will change to

$$F_Y^{t_0}(t_1) = \mathbb{E}_{t_1}^Y[X_Y(T) - X_Y(t_0)] = X_Y(t_1) - X_Y(t_0).$$

In order to make $F^{t_0}$ a zero price contract at time $t_1$, one should subtract $X_Y(t_1) - X_Y(t_0)$ units of the asset $Y$ from it. This technically creates a new forward contract $F^{t_1}$ that has a zero price at time $t_1$. The relationship between $F^{t_1}$ and $F^{t_0}$ is the following

$$\begin{aligned} F^{t_1} &= F^{t_0} - [X_Y(t_1) - X_Y(t_0)] \cdot Y \\ &= X - X_Y(t_0) \cdot Y - X_Y(t_1) \cdot Y + X_Y(t_0) \cdot Y = X - X_Y(t_1) \cdot Y. \end{aligned}$$

One can continue this procedure for other times $t_k$. Table 1.5 shows the result of this procedure between times $t_{k-1}$ and $t_k$.

Table 1.5: Splitting the payments.

| | Time $t = t_{k-1}$ | Time $t = t_k$ |
|---|---|---|
| Asset $F^{t_{k-1}}$ | 1 | 0 |
| Asset $F^{t_k}$ | 0 | 1 |
| Asset $X$ | 0 | 0 |
| Asset $Y$ | 0 | $X_Y(t_k) - X_Y(t_{k-1})$ |

In contrast to the forward contract, this procedure does not wait until its expiration $T$, but rather settles the changes of the price of the forward contract daily. The forward contract $F^{t_{k-1}}$ from the previous time $t_{k-1}$ is replaced by a new forward contract $F^{t_k}$ at time $t_k$ so that $F^{t_k}$ has a zero price. The difference between the prices of $F^{t_{k-1}}$ and $F^{t_k}$ is settled in the asset $Y$. At the end of this procedure, one would collect

$$\sum_{k=1}^{n}[X_Y(t_k) - X_Y(t_{k-1})] = X_Y(T) - X_Y(0)$$

units of an asset $Y$. Splitting the payments is a principle of a contract known as **futures.** A futures contract is defined as a series of payments

$$\boxed{\sum_{k=1}^{n}[\mathrm{Fut}(t_k, T) - \mathrm{Fut}(t_{k-1}, T)] \cdot Y_{t_k}} \tag{1.78}$$

that are settled in the asset $Y$ at the corresponding times $t_k$. The **futures price** $\mathrm{Fut}(t_m, T)$ is a number that makes the series of the remaining payments

$$\sum_{k=m+1}^{n}[\mathrm{Fut}(t_k, T) - \mathrm{Fut}(t_{k-1}, T)] \cdot Y_{t_k}$$

have a zero price at time $t_m$. At time $t = T$, $\mathrm{Fut}(T, T)$ agrees with the price $X_Y(T)$, the number of units of an asset $Y$ required to obtain a unit of an asset $X$.

Let us determine $\mathrm{Fut}(t_m, T)$ when $X$ and $Y$ are two no-arbitrage assets. At time $t_{n-1}$, the futures contract has only one payment left, namely

$$[\mathrm{Fut}(T, T) - \mathrm{Fut}(t_{n-1}, T)] \cdot Y(T) = [X_Y(T) - \mathrm{Fut}(t_{n-1}, T)] \cdot Y(T).$$

If the price of this contract be zero at time $t_{n-1}$, we must have

$$0 = \mathbb{E}^Y_{t_{n-1}}[X_Y(T) - \mathrm{Fut}(t_{n-1}, T)] = X_Y(t_{n-1}) - \mathrm{Fut}(t_{n-1}, T)$$

from the martingale property of $X_Y(t)$. We conclude that

$$\mathrm{Fut}(t_{n-1}, T) = X_Y(t_{n-1}).$$

Repeating this argument, we obtain

$$\mathrm{Fut}(t_m, T) = X_Y(t_m)$$

at all times $t_k$. Thus in the case when both assets $X$ and $Y$ are no-arbitrage assets, the forward and the futures price agree:

$$\mathrm{Fut}(t, T) = \mathrm{For}(t, T) = X_Y(t),$$

and futures is the same as the forward contract. However, by splitting the payments, one minimizes the default risk of the counter party.

One can avoid the counter party risk completely by trading such contracts on an exchange. Members of the exchange are required to deposit enough funds to cover for all their potential losses that may happen within one day. This deposit is known as a **margin account.** When the funds in the margin account become critically low, the member receives a margin call, a request to add more funds. If the member fails to do so, his positions are closed. Closing the existing positions does not cost anything as the prices of the futures contracts are set to zero continuously.

The most typical futures contracts are settled in currencies, rather than in a no-arbitrage asset. It slightly changes the situation since we also need to take into account the time value of money. Let us assume that the asset $X$ is a stock $S$, and the asset $Y$ is a dollar \$. The futures contract is defined in this case as a series of payments of the following form

$$\sum_{k=1}^{n}[\mathrm{Fut}(t_k, T) - \mathrm{Fut}(t_{k-1}, T)] \cdot \$(t_k). \tag{1.79}$$

$\mathrm{Fut}(t_m, T)$ is the value that makes the price of the remaining payments

$$\sum_{k=m+1}^{n}[\mathrm{Fut}(t_k, T) - \mathrm{Fut}(t_{k-1}, T)] \cdot \$(t_k).$$

to be zero at time $t_m$. Equation (1.79) is written in terms of an arbitrage asset \$. However, the investor would immediately convert the dollar position into a position in the money market $M$. Assume that the price of the money market $M_\$(0)$ starts at one, so we have $M_\$(0) = 1$. From the relationship

$$M_\$(t_k) \cdot \$(t_k) = M(t_k),$$

we can write

$$\$(t_k) = \frac{1}{M_\$(t_k)} \cdot M(t_k).$$

The dollar \$ at time $t_k$ can be exchanged for $\frac{1}{M_\$(t_k)}$ number of units of the money market $M$. Thus the payoff of the futures contract can be reexpressed as

$$\sum_{k=1}^{n} [\mathrm{Fut}(t_k, T) - \mathrm{Fut}(t_{k-1}, T)] \cdot \frac{1}{M_\$(t_k)} \cdot M_{t_k}. \tag{1.80}$$

Note that this makes the money market $M$ a natural reference asset for computing the price of the futures contract. Let us determine $\mathrm{Fut}(t_m, T)$. At the terminal time $t_n = T$, $\mathrm{Fut}(T, T)$ agrees with the dollar price of the stock $S_\$(T)$. At time $t_{n-1}$ the futures contract has only a single payment

$$[\mathrm{Fut}(T, T) - \mathrm{Fut}(t_{n-1}, T)] \cdot \frac{1}{M_\$(T)} \cdot M_T = [S_\$(T) - \mathrm{Fut}(t_{n-1}, T)] \cdot \frac{1}{M_\$(T)} \cdot M_T.$$

Should the price of this payment be zero at time $t_{n-1}$, we must have

$$\begin{aligned} 0 = \mathbb{E}^M_{t_{n-1}} &\left[ [S_\$(T) - \mathrm{Fut}(t_{n-1}, T)] \cdot \frac{1}{M_\$(T)} \right] \\ &= \frac{1}{M_\$(T)} \cdot \left[ \mathbb{E}^M_{t_{n-1}}[S_\$(T)] - \mathrm{Fut}(t_{n-1}, T) \right]. \end{aligned}$$

We have used the fact that the price of the money market account $M_\$(T)$ is already known at the prior time $t_{n-1}$. The reason is that the interest rate that corresponds to the time interval $[t_{n-1}, t_n]$ is set at time $t_{n-1}$, so the investor knows the price $M_\$(t_n)$ of the money market account one period ahead. Therefore

$$\mathrm{Fut}(t_{n-1}, T) = \mathbb{E}^M_{t_{n-1}}[S_\$(T)].$$

Repeating this procedure for the previous times $t$, we get

$$\boxed{\mathrm{Fut}(t, T) = \mathbb{E}^M_t[S_\$(T)].} \tag{1.81}$$

Let us compare the futures price $\mathrm{Fut}(t, T)$ with $\mathrm{For}(t, T)$, the price of the corresponding forward contract. The forward contract $F$ when written on a stock $S$ and a dollar \$ pays off

$$F(T) = S(T) - K \cdot \$(T).$$

This payoff can be rewritten in terms of a bond $B^T$ that delivers a dollar \$ at time $T$ as

$$F(T) = S(T) - K \cdot B^T(T).$$

The forward price is a number $\mathrm{For}(t,T)$ that corresponds to a choice of $K$ such that the price of the forward contract $F$ is zero at time $T$. Thus $\mathrm{For}(t,T)$ satisfies the equation

$$0 = \mathbb{E}_t^T[S_{B^T}(T) - \mathrm{For}(t,T)].$$

The natural choice of the reference asset is a bond $B^T$. Solving for $\mathrm{For}(t,T)$, we get

$$\boxed{\mathrm{For}(t,T) = \mathbb{E}_t^T[S_\$(T)].} \tag{1.82}$$

We used a simple relationship $S_\$(T) = S_{B^T}(T) \cdot B_\$^T(T) = S_{B^T}(T)$.

Both the futures price $\mathrm{Fut}(t,T)$ and the forward price $\mathrm{For}(t,T)$ are expectations of the terminal price of the stock $S_\$(T)$, but under different probability measures. The futures price is computed under the risk-neutral measure $\mathbb{P}^M$, while the forward price is computed under the T-forward measure $\mathbb{P}^T$. We have already seen that when the interest rate $r(t)$ is deterministic, the two measures agree: $\mathbb{P}^M = \mathbb{P}^T$. In this case, the futures price and the forward price agree.

When the interest rate $r(t)$ is stochastic, the two measures $\mathbb{P}^M$ and $\mathbb{P}^T$ are in general different, and the futures price may be different from the forward price. Let us compute the difference between them:

$$\begin{aligned}
\mathrm{Fut}(0,T) &- \mathrm{For}(0,T) = && (1.83)\\
&= \mathbb{E}^M S_\$(T) - \mathbb{E}^T S_\$(T) \\
&= \mathbb{E}^T\left[S_\$(T)\cdot\frac{M_{B^T}(T)}{M_{B^T}(0)}\right] - \frac{\mathbb{E}^T M_{B^T}(T)}{M_{B^T}(0)}\cdot\mathbb{E}^T S_\$(T) \\
&= B_\$^T(0)\Big[\mathbb{E}^T\left[S_\$(T)\cdot M_\$(T)\right] - \mathbb{E}^T\left[S_\$(T)\right]\cdot\mathbb{E}^T\left[M_\$(T)\right]\Big] \\
&= B_\$^T(0)\cdot\mathrm{cov}^T(S_\$(T), M_\$(T)) \\
&= B_\$^T(0)\cdot\mathrm{cov}^T\left(S_\$(T), \exp\left(\textstyle\int_0^T r(t)dt\right)\right).
\end{aligned}$$

Thus the difference between $\mathrm{Fut}(0,T)$ and $\mathrm{For}(0,T)$ is proportional to the covariance between the stock price $S_\$(T)$ and the price of the money market account $M_\$(T)$. The covariance is computed in the T-forward measure $\mathbb{P}^T$ that corresponds to the bond $B^T$ as choice of the reference asset. The price of the money market $M_\$(T)$ is directly related to the interest rate $r(t)$: the higher is the interest rate, the higher is the price of the money market.

When the covariance between $S_\$(T)$ and $M_\$(T)$ is positive, the futures price is higher than the forward price. This can be explained by the following argument. In the scenarios when the stock price $S_\$(T)$ ends up above the initial stock price $S_\$(0)$, the corresponding price of the money market $M_\$(T)$ will also tend to increase more than in the scenarios when the stock price $S_\$(T)$ ends up lower than $S_\$(0)$. This follows from the positive correlation of $S_\$(T)$ and $M_\$(T)$. When the price of the stock goes up, the holder of the futures contract will be receiving a positive cash flow, and this cash flow will tend to earn a higher interest rate $r(t)$ on those scenarios. On the other hand, when the stock goes down, the holder of the futures contract will be receiving a negative cash flow, and this cash flow will tend to earn a lower interest rate $r(t)$ on those scenarios. The fact that the resulting cash flow from the futures contracts earns a favorable interest means that the futures price should be higher than the corresponding forward price. In contrast to the futures contract, the forward contract is settled in one single payment at its maturity time, and thus it cannot benefit from varying interest rate.

The reader can check that the difference between the futures price and the forward price can also be expressed as

$$\begin{aligned}\mathrm{Fut}(0,T)-\mathrm{For}(0,T) &= -B_{\$}^{T}(0)\cdot \mathrm{cov}^{M}(S_{\$}(T),\tfrac{1}{M_{\$}(T)}) \\ &= -B_{\$}^{T}(0)\cdot \mathrm{cov}^{M}\left(S_{\$}(T),\exp\left(-\textstyle\int_0^T r(t)dt\right)\right)\end{aligned}$$

if we use the risk-neutral measure $\mathbb{P}^M$ that corresponds to the money market $M$ as a reference asset. The idea is to follow the computation in (1.83), but apply the change of measure from $\mathbb{P}^T$ to $\mathbb{P}^M$ in the third line of the equation.

# References and Further Reading

The concept of Arrow–Debreu securities traces back to Arrow and Debreu [2]. The idea of pricing financial securities by a no-arbitrage argument was already present in the papers of Black and Scholes [8] and Merton [60]. However, the theory of pricing under the martingale measure was fully developed only in later papers of Harrison and Kreps [36] and Harrison and Pliska [37]. Different formulations of the absence of arbitrage and the existence of a martingale measure appear in Delbaen and Schachermayer [19]. An extensive survey of the mathematics of arbitrage appears in the monograph Delbaen and Schachermayer [20].

The fact that one may use different reference assets for pricing appeared early in the relevant literature. Margrabe [59] was the first to use a stock measure for pricing an exchange option, a contract written on two stocks. Jamshidian [48] used bonds as a numeraire in pricing problems of the fixed income markets, introducing the T-forward measure. A more systematic theory of the change of numeraire was developed in Geman et al. [32]. The change of numeraire is now a mainstream technique used in the finance theory, as illustrated in the papers of Gourieroux et al. [35], Long [57], Papell and Theodoridis [67], Brekke [11], Flemming et al. [26], Schroder [74], Platen [69], Johansson [50], Karatzas and Kardaras [52], Platen [68], and Filipovic citeFilipovic. The distinction between the forward and futures contracts was pointed out by Margrabe [58] and Black [7].

There are many additional books on quantitative finance that may be useful for the reader who is interested in a more thorough study of the field. An overview of many financial products is given in Hull [44], which also serves as an introduction to option pricing for practitioners. The book by Baxter and Rennie [4] serves as a very intuitive introduction to contingent pricing in continuous time. An incomplete list of quantitative finance monographs includes Shreve [76], Merton [62], Bjork [6], Musiela and Rutkowski [64], Duffie [22], Dana and Jeanblanc [18], Jeanblanc et al. [49], Karatzas and Shreve [53], Shiryaev [75], Joshi [51], Wilmott [85], Cerny [16], or Neftci [65].

# Chapter 2

# Diffusion Models

This chapter introduces diffusion models. Under very broad conditions, all no-arbitrage models of a continuous price evolution are diffusion models. In other words, every continuous evolution of the price can be expressed as an Ito's integral. This result is known as a Martingale Representation Theorem.

Diffusion models of price use Brownian motion to represent market noise. Since the market noise itself can take negative values, it does not serve as a good model for the prices. However, we can take the corresponding stochastic exponential which is a positive martingale, and thus is perfectly suitable for a no-arbitrage model of a price process. The simplest model assumes a constant volatility that leads to a geometric Brownian motion. While most of the real price processes do not have constant volatility, this assumption still results in reasonable models for prices and hedges of complex financial instruments. Moreover, the prices of many financial products in the geometric Brownian motion model admit closed form solutions, and thus they are easy to use.

In order to compute the prices of financial derivatives, we need to determine the martingale measures that correspond to all the assets relevant to the given contract. For instance, the Black–Scholes formula for the price of the European call option uses both probability measures $\mathbb{P}^X$ and $\mathbb{P}^Y$ that are associated with the assets $X$ and $Y$. The probability measure $\mathbb{P}^X$ can be determined from the evolution of the inverse price $Y_X(t)$, and this price has to be a $\mathbb{P}^X$ martingale. It turns out that the evolution of the inverse price $Y_X(t)$ is also a geometric Brownian motion, but the market noise $W^X$ is associated with the reference asset $X$.

Diffusion models have one important property: every no-arbitrage asset comes with its own market noise. An asset $Y$ has a market noise $W^Y$, and an asset $X$ has a market noise $W^X$. Although $W^Y$ and $W^X$ are perfectly correlated in the geometric Brownian motion model, we can always identify the market noise that comes with each individual asset. Even more complicated assets, such as a power option, or an average asset, come with its own market noise. This fact will be used for pricing barrier, lookback, and Asian options in the subsequent chapters.

The first section introduces the geometric Brownian motion model, and studies the evolution of the prices $X_Y$ and $Y_X$ under the corresponding martingale measures $\mathbb{P}^Y$ and $\mathbb{P}^X$. We also show that the measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ have the interpretation of how costly a given event is if settled in terms of the asset $Y$, or in terms of the asset $X$, respectively. The second section introduces general European contracts. European contracts are contracts on two assets that are defined by the payoff function, which can be expressed in terms of each reference asset $Y$ or $X$. The two payoff functions are related by a formula known as a perspective mapping. Some contracts remain the same if the roles of the assets $Y$ and $X$ is switched in the payoff function; for instance the best of the two assets defined as $\max(X(T), Y(T))$ is the same as $\max(Y(T), X(T))$. The best of the two assets naturally leads to European call and put options with the payoff

$$(X(T) - K \cdot Y(T))^+ = \max(X(T), K \cdot Y(T)) - K \cdot Y(T).$$

We give examples of European call and put options that appear in different markets: a stock option, a currency option, an exchange option, or a caplet. Their prices and the hedging portfolios are given by the Black–Scholes formula. We compute all prices in terms of the no-arbitrage assets so that we can employ the First Fundamental Theorem of Asset Pricing directly. In order to get the prices in terms of a dollar, an arbitrage asset, we can trivially apply the change of numeraire formula to the prices computed with respect to no-arbitrage assets.

The price of a contingent claim can be computed by two alternative methods: by computing the conditional expectation, or by solving the associated partial differential equation. The case of European options is usually simple enough to obtain closed form formulas, but both approaches also work for more complicated products when no close formula is known. The conditional expectation can be approximated by Monte Carlo methods, and the partial differential equation can be solved numerically by applying finite difference techniques.

The primary goal of contingent pricing is to find the dollar price of a given contract. **Our text suggests to compute the price of a contingent claim with respect to a no-arbitrage asset first, such as a corresponding bond, and then convert it to the dollar price using the change of numeraire. This approach is valid in general, and it has clear computational advantages when the contingent claim is more complex, such as in the case of exotic options.** However, the dollar prices of European claims also satisfy a certain and more complicated partial differential equation that is obtained by discounting to the dollar prices of the underlying assets. But this partial differential equation does not hold in general, it assumes a deterministic evolution of the interest rate. We mention it in our text since the partial differential equation in terms of dollars is the most widely used in practice. For simple contracts, such as for European options, it does not make a difference to compute the prices under different reference assets (arbitrage or no-arbitrage) since the price of the contract is simple to determine.

The only loss when computing the dollar prices directly from the corresponding partial differential equation approach is that the approach does not apply to stochastic interest rates. In that case one should compute the prices in terms of the bond, and convert it to dollar prices by changing the numeraire.

For more complex products, such as for barrier, lookback, or Asian options, using the no-arbitrage asset as a numeraire leads to significant computational advantages. On the other hand, American options have to use dollar values in order to compare the intrinsic and the continuation values, and the partial differential equation in terms of dollars has to be used. In the case of the American option, it is the setup of the contract that forces us to use the partial differential equation in terms of a dollar.

We also discuss how to construct the hedging portfolios for European contracts. The hedging must always be done in the two underlying no-arbitrage assets. We determine the hedging positions in both assets. We can also get a similar expression for the hedging positions in terms of the dollar price functions. The hedging positions for European call options are bounded in both assets; the position in the asset $X$ is always between zero and one, and the hedging position in the asset $Y$ is always between minus the strike $K$ and zero.

We also briefly introduce stochastic volatility models. The price of the contract is still considered to be Markov, but it depends on two parameters: the price $X_Y(t)$ of the asset $X$ and stochastic volatility $\xi(t)$. The resulting partial differential equation for the price of the derivative security becomes two dimensional in space. The chapter is concluded with an example of a European option contract in the foreign exchange market which is just a special case of the general approach presented in the previous text.

## 2.1 Geometric Brownian Motion

Assume that the two assets $X$ and $Y$ are no-arbitrage assets. We have seen that the price $X_Y(t)$ must be a $\mathbb{P}^Y$ martingale in order to prevent any arbitrage opportunity. In continuous time, a general martingale can be written as a sum of a martingale with continuous paths and a purely discontinuous martingale:

$$\mathcal{M}(t) = \mathcal{M}^c(t) + \mathcal{M}^d(t). \tag{2.1}$$

Continuous martingales adapted to a filtration $\mathcal{F}_t^W$ generated by a Brownian motion $W$ are in fact diffusions; they can be represented as stochastic integrals with respect to Brownian motion. Thus

$$\mathcal{M}^c(t) = \mathcal{M}^c(0) + \int_0^t \phi(s)dW(s), \tag{2.2}$$

where $\phi(t)$ is adapted to $\mathcal{F}_t^W$. This result is known as a **Martingale Representation Theorem**.

This chapter focuses on price models with continuous paths. The process $X_Y(t)$ must have the form

$$dX_Y(t) = \phi(t)dW(t).$$

Let us start with the simple but very popular model when

$$\phi(t) = \sigma X_Y(t).$$

The price process $X_Y(t)$ follows

$$dX_Y(t) = \sigma X_Y(t)dW^Y(t), \tag{2.3}$$

which is known as a **geometric Brownian motion.** The parameter $\sigma$ is referred to as **volatility**. Volatility is inherent to diffusion models. Similar to price, volatility is a pairwise relationship between two assets $X$ and $Y$. The price $X_Y$ of the asset $X$ with respect to a reference asset $Y$ may have very different volatility than the price $X_Z$ with respect to a different reference asset $Z$. For instance, a typical dollar stock price $S_\$$ is more volatile than a stock price $S_I$ taken with respect to a market index $I$. Sometimes we will denote by $\sigma_{xy}$ the volatility that corresponds to the assets $X$ and $Y$.

A natural question is how the measure $\mathbb{P}^Y$ is determined. Under $\mathbb{P}^Y$, the driving process $W^Y(t)$ is a Brownian motion. Also the above stochastic differential equation has the following solution:

$$\boxed{X_Y(t) = X_Y(0) \cdot \exp\left(\sigma W^Y(t) - \tfrac{1}{2}\sigma^2 t\right).} \tag{2.4}$$

Note that $X_Y(t)$ is a $\mathbb{P}^Y$ martingale.

In order to compute the prices of European options and other derivative securities, we also need to determine the probability measure $\mathbb{P}^X$. The role of $X$ and $Y$ should be exchangeable in models that preserve the symmetry between both assets. Mathematically, this requirement translates to

$$\mathcal{L}_t^Y\left(\frac{X_Y(T)}{X_Y(t)}\right) = \mathcal{L}_t^X\left(\frac{Y_X(T)}{Y_X(t)}\right), \tag{2.5}$$

meaning that the price increment $\frac{X_Y(T)}{X_Y(t)}$ under the probability measure $\mathbb{P}^Y$ should have the same distribution as the price increment $\frac{Y_X(T)}{Y_X(t)}$ under the probability measure $\mathbb{P}^X$. Therefore we need to have a description of the dynamics of the inverse price, $Y_X(t)$, that would be analogous to the dynamics of the original price $X_Y(t)$. Ideally, the evolution of this price should have the same form as (2.3), but the dynamics are already determined by Ito's formula (see Appendix):

$$\begin{aligned} dY_X(t) = dX_Y(t)^{-1} &= -X_Y(t)^{-2}dX_Y(t) + \tfrac{1}{2}\cdot 2X_Y(t)^{-3}d^2X_Y(t) \\ &= -\sigma Y_X(t)dW^Y(t) + \sigma^2 Y_X(t)dt \\ &= \sigma Y_X(t)\cdot\left(-dW^Y(t) + \sigma dt\right). \end{aligned} \tag{2.6}$$

Given the exchangeability argument of $X$ and $Y$, we should also have

$$dY_X(t) = \sigma Y_X(t) dW^X(t), \tag{2.7}$$

which is the same stochastic differential equation as (2.3) with $X$ and $Y$ flipped, and with a different Brownian motion $W^X(t)$ under the measure $\mathbb{P}^X$. The solution of the above stochastic differential equation is given by

$$\boxed{Y_X(t) = Y_X(0) \cdot \exp\left(\sigma W^X(t) - \tfrac{1}{2}\sigma^2 t\right).} \tag{2.8}$$

In diffusion models, each reference asset $Y$ has its own market noise that is represented by one or several Brownian motions $W^{i,Y}(t)$. Other reference assets, such as an asset $X$, have different market noise that is represented by Brownian motions $W^{i,X}(t)$. Obviously, the Brownian motions $W^Y$ and $W^X$ are related. In the above case, we just have one Brownian motion for each asset, and the relationship between $W^X(t)$ and $W^Y(t)$ follows from the equation

$$dY_X(t) = \sigma Y_X(t) \cdot \left(-dW^Y(t) + \sigma dt\right) = \sigma Y_X(t) dW^X(t). \tag{2.9}$$

Thus we must have

$$dW^X(t) = -dW^Y(t) + \sigma dt,$$

or in other words,

$$W^X(t) = -W^Y(t) + \sigma t. \tag{2.10}$$

Note that a symmetric relationship holds as well

$$W^Y(t) = -W^X(t) + \sigma t. \tag{2.11}$$

**Remark 2.1** *Some authors define $dW^X(t)$ as $dW^Y(t) + \sigma dt$ which is an equivalent definition since the Brownian motion is symmetric and thus $dW^Y(t)$ has the same distribution as $-dW^Y(t)$. However, such a definition would break the symmetry of the price formulas for $X$ and $Y$, and thus it is more appropriate to use $dW^X(t) = -dW^Y(t) + \sigma dt$.*

The two Brownian motions $W^Y(t)$ and $W^X(t)$ are perfectly correlated with a correlation coefficient of -1:

$$dW^Y(t) \cdot dW^X(t) = -1 \cdot dt.$$

This makes sense since when $X_Y(t)$ goes up, the inverse price $Y_X(t)$ goes down, and vice versa.

From the financial representation of the Radon–Nikodým derivative we have

$$\mathbb{Z}(t) = \frac{d\mathbb{P}_t^X}{d\mathbb{P}_t^Y} = \frac{X_Y(t)}{X_Y(0)} = \exp\left(\sigma W^Y(t) - \tfrac{1}{2}\sigma^2 t\right) = \exp\left(-\sigma W^X(t) + \tfrac{1}{2}\sigma^2 t\right). \tag{2.12}$$

The concept of equivalent treatment of both $X$ and $Y$ is also supported by the following theorem.

**Theorem 2.2 (Girsanov.)** *Let $W^Y(t)$ be a $\mathbb{P}^Y$ Brownian motion. Then $W^X(t) = -W^Y(t) + \sigma t$ is a $\mathbb{P}^X$ Brownian motion, where $\mathbb{Z}(t) = \frac{d\mathbb{P}^X_t}{d\mathbb{P}^Y_t} = \frac{X_Y(t)}{X_Y(0)} = \exp\left(\sigma W^Y(t) - \frac{1}{2}\sigma^2 t\right)$.*

**Remark 2.3** *The two measures $\mathbb{P}^Y$ and $\mathbb{P}^X$ may disagree on the drift of the Brownian motion. More specifically,*

$$\mathbb{E}^Y[W^X(t)] = \mathbb{E}^Y[-W^Y(t) + \sigma t] = \sigma t,$$

*but*

$$\mathbb{E}^X[W^X(t)] = 0.$$

*The last statement can be proved by a change of the measure argument (1.63)*

$$\begin{aligned}
\mathbb{E}^X[W^X(t)] &= \mathbb{E}^X[-W^Y(t) + \sigma t] = \mathbb{E}^Y[(-W^Y(t))Z(t)] + \sigma t \\
&= \mathbb{E}^Y[-W^Y(t) \cdot \exp\left(\sigma W^Y(t) - \tfrac{1}{2}\sigma^2 t\right)] + \sigma t \\
&= -\exp(-\tfrac{1}{2}\sigma^2 t) \cdot \mathbb{E}^Y[W^Y(t) \cdot \exp\left(\sigma W^Y(t)\right)] + \sigma t \\
&= -\exp(-\tfrac{1}{2}\sigma^2 t) \cdot \tfrac{d}{d\sigma}\mathbb{E}^Y[\exp\left(\sigma W^Y(t)\right)] + \sigma t \\
&= -\exp(-\tfrac{1}{2}\sigma^2 t) \cdot \tfrac{d}{d\sigma}[\exp(\tfrac{1}{2}\sigma^2 t)] + \sigma t \\
&= 0.
\end{aligned}$$

Note that we have

$$\frac{d^2 X_Y(t)}{X_Y(t)^2} = \sigma^2 dt, \tag{2.13}$$

as well as

$$\frac{d^2 Y_X(t)}{Y_X(t)^2} = \sigma^2 dt, \tag{2.14}$$

and thus the volatility of $X_Y(t)$ is the same as the volatility of $Y_X(t)$. It does not matter which of the two assets, $X$ or $Y$, is chosen as a numeraire. For instance the volatility of the dollar/euro exchange rate is the same as the volatility of the euro/dollar exchange rate. This is true even when the volatility is stochastic.

Having the closed form expressions for the price $X_Y(T)$ from Equation (2.3) and for the price $Y_X(T)$ from Equation (2.8), we can determine the prices of digital options that pay off either $\mathbb{I}_A(\omega)$ units of an asset $Y$ at time $T$, or $\mathbb{I}_A(\omega)$ units of an asset $X$ at the same time. Let us consider events $A$ of the form

$$A = \{\omega \in \Omega : X_Y(T, \omega) \geq K\}$$

for a given constant $K$. $A$ is a set of scenarios where the terminal price of $X_Y(T)$ exceeds a level $K$. Let us determine the price of a contract $U$ that pays off

$$U(T) = \mathbb{I}_A(\omega) \cdot Y(T).$$

Since the price of this contract is a martingale under the $\mathbb{P}^Y$ measure, we have

$$U_Y(t) = \mathbb{E}^Y_t[\mathbb{I}_A(\omega)] = \mathbb{P}^Y_t(A).$$

The event

$$A = \{X_Y(T) \geq K\}$$

is equivalent to

$$X_Y(t) \cdot \exp\left(\sigma(W^Y(T) - W^Y(t)) - \tfrac{1}{2}\sigma^2(T-t)\right) \geq K,$$

or in other words

$$-\frac{W^Y(T) - W^Y(t)}{\sqrt{T-t}} \leq \frac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) - \tfrac{1}{2}\sigma\sqrt{T-t}.$$

Since $-\frac{W^Y(T)-W^Y(t)}{\sqrt{T-t}}$ has a normal distribution with zero mean and a unit variance $N(0,1)$ under the probability measure $\mathbb{P}^Y$ , the probability of the event $A$ is given by

$$\mathbb{P}_t^Y(A) = \mathbb{P}_t^Y(X_Y(T) \geq K) = N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) - \tfrac{1}{2}\sigma\sqrt{T-t}\right), \tag{2.15}$$

where $N(\cdot)$ is a cumulative distribution function of a standard normal variable

$$N(x) = \int_{-\infty}^{x} \tfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}\, dy.$$

We can determine the price of the digital option $V$ that pays off $\mathbb{I}_A(\omega)$ units of $X$ at time $T$ in a similar fashion. At time $T$ we have

$$V(T) = \mathbb{I}_A(\omega) \cdot X(T).$$

The price $V_X(t)$ is a $\mathbb{P}^X$ martingale, and thus

$$V_X(t) = \mathbb{E}_t^X[\mathbb{I}_A(\omega)] = \mathbb{P}_t^X(A).$$

The event

$$A = \{X_Y(T) \geq K\}$$

is equivalent to

$$X_Y(t) \cdot \exp\left(-\sigma(W^X(T) - W^X(t)) + \tfrac{1}{2}\sigma^2(T-t)\right) \geq K.$$

Here we used the fact that

$$\begin{aligned} X_Y(T) = \frac{1}{Y_X(T)} &= \frac{1}{Y_X(t) \cdot \exp\left(\sigma(W^X(T) - W^X(t)) - \frac{1}{2}\sigma^2(T-t)\right)} \\ &= X_Y(t) \cdot \exp\left(-\sigma(W^X(T) - W^X(t)) + \tfrac{1}{2}\sigma^2(T-t)\right). \end{aligned}$$

We need to express the price of $X_Y(T)$ in terms of the Brownian motion $W^X(t)$ in order to determine the probability of the event $A$ using the $\mathbb{P}^X$ measure. The event $A$ is equivalent to

$$\frac{W^X(T) - W^X(t)}{\sqrt{T-t}} \leq \frac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}.$$

Therefore

$$\mathbb{P}_t^X(A) = \mathbb{P}_t^X(X_Y(T) \geq K) = N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right). \tag{2.16}$$

**Remark 2.4** *It is interesting to note that when* $K = X_Y(0)$*, we have*

$$\mathbb{P}^Y(X_Y(T) \geq X_Y(0)) = N\left(-\tfrac{1}{2}\sigma\sqrt{T}\right) < \tfrac{1}{2},$$

*and*

$$\mathbb{P}^X(X_Y(T) \geq X_Y(0)) = N\left(\tfrac{1}{2}\sigma\sqrt{T}\right) > \tfrac{1}{2}.$$

*A delivery of a unit of* $Y$ *when the price* $X_Y$ *of the asset* $X$ *moves up requires less than a* $\frac{1}{2}$ *unit of* $Y$ *to start with. On the other hand, a delivery of a unit of* $X$ *on the same event requires more than a* $\frac{1}{2}$ *unit of an asset* $X$*. In this sense, the asset* $Y$ *is "cheaper" (it requires a smaller fraction of the underlying asset) to deliver than the asset* $X$ *on the up movement of the price* $X_Y$*.*

## 2.2 General European Contracts

A general European-type contract pays off either $f^Y(X_Y(T))$ units of an asset $Y$, or $f^X(Y_X(T))$ units of an asset $X$ at time $T$. In order that these two payoffs correspond to the same contract, we must have

$$f^Y(X_Y(T)) \cdot Y = f^X(Y_X(T)) \cdot X$$

or in other words,

$$f^Y(X_Y(T)) \cdot Y = f^X\left(\frac{1}{X_Y(T)}\right) \cdot X_Y(T) \cdot Y.$$

Therefore the two payoff functions $f^Y$ and $f^X$ are linked by the following symmetric relationship

$$\boxed{f^Y(x) = f^X\left(\tfrac{1}{x}\right) \cdot x,} \quad \text{or} \quad \boxed{f^X(x) = f^Y\left(\tfrac{1}{x}\right) \cdot x,} \tag{2.17}$$

which is valid for $0 < x < \infty$, meaning that neither the asset $X$ nor the asset $Y$ is worthless. Note that the payoff function depends on a choice of the reference asset. The formulas that link functions $f^Y$ and $f^X$ are known as a **perspective mapping.** A financial contract with a non-negative payoff function $f^Y(x)$ is known as an **option.** Note that $f^Y(x) \geq 0$ is equivalent to $f^X(x) \geq 0$, so the definition of the option does not depend on the choice of the reference asset. An option of this type is also known as a **plain vanilla option.** The perspective mapping also preserves convexity; $f^Y(x)$ is convex if and only if $f^X(x)$ is convex.

**Example 2.5 (The best asset and the worst asset)** *The simplest contract on two assets one can think of is the best of the two assets, or the worst of the two assets. The best of the two assets contract pays off* $\max(X(T), Y(T))$ *at time* $T$*; the worst of the two assets contract pays off* $\min(X(T), Y(T))$ *at time* $T$*. These contracts are completely symmetric since*

$$\max(X(T), Y(T)) = \max(Y(T), X(T)),$$

*and*

$$\min(X(T), Y(T)) = \min(Y(T), X(T)).$$

*When the best of the two assets contract is settled in the asset $Y$, the contract pays off* $\max\left(X_Y(T), 1\right)$ *units of* $Y$*. Similarly, when the best of the two assets contract is settled in the asset* $X$*, the contract pays off* $\max\left(Y_X(T), 1\right)$ *units of* $X$*. The payoff functions for the best of the two assets are thus given by*

$$f^Y(x) = \max(x, 1),$$

*and*

$$f^X(x) = f^Y(\tfrac{1}{x}) \cdot x = \max(\tfrac{1}{x}, 1) \cdot x = \max(1, x).$$

*Note that we have* $f^X(x) = f^Y(x)$*. Analogously, the payoff functions for the worst of the two assets are given by*

$$f^Y(x) = \min(x, 1),$$

$$f^X(x) = f^Y(\tfrac{1}{x}) \cdot x = \min(x, 1).$$

*Note that the payoff of the best asset contract can be re-expressed in the following form*

$$\max(X(T), Y(T)) = (X(T) - Y(T))^+ + Y(T) = (Y(T) - X(T))^+ + X(T),$$

*where* $x^+ = \max(x, 0)$*, leading us to contracts known as the call and the put options.*

The most typical traded contract that has the feature of paying the best asset is a **convertible bond.** One of the payments of the convertible bond is $\max(S(T), K \cdot B^T(T))$, so the holder of this contract can choose between the equity position in the asset $S$, and $K$ units of the bond $B^T$ at the expiration time $T$.

However, the logic of the financial markets is to allow for maximal leverage, and in this respect, the contract that delivers the best asset is not ideal as it ties down a portion of the capital of the investor that can be used otherwise. Instead, one can trade just the differences between the best asset and the asset itself, which requires significantly less capital. The contract on the difference of the best asset and the asset itself is known as a call option. Formally, a **European call option** $V^{EC}(X, K \cdot Y, T)$ is a contract that pays off

$$\boxed{(X(T) - K \cdot Y(T))^+,} \tag{2.18}$$

where $X$ and $Y$ are two assets. The constant $K$ is known as the **strike.** The relationship between the European call option and the contract that delivers the best asset is given by

$$\max(X(T), K \cdot Y(T)) = (X(T) - K \cdot Y(T))^+ + K \cdot Y(T).$$

In the contract that delivers we may rescale one of the assets by a factor of $K$ to achieve a better proportionality of the assets $X$ and $Y$. Note that the European call option is a combination of two digital options

$$\boxed{(X(T) - K \cdot Y(T))^+ = \mathbb{I}(X_Y(T) \geq K) \cdot X - K \cdot \mathbb{I}(X_Y(T) \geq K) \cdot Y.} \tag{2.19}$$

The first digital option pays off $\mathbb{I}(X_Y(T) \geq K)$ units of the asset $X$, the second digital option pays off $\mathbb{I}(X_Y(T) \geq K)$ units of the asset $Y$.

A closely related contract to a European call option is a **European put option** $V^{EP}(K \cdot Y, X, T)$ with a payoff

$$\boxed{(K \cdot Y(T) - X(T))^+.} \tag{2.20}$$

The put option is also related to the contract that delivers the best asset by

$$\max(X(T), K \cdot Y(T)) = (K \cdot Y(T) - X(T))^+ + X(T).$$

The only difference between the call and the put option is which of the two available assets is chosen to be subtracted from the payoff of the contract on the best asset. Since this choice is arbitrary, the call option on assets $X$ and $K \cdot Y$ is the same contract as a put option on assets $K \cdot Y$ and $X$. This relationship is known as the **put-call duality:**

$$\boxed{V^{EC}(X, K \cdot Y, T) = V^{EP}(K \cdot Y, X, T) = K \cdot V^{EP}(Y, \tfrac{X}{K}, T).} \tag{2.21}$$

Another simple relationship between European call and European put options is a **put-call parity.** Note that

$$X(T) - K \cdot Y(T) = (X(T) - K \cdot Y(T))^+ - (K \cdot Y(T) - X(T))^+, \tag{2.22}$$

where $X(T) - K \cdot Y(T)$ is a payoff of a forward contract $F(X, K \cdot Y, T)$. The relationship between the forward contract and the corresponding call and put options holds at all times $t \leq T$:

$$\boxed{F(X, K \cdot Y, T) = V^{EC}(X, K \cdot Y, T) - V^{EP}(X, K \cdot Y, T).} \tag{2.23}$$

A European call option payoff can be written in the following equivalent ways

$$(X(T) - K \cdot Y(T))^+ = (X_Y(T) - K)^+ \cdot Y(T) = (1 - K \cdot Y_X(T))^+ \cdot X(T). \tag{2.24}$$

When the European call option is settled in the asset $Y$, the payoff is given by

$$\boxed{(X_Y(T) - K)^+ \cdot Y} \tag{2.25}$$

which corresponds to a payoff function $f^Y(x) = (x - K)^+$. The holder receives $(X_Y(T) - K)^+$ units of $Y$ at time $T$. Similarly, the European call option settled in the asset $X$ has the payoff

$$\boxed{(1 - K \cdot Y_X(T))^+ \cdot X} \tag{2.26}$$

which corresponds to a payoff function $f^X(x) = (1 - K \cdot x)^+$. Note that $f^X(\frac{1}{x}) \cdot x = (1 - K \cdot \frac{1}{x})^+ \cdot x = (x - K)^+ = f^Y(x)$. The holder receives $(1 - K \cdot Y_X(T))^+$ units of $X$ at time $T$.

European-type contracts can always be expressed in terms of two no-arbitrage assets.

**Remark 2.6 (European option as a contract on two no-arbitrage assets)**

*A European option can always be expressed as a contract on two no-arbitrage assets. The payoff of a European option is defined as $f^Y(X_Y(T))$ units of an asset $Y$, or $f^X(Y_X(T))$ units of an asset $X$ at time $T$ for general assets with positive price $X$ and $Y$. When $X$ or $Y$ is an arbitrage asset, such as the dollar \$, we can substitute an arbitrage asset $X$ (or $Y$) with a corresponding no-arbitrage asset $U$ or $V$ that delivers a unit of an asset $X$ or an asset $Y$ at time $T$. In particular, we have*

$$U(T) = X(T), \quad V(T) = Y(T).$$

*Thus the European option payoff can be re-expressed as $f^Y(U_V(T)) = f^V(U_V(T))$ units of an asset $V$, or $f^X(V_U(T)) = f^U(V_U(T))$ units of an asset $U$ at time $T$ for two no-arbitrage assets $U$ and $V$. This substitution is not possible when there is no fixed delivery of the option payoff such as in the case of American options.*

**Example 2.7 (European call option in different markets)**

**Stock option** *When the asset $X$ is a stock $S$, and the asset $Y$ is a dollar \$, we have a European stock option*

$$(S(T) - K \cdot \$(T))^+. \tag{2.27}$$

*Note that the existing literature typically omits the fact that the strike is in fact multiplied by the dollar \$. This notation means that the holder of the option has the right to increase his position in the stock $S$ by one unit, and decrease his position in the dollar \$ by $K$ units at time $T$.*

*Should the contract be settled in dollars, one can write the payoff as*

$$(S_\$(T) - K)^+ \cdot \$(T). \tag{2.28}$$

*The holder receives $(S_\$(T) - K)^+$ units of the dollar \$ at time $T$. As noted earlier, the European option on a stock may also be settled in terms of a bond $B^T$, a contract that delivers 1 \$ at time $T$, so that $B^T(T) = 1\$(T)$. In this case, the payoff of the option may be written as*

$$(S(T) - K \cdot B^T(T))^+. \tag{2.29}$$

*This fact is useful in the pricing of this option. In contrast to the dollar, the bond does not create arbitrage opportunities in time. Therefore it can be used* *as a natural reference asset for pricing this option. The option can be settled entirely in the bond*

$$(S_{B^T}(T) - K)^+ \cdot B^T(T), \tag{2.30}$$

*or in the stock*

$$\left(1 - K \cdot B_S^T(T)\right)^+ \cdot S(T). \tag{2.31}$$

**Exchange option** *When the asset $X$ is a stock $S^1$, and the asset $Y$ is another stock $S^2$, the corresponding European call option*

$$(S^1(T) - K \cdot S^2(T))^+ \tag{2.32}$$

*is known as an* **exchange option.** *The natural reference asset for pricing this option is either the stock $S^1$, or the stock $S^2$. Adding another reference asset, such as a dollar \$, for pricing this option would only increase the dimensionality of the problem.*

**Currency option** *When the asset $X$ is a euro €, and the asset $Y$ is a dollar \$ (or any other currencies), we have a European currency option*

$$(€(T) - K \cdot \$(T))^+. \tag{2.33}$$

*A European currency option can be settled in the dollar or in the euro only*

$$(€(T) - K \cdot \$(T))^+ = (€_\$(T) - K)^+ \cdot \$(T) = (1 - K \cdot \$_€(T))^+ \cdot €(T).$$

*In order to express the payoff in terms of no-arbitrage assets only, we can take a foreign bond $B^{€,T}$ that delivers a unit of a foreign currency € at time $T$, and a domestic bond $B^T$ that delivers a unit of a domestic currency \$ at time $T$. The payoff of the currency option is equivalent to*

$$(€(T) - K \cdot \$(T))^+ = (B^{€,T}(T) - K \cdot B^T(T))^+.$$

**Caplet** *A caplet is an option on a LIBOR that pays off*

$$(L(T,T) - K)^+ \cdot \$(T + \delta). \tag{2.34}$$

*The LIBOR $L(T,T)$ is observed at time $T$, but the contract is settled at a later time $T+\delta$ in a dollar \$. Here it is not entirely obvious what the corresponding assets $X$ and $Y$ should be. But from the definition of the LIBOR*

$$L(T,T) = \frac{B_\$^T(T) - B_\$^{T+\delta}(T)}{\delta B_\$^{T+\delta}(T)} = [B^T - B^{T+\delta}]_{\delta B^{T+\delta}}(T),$$

*and using the fact that $B^{T+\delta}(T+\delta) = \$(T+\delta)$, we can rewrite the payoff as*

$$\begin{aligned}
&(L(T,T) - K)^+ \cdot \$(T+\delta) \\
&\quad = \left([B^T - B^{T+\delta}]_{\delta B^{T+\delta}}(T) - K\right)^+ \cdot B^{T+\delta}(T+\delta) \\
&\qquad = \tfrac{1}{\delta} \cdot \left([B^T - B^{T+\delta}](T) - K \cdot \delta B^{T+\delta}(T+\delta)\right)^+.
\end{aligned}$$

*Thus the asset $X$ is a combination of two bonds $[B^T - B^{T+\delta}]$, and the asset $Y$ is $\delta B^{T+\delta}$.*

**Remark 2.8 (European call option price)** *We have already seen that a European call option is just a combination of two digital options*

$$(X(T) - K \cdot Y(T))^+ = \mathbb{I}(X_Y(T) \geq K) \cdot X(T) - K \cdot \mathbb{I}(X_Y(T) \geq K) \cdot Y(T). \quad (2.35)$$

*The first digital option costs $\mathbb{P}_t^X(X_Y(T) \geq K)$ units of the asset $X$, the second digital option costs $\mathbb{P}^Y(X_Y(T) \geq K)$ units of the asset $Y$. Therefore we have the following result:*

***Theorem 2.9 (Black–Scholes formula)*** *The price of a European option contract $V^{EC}(X, K \cdot Y, T)$ with the payoff $(X(T) - K \cdot Y(T))^+$ is given by*

$$\boxed{V^{EC}(X, K \cdot Y, T) = \mathbb{P}_t^X(X_Y(T) \geq K) \cdot X - K\mathbb{P}_t^Y(X_Y(T) \geq K) \cdot Y.} \quad (2.36)$$

*Recall from the previous section (Equations (2.16) and (2.15)) that for the geometric Brownian motion model, we have*

$$\mathbb{P}_t^X(X_Y(T) \geq K) = N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right),$$

*and*

$$\mathbb{P}_t^Y(X_Y(T) \geq K) = N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) - \tfrac{1}{2}\sigma\sqrt{T-t}\right).$$

*Thus in the geometric Brownian motion model, the Black–Scholes formula simplifies to*

$$\boxed{V^{EC}(X, K \cdot Y, T) = [N(d_+)] \cdot X(t) + [-K \cdot N(d_-)] \cdot Y(t),} \quad (2.37)$$

*where*

$$\boxed{d_\pm = \tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{1}{K} \cdot X_Y(t)\right) \pm \tfrac{1}{2}\sigma\sqrt{T-t}.} \quad (2.38)$$

**Remark 2.10 (Money as a reference asset)** *We have seen that European-type contracts can be expressed in terms of two no-arbitrage assets $X$ and $Y$ which also serve as natural reference assets for pricing a given European option $V$. Thus for pricing a general European contract, one first determines the price $V_Y(t)$ or $V_X(t)$ in terms of the no-arbitrage assets $Y$ or $X$. The dollar price $V_\$(t)$ follows immediately from the change of numeraire formula*

$$V_\$(t) = V_Y(t) \cdot Y_\$(t) = V_X(t) \cdot X_\$(t).$$

*Let us illustrate how to compute the dollar price of a European call option on a stock and a dollar with a payoff*

$$V^{EC}(T) = (S(T) - K \cdot \$(T))^+.$$

*Since a dollar does not have a martingale measure $\mathbb{P}^\$$, we have to compute the price of the European call option using the First Fundamental Theorem of Asset Pricing either in terms of a stock $S$, or a bond $B^T$. This leads to the Black–Scholes formula (2.37), which takes the following form:*

$$V^{EC}(t) = [N(d_+)] \cdot S(t) + [-K \cdot N(d_-)] \cdot B^T(t),$$

*where*

$$d_{\pm} = \tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{1}{K} \cdot S_{B^T}(t)\right) \pm \tfrac{1}{2}\sigma\sqrt{T-t}.$$

*We can rewrite the Black–Scholes formula in terms of prices with respect to a bond $B^T$ as*

$$V^{EC}_{B^T}(t) = N\left(d_+\right) \cdot S_{B^T}(t) - K \cdot N\left(d_-\right).$$

*Multiplying the above equation by the dollar price of the bond $B^T_\$(t)$ and using the change of numeraire formula, we obtain the dollar price of the European call option*

$$\begin{aligned} V^{EC}_\$(t) &= V^{EC}_{B^T}(t) \cdot B^T_\$(t) \\ &= N\left(d_+\right) \cdot S_{B^T}(t) \cdot B^T_\$(t) - K \cdot N\left(d_-\right) \cdot B^T_\$(t) \\ &= N\left(d_+\right) \cdot S_\$(t) - K \cdot N\left(d_-\right) \cdot B^T_\$(t). \end{aligned}$$

*The formula for $d_{\pm}$ can also be expressed in terms of dollar prices as*

$$d_{\pm} = \tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{1}{K} \cdot S_\$(t) \cdot \$_{B^T}(t)\right) \pm \tfrac{1}{2}\sigma\sqrt{T-t}.$$

*If we further assume a deterministic term structure evolution with a constant interest rate $r$,*

$$B^T(t) = e^{-r(T-t)} \cdot \$(t),$$

*the above relationships simplify to*

$$V^{EC}_\$(t) = S_\$(t) \cdot N\left(d_+\right) - K \cdot e^{-r(T-t)} \cdot N\left(d_-\right), \tag{2.39}$$

*with*

$$d_{\pm} = \tfrac{1}{\sigma\sqrt{T-t}} \cdot \left[\log\left(\tfrac{1}{K} \cdot S_\$(t)\right) + \left(r \pm \tfrac{1}{2}\sigma^2\right)(T-t)\right]. \tag{2.40}$$

*This is the Black–Scholes formula expressed in terms of the dollar prices. Note that we had to assume a deterministic interest rate $r$ in order to simplify the Black–Scholes formula (2.37) that applies also to stochastic interest rates.*

Table 2.1 summarizes payoffs of various contracts. Options with the power and the logarithmic payoff do not appear directly on the market, but they are related to barrier and lookback options as we will see in the following text. Note that the payoff function $f^Y(x)$ that corresponds to $Y$ being chosen as a reference asset may have a different form than the payoff function $f^X(x)$ that corresponds to $X$ being chosen as a reference asset. But the two payoff functions $f^Y$ and $f^X$ represent the same contract. One can think of switching roles of the assets $X$ and $Y$, in which case we would get a new contract with a payoff $f^X(X_Y(T))$ units of $Y$. This is a **dual contract** to the original contract that pays off $f^Y(X_Y(T))$ units of $Y$. When we know the price of an original contract, we also know the price of the dual contract by switching the roles of $X$ and $Y$.

We have already seen that the call option with a payoff $f^Y(x) = (x-K)^+$ is a dual contract to the put option with a payoff $f^X(x) = (1-K\cdot x)^+$. The contract that pays off the best asset $\max(X(T), Y(T))$ is dual to itself as $f^Y(x) = f^X(x) = \max(x, 1)$. The role of $X$ and $Y$ can be switched and it does not change the contract as

Table 2.1: Contracts on two assets.

| Contract | Payoff | $f^Y(x)$ | $f^X(x)$ |
|---|---|---|---|
| Digital | $\mathbb{I}_A(X_Y(T)) \cdot Y(T)$ | $\mathbb{I}_A(x)$ | $\mathbb{I}_A(\frac{1}{x}) \cdot x$ |
| Best Asset | $\max(X(T), K \cdot Y(T))$ | $\max(x, K)$ | $\max(K \cdot x, 1)$ |
| Worst Asset | $\min(X(T), K \cdot Y(T))$ | $\min(x, K)$ | $\min(K \cdot x, 1)$ |
| Call | $(X(T) - K \cdot Y(T))^+$ | $(x - K)^+$ | $(1 - K \cdot x)^+$ |
| Put | $(K \cdot Y(T) - X(T))^+$ | $(K - x)^+$ | $(K \cdot x - 1)^+$ |
| Forward | $X(T) - K \cdot Y(T)$ | $x - K$ | $1 - K \cdot x$ |
| Power | $[X_Y(T)]^\alpha \cdot Y(T)$ | $x^\alpha$ | $x^{1-\alpha}$ |
| Logarithm | $\log(X_Y(T)) \cdot Y(T)$ | $\log(x)$ | $-x \cdot \log(x)$ |

$\max(X(T), Y(T)) = \max(Y(T), X(T))$. Similarly, the worst asset $\min(X(T), Y(T))$ is also dual to itself. The following example illustrates the concept of the dual contracts of the power options.

**Example 2.11 (Dual contracts of power options) A power option** $R^{(\alpha)}$ *pays off*

$$R^{(\alpha)}(T) = [X_Y(T)]^\alpha \cdot Y(T).$$

*Power options are useful in pricing barrier and lookback options. The dual contract switches the roles of the assets $X$ and $Y$; it pays off*

$$[Y_X(T)]^\alpha \cdot X(T).$$

*This can be rewritten as*

$$[Y_X(T)]^\alpha \cdot X(T) = [X_Y(T)]^{-\alpha} \cdot X_Y(T) \cdot Y_T = [X_Y(T)]^{1-\alpha} \cdot Y(T).$$

*Thus the payoff function $x^\alpha$ has a dual payoff function $x^{1-\alpha}$.*

*Note that when $\alpha = 0$, the corresponding power option $R^{(0)}$ coincides with the asset $Y$. When $\alpha = 1$, the corresponding power option $R^{(1)}$ is the asset $X$. This suggests that for $\alpha \in (0,1)$, the resulting power option $R^{(\alpha)}$ creates an asset that is a combination of the assets $X$ and $Y$. When $\alpha > 1$, the power option $R^{(\alpha)}$ leverages the position in the asset $X$. Similarly, when $\alpha < 0$, the power option $R^{(\alpha)}$ leverages the position in the asset $Y$. This is supported by the following argument. When $X$ is comparable to $Y$ in terms of price, meaning $X_Y(T) \approx 1$, we have*

$$[X_Y(T)]^\alpha \approx 1 + \alpha(X_Y(T) - 1) = (1 - \alpha) + \alpha X_Y(T)$$

*according to the first order Taylor expansion around 1. Rewriting this relationship in terms of the assets, we have*

$$R^{(\alpha)}(T) \approx (1 - \alpha) \cdot Y(T) + \alpha \cdot X(T). \tag{2.41}$$

*Clearly, when $\alpha \in (0,1)$, the power option is approximately a linear combination of the assets $X$ and $Y$ with positive weights. In particular, the power option $R^{(1/2)}$ corresponding to a* **square root asset** *is approximately just an average of the two assets $X$ and $Y$. The square root asset is the only power option that is dual to itself, meaning that one can swap the roles of the assets $X$ and $Y$ without changing the contract.*

*When $\alpha > 1$, the power option corresponds to having a long position in the asset $X$, and a short position in the asset $Y$. When $\alpha < 0$, the situation is reversed, and the power option represents a long position in the asset $Y$, and a short position in the asset $X$. The hedging position $\Delta^X(t)$ of the power option indeed has the same sign as $\alpha$, and the hedging position $\Delta^Y(t)$ has the same sign as $1-\alpha$. This confirms that the approximation from (2.41) is reasonable.*

## 2.3 Price as an Expectation

For pricing a general European claim $V$, we can use either reference asset $Y$ or $X$ in order to determine the price of $V$:

$$V = V_Y(t) \cdot Y = V_X(t) \cdot X.$$

In Markovian models, which include geometric Brownian motion, we can express these prices in terms of the price functions $u^Y$ and $u^X$ defined as

$$V_Y(t) = u^Y(t, X_Y(t)), \qquad V_X(t) = u^X(t, Y_X(t)).$$

The functions $u^Y$ and $u^X$ are linked by

$$u^Y(t, X_Y(t)) = u^X(t, Y_X(t)) \cdot X_Y(t),$$

or by

$$u^X(t, Y_X(t)) = u^Y(t, X_Y(t)) \cdot Y_X(t).$$

Therefore we have the following symmetric relationship

$$\boxed{u^Y(t,x) = u^X(t, \tfrac{1}{x}) \cdot x,} \quad \text{or} \quad \boxed{u^X(t,x) = u^Y(t, \tfrac{1}{x}) \cdot x} \tag{2.42}$$

for $0 < x < \infty$, which is known as a **perspective mapping.** Note that we have $u^Y(T,x) = f^Y(x)$, and $u^X(T,x) = f^X(x)$, so the terminal price of the contract agrees with the payoff function. We have already seen that $f^Y(x) = f^X(\frac{1}{x}) \cdot x$, and $f^X(x) = f^Y(\frac{1}{x}) \cdot x$, which is just a special case of the relationship between the prices $u^Y(t,x)$, and $u^X(t,x)$.

Recall that the payoff of European options can always be written in terms of two no-arbitrage assets: $U$ that agrees to deliver an asset $X$ at time $T$, and $V$ that agrees to deliver an asset $Y$ at time $T$. It is easy to see that the contract to deliver a no-arbitrage asset is the asset itself, so the substitution of the underlying for a no-arbitrage asset makes sense only when one of the underlying assets is an arbitrage asset, such as in the case of the dollar or other currencies. Therefore without loss of generality, we may assume that the European option is settled in terms of two no-arbitrage assets.

Given that European options can be expressed in terms of two no-arbitrage assets, the First Fundamental Theorem of Asset Pricing states that the price of $V$ in terms of the reference asset $Y$ is a $\mathbb{P}^Y$ martingale, and the price of $V$ in terms of the reference asset $X$ is a $\mathbb{P}^X$ martingale. This gives us a stochastic representation of the contingent claim price

$$V_Y(t) = \mathbb{E}_t^Y\left[V_Y(T)\right] = \mathbb{E}_t^Y\left[f^Y\left(X_Y(T)\right)\right], \tag{2.43}$$

when the asset $Y$ is used as a numeraire, and

$$V_X(t) = \mathbb{E}_t^X\left[V_X(T)\right] = \mathbb{E}_t^X\left[f^X\left(Y_X(T)\right)\right], \tag{2.44}$$

when the asset $X$ is used as a numeraire. The number of units $\mathbb{E}_t^Y\left[f^Y\left(X_Y(T)\right)\right]$ of $Y$ that is needed in order to acquire the contract $V$ is the price of the contract in terms of the reference asset $Y$. Similarly, the number of units $\mathbb{E}_t^X\left[f^X\left(Y_X(T)\right)\right]$ of $X$ that is needed in order to acquire the contract $V$ is the price of the contract in terms of the reference asset $X$.

When the prices $V_Y(t)$ and $V_X(t)$ are Markovian in the prices $X_Y(t)$ and $Y_X(t)$, the price functions $u^Y$ and $u^X$ have the following representations

$$u^Y(t,x) = \mathbb{E}^Y\left[f^Y\left(X_Y(t)\right)|X_Y(t) = x\right], \tag{2.45}$$

and

$$u^X(t,x) = \mathbb{E}^X\left[f^X\left(Y_X(t)\right)|Y_X(t) = x\right]. \tag{2.46}$$

The price processes $V_Y$ and $V_X$ are indeed Markovian in the geometric Brownian motion model.

When the price processes $X_Y(t)$ and $Y_X(t)$ are geometric Brownian motions, we can compute the price functions $u^Y$ and $u^X$ directly by computing the conditional expected value. For the function $u$ we have

$$\begin{aligned} u^Y(t,x) &= \mathbb{E}^Y\left[f^Y\left(X_Y(T)\right)|X_Y(t) = x\right] \qquad (2.47)\\ &= \mathbb{E}^Y\left[f^Y\left(X_Y(t)\cdot\exp\left(\sigma W^Y(T-t) - \tfrac{1}{2}\sigma^2(T-t)\right)\right)|X_Y(t) = x\right]\\ &= \mathbb{E}^Y\left[f^Y\left(x\cdot\exp\left(\sigma W^Y(T-t) - \tfrac{1}{2}\sigma^2(T-t)\right)\right)|X_Y(t) = x\right]\\ &= \int_{-\infty}^{\infty} f^Y\left(x\cdot\exp\left(\sigma y\sqrt{T-t} - \tfrac{1}{2}\sigma^2(T-t)\right)\right)\cdot\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{y^2}{2}\right)dy. \end{aligned}$$

We have used the fact that

$$X_Y(T) = X_Y(t)\cdot\exp\left(\sigma W^Y(T-t) - \tfrac{1}{2}\sigma^2(T-t)\right),$$

and that $\frac{W^Y(T-t)}{\sqrt{T-t}}$ has a normal distribution $N(0,1)$.

Similarly, the function $u^X$ can be determined from the following formula:

$$\begin{aligned} u^X(t,x) &= \mathbb{E}^X\left[f^X\left(Y_X(T)\right)|Y_X(t)=x\right] \qquad (2.48) \\ &= \mathbb{E}^X\left[f^X\left(Y_X(t)\cdot\exp\left(\sigma W^X(T-t)-\tfrac{1}{2}\sigma^2(T-t)\right)\right)|Y_X(t)=x\right] \\ &= \mathbb{E}^X\left[f^X\left(x\cdot\exp\left(\sigma W^X(T-t)-\tfrac{1}{2}\sigma^2(T-t)\right)\right)|Y_X(t)=x\right] \\ &= \int_{-\infty}^{\infty} f^X\left(x\cdot\exp\left(\sigma y\sqrt{T-t}-\tfrac{1}{2}\sigma^2(T-t)\right)\right)\cdot\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{y^2}{2}\right)dy. \end{aligned}$$

**Example 2.12** *Consider a European call option with a payoff $(X(T)-K\cdot Y(T))^+$. When $Y$ is chosen as a reference asset, the payoff function is given by $f^Y(x)=(x-K)^+$. Thus we have*

$$\begin{aligned} u^Y(t,x) &= \int_{-\infty}^{\infty} f^Y\left(x\cdot\exp\left(\sigma y\sqrt{T-t}-\tfrac{1}{2}\sigma^2(T-t)\right)\right)\cdot\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{y^2}{2}\right)dy \\ &= \int_{-\infty}^{\infty}\left(x\cdot\exp\left(\sigma y\sqrt{T-t}-\tfrac{1}{2}\sigma^2(T-t)\right)-K\right)^+ \\ &\qquad\qquad \times\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{y^2}{2}\right)dy \\ &= x\cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}}\cdot\left[\log\left(\tfrac{x}{K}\right)+\tfrac{1}{2}\sigma^2(T-t)\right]\right) \\ &\qquad -K\cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}}\cdot\left[\log\left(\tfrac{x}{K}\right)-\tfrac{1}{2}\sigma^2(T-t)\right]\right). \end{aligned}$$

*When $X$ is chosen as a reference asset, the payoff function is given by $f^X(x)=f^Y(\frac{1}{x})\cdot x=(1-K\cdot x)^+$, and thus we have*

$$\begin{aligned} u^X(t,x) &= \int_{-\infty}^{\infty} f^X\left(x\cdot\exp\left(\sigma y\sqrt{T-t}-\tfrac{1}{2}\sigma^2(T-t)\right)\right)\cdot\tfrac{1}{\sqrt{2\pi}}\cdot\exp\left(-\tfrac{y^2}{2}\right)dy \\ &= \int_{-\infty}^{\infty}\left(1-K\cdot x\cdot\exp\left(\sigma y\sqrt{T-t}-\tfrac{1}{2}\sigma^2(T-t)\right)\right)^+ \\ &\qquad\qquad \times\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{y^2}{2}\right)dy \\ &= N\left(\tfrac{1}{\sigma\sqrt{T-t}}\cdot\left[\log\left(\tfrac{1}{K\cdot x}\right)+\tfrac{1}{2}\sigma^2(T-t)\right]\right) \\ &\qquad -K\cdot x\cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}}\cdot\left[\log\left(\tfrac{1}{K\cdot x}\right)-\tfrac{1}{2}\sigma^2(T-t)\right]\right). \end{aligned}$$

*The reader may check that the price functions $u^Y$ and $u^X$ indeed satisfy $u^X(t,x)=u^Y(t,\frac{1}{x})\cdot x$.*

## 2.4 Connections with Partial Differential Equations

Let us assume that the price $X_Y(t)$ follows the geometric Brownian motion model

$$dX_Y(t)=\sigma X_Y(t)dW^Y(t).$$

We point out in this section that the price functions $u^Y$ and $u^X$ satisfy a certain partial differential equation.

**Theorem 2.13** *The price function* $u^Y(t,x) = \mathbb{E}^Y\left[f^Y\left(X_Y(T)\right)|X_Y(t) = x\right]$ *satisfies the partial differential equation*

$$\boxed{u_t^Y(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^Y(t,x) = 0} \tag{2.49}$$

*with the terminal condition*

$$u^Y(T,x) = f^Y(x), \tag{2.50}$$

*and the boundary condition*

$$u^Y(t,0) = f^Y(0). \tag{2.51}$$

*The price function* $u^X(t,x) = \mathbb{E}^X\left[f^X\left(Y_X(T)\right)|Y_X(t) = x\right]$ *satisfies the partial differential equation*

$$\boxed{u_t^X(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^X(t,x) = 0} \tag{2.52}$$

*with the terminal condition*

$$u^X(T,x) = f^X(x), \tag{2.53}$$

*and the boundary condition*

$$u^X(t,0) = f^X(0). \tag{2.54}$$

**Remark 2.14** *The partial differential equations (2.49) and (2.52) are also known as the* **Black–Scholes partial differential equations.**

Proof: Let

$$u^Y(t,x) = \mathbb{E}^Y\left[f^Y\left(X_Y(T)\right)|X_Y(t) = x\right]$$

be the price of the contract with respect to the reference asset $Y$. According to Ito's formula, the option price has the following dynamics:

$$\begin{aligned} du^Y\left(t,X_Y(t)\right) &= u_t^Y\left(t,X_Y(t)\right)dt + u_x^Y\left(t,X_Y(t)\right)dX_Y(t) \\ &\quad + \tfrac{1}{2}u_{xx}^Y\left(t,X_Y(t)\right)d^2X_Y(t) \\ &= \left[u_t^Y\left(t,X_Y(t)\right) + \tfrac{1}{2}\sigma^2 X_Y(t)^2 u_{xx}^Y\left(t,X_Y(t)\right)\right]dt \\ &\quad + u_x^Y\left(t,X_Y(t)\right)dX_Y(t). \end{aligned}$$

Since $u^Y\left(t,X_Y(t)\right)$ is a $\mathbb{P}^Y$ martingale, the $dt$ term of this equation must vanish for all values of $X_Y(t)$, and thus the following partial differential equation for the price of the option must hold:

$$u_t^Y(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^Y(t,x) = 0,$$

with the terminal condition

$$u^Y(T,x) = f^Y(x).$$

The case when $x = 0$ represents the situation when $X_Y(t) = 0$ (the asset $X$ becomes worthless), and thus the value of $X_Y(T)$ will also be zero. Thus the payoff of the option will be $f^Y(0)$ units of an asset $Y$ at time $T$. Thus the value of the contract at time $t$ is $u^Y(t,0) = f^Y(0)$.

We can apply the same technique using the no-arbitrage asset $X$ as a numeraire when the payoff of the contract is $f^X\left(Y_X(T)\right)$ units of an asset $X$, leading to the partial differential equation (2.52).

**Remark 2.15 (The prices of $X$ and $Y$ satisfy the Black–Scholes partial differential equation)**

*Partial differential equation (2.49)*

$$u_t^Y(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^Y(t,x) = 0$$

*has two trivial solutions that correspond to the payoff functions $f^Y(x) = 1$ and $f^Y(x) = x$. When the payoff function is $f^Y(x) = 1$, the price function $u^Y(t,x)$ is also identically equal to one, and the partial differential equation (2.49) is satisfied. In financial terms, the payoff function $f^Y(x) = 1$ corresponds to the delivery of a unit of an asset $Y$ at time $T$. This is a contract to deliver an asset $Y$, and its price at any given time $t \leq T$ is one unit of an asset $Y$. Thus we have $u^Y(t,x) = 1$ as a solution. When the payoff function is $f^Y(x) = x$, the price function $u^Y(t,x)$ is also equal to $x$, and the partial differential equation (2.49) is satisfied. In financial terms, the payoff function $f^Y(x) = x$ corresponds to the delivery of a unit of an asset $X$ at time $T$ (it is $X_Y(t)$ units of an asset $Y$). This is a contract to deliver an asset $X$ at time $T$ and its price at any given time $t \leq T$ is one unit of an asset $X$. Thus we have $u^Y(t,x) = x$ as a solution.*

*Similarly, the partial differential equation (2.52)*

$$u_t^X(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^X(t,x) = 0$$

*also has two trivial solutions that correspond to the payoff functions $f^X(x) = 1$ and $f^X(x) = x$. In financial terms, the payoff function $f^X(x) = 1$ corresponds to the delivery of an asset $X$, the payoff function $f^X(x) = x$ corresponds to the delivery of an asset $Y$.*

**Example 2.16** *The European option $V$ with a payoff $V(T) = (X(T) - K \cdot Y(T))^+$ has an associated payoff function $f^Y(x) = (x-K)^+$, or $f^X(x) = (1 - K \cdot x)^+$. The $V_Y(t) = u^Y(t, X_Y(t))$ price satisfies the partial differential equation (2.49) and the $V_X(t) = u^X(t, Y_X(t))$ price satisfies the partial differential equation (2.52). When the asset $X$ becomes worthless, or in other words when $X_Y(t) = 0$, the option will also be worthless as $f^Y(0) = 0$, giving us the boundary condition $u^Y(t,0) = 0$. The asset $X$ will not serve as a reference asset in this case, but the price of the contract can still be expressed in terms of the asset $Y$. On the other hand, when the asset $Y$ becomes worthless, $Y_X(t) = 0$, the option will pay off a unit of the asset $X$, which corresponds to $f^X(0) = 1$. This gives the boundary condition $u^X(t,0) = 1$, the asset $X$ can still be used as a numeraire. Note that the boundary conditions when one of the prices is zero do not have a perspective mapping counterpart as the perspective mapping applies only to cases when the prices are positive. When one of the assets becomes worthless, it still makes sense to use the remaining asset with a positive price as a numeraire, but the pricing problem cannot be solved using the worthless asset.*

## 2.5 Money as a Reference Asset

It is also possible to write the Black–Scholes partial differential equation in terms of the dollar \$ as a reference asset. Let $X$ be a stock $S$, and $Y$ be a bond $B^T$. A contract $V$ that pays off $f^T(S_{B^T}(T))$ units of a bond $B^T$ at time $T$ can equivalently be expressed as

$$V(T) = f^T(S_{B^T}(T)) \cdot B^T(T) = f^\$(S_\$(T)) \cdot \$(T),$$

a contract that pays off $f^\$(S_\$(T))$ units of a dollar \$ at time $T$. The payoff functions in terms of a bond $B^T$ and a dollar \$ agree: $f^T(x) = f^\$(x)$. The contract $V$ at time $t$ can be also expressed in the following equivalent ways:

$$V(t) = V_{B^T}(t) \cdot B^T(t) = V_\$(t) \cdot \$(t) = V_S(t) \cdot S(t).$$

Let $u^T(t, S_{B^T}(t)) = V_{B^T}(t)$ be the price of the contract $V$ in terms of a bond $B^T$, and let

$$v^\$(t, S_\$(t)) = V_\$(t) \tag{2.55}$$

be the price of the contract $V$ in terms of a dollar \$. We are using a different letter $v$ for the dollar price in order to distinguish it from the prices $u$ that use only no-arbitrage assets. Let us also assume $B^T_\$(t) = e^{-r(T-t)}$. Since

$$V(t) = u^T(t, S_{B^T}(t)) \cdot B^T(t) = v^\$(t, S_\$(t)) \cdot \$(t),$$

we get the following relationship between $u^T$ and $v^\$$:

$$\boxed{v^\$(t, x) = e^{-r(T-t)} \cdot u^T(t, e^{r(T-t)}x),} \tag{2.56}$$

and

$$\boxed{u^T(t, x) = e^{r(T-t)} \cdot v^\$(t, e^{-r(T-t)}x).} \tag{2.57}$$

We have seen that the price function $u^T$ satisfies the partial differential equation

$$u^T_t(t, x) + \tfrac{1}{2}\sigma^2 x^2 u^T_{xx}(t, x) = 0.$$

Using the relationship between the functions $u^T$ and $v^\$$, we find that

$$\begin{aligned} u^T_t(t, x) = e^{r(T-t)} \cdot \Bigg( & -rv^\$(t, e^{-r(T-t)}x) + v^\$_t(t, e^{-r(T-t)}x) \\ & + r\left(e^{-r(T-t)}x\right) v^\$_x(t, e^{-r(T-t)}x) \Bigg), \end{aligned}$$

and

$$u^T_{xx}(t, x) = e^{r(T-t)} \left(e^{-r(T-t)}\right)^2 v^\$_{xx}(t, e^{-r(T-t)}x).$$

After substitution of $x$ for $e^{-r(T-t)}x$, we conclude that $v^\$$ satisfies the following partial differential equation

$$\boxed{-rv^\$(t, x) + v^\$_t(t, x) + rxv^\$_x(t, x) + \tfrac{1}{2}\sigma^2 x^2 v^\$_{xx}(t, x) = 0.} \tag{2.58}$$

The terminal condition is given by

$$v^{\$}(T,x) = f^T(x) = f^{\$}(x), \tag{2.59}$$

and the boundary condition is

$$v^{\$}(t,0) = e^{-r(T-t)} \cdot u^T(t,0) = e^{-r(T-t)} \cdot f^T(0). \tag{2.60}$$

The Black–Scholes partial differential equation in the form of (2.58) is widely used since it directly determines the price of a contract in terms of a dollar. **However, the partial differential equation (2.58) has two limitations. First, it applies only when the interest rate $r$ is deterministic. Second, its form is more complicated than the Black–Scholes partial differential equation (2.49) obtained for two no-arbitrage assets $S$ and $B^T$.** The pricing of European options is still relatively straightforward, so the advantage of using no-arbitrage assets for pricing is small. Therefore using no-arbitrage assets in pricing is more important for complex financial products, such as exotic options.

**Example 2.17** *We have seen that the price of the European call option with a payoff $(S(T) - K \cdot B^T(T))^+$ is given by*

$$\begin{aligned} u^T(t,x) = x \cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \left[\log\left(\tfrac{x}{K}\right) + \tfrac{1}{2}\sigma^2(T-t)\right]\right) \\ - K \cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \left[\log\left(\tfrac{x}{K}\right) - \tfrac{1}{2}\sigma^2(T-t)\right]\right). \end{aligned} \tag{2.61}$$

*Using the relationship $v^{\$}(t,x) = e^{-r(T-t)} \cdot u^T(t, e^{r(T-t)}x)$, we can express the dollar price of the option as*

$$\begin{aligned} v^{\$}(t,x) = x \cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \left[\log\left(\tfrac{x}{K}\right) + (r + \tfrac{1}{2}\sigma^2)(T-t)\right]\right) \\ - Ke^{-r(T-t)} \cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \left[\log\left(\tfrac{x}{K}\right) + (r - \tfrac{1}{2}\sigma^2)(T-t)\right]\right). \end{aligned} \tag{2.62}$$

*This is the best-known form of the Black–Scholes formula. One can verify that $v^{\$}(t,x)$ from (2.62) satisfies the Black–Scholes partial differential equation (2.58).*

Similarly, we can define the price function $v^S$ in terms of \$ and $S$ as a reference asset by

$$V(t) = v^S(t, \$_S(t)) \cdot S(t). \tag{2.63}$$

The relationship between $V^S$ and the price function $u^S$ defined as

$$V(t) = u^S(t, B_S^T(t)) \cdot S(t) \tag{2.64}$$

is given by

$$\boxed{v^S(t,x) = u^S(t, e^{-r(T-t)}x),} \qquad \boxed{u^S(t,x) = v^S(t, e^{r(T-t)}x).} \tag{2.65}$$

Using the relationship between the price functions $v^S$ and $u^S$, we can obtain a partial differential equation for $v^S$. Since $u^S$ satisfies the partial differential equation

$$u_t^S(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^S(t,x) = 0,$$

the function $v^S$ satisfies the partial differential equation

$$\boxed{v_t^S(t,x) - rxv_x^S(t,x) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}^S(t,x) = 0.} \tag{2.66}$$

The terminal condition is

$$v^S(T,x) = u^S(T,x) = f^S(x), \tag{2.67}$$

and the boundary condition is

$$v^S(t,0) = u^S(t,0) = f^S(0). \tag{2.68}$$

## 2.6 Hedging

Let us determine the hedging portfolio for a general European option contract $V$.

**Theorem 2.18** *The hedging portfolio $P(t)$ of the European option is given by*

$$\boxed{P(t) = \Big[u_x^Y\left(t, X_Y(t)\right)\Big] \cdot X + \Big[u^Y(t, X_Y(t)) - u_x^Y(t, X_Y(t)) \cdot X_Y(t)\Big] \cdot Y,} \tag{2.69}$$

*or equivalently by*

$$\boxed{P(t) = \Big[u^X(t, Y_X(t)) - u_x^X(t, Y_X(t)) \cdot Y_X(t)\Big] \cdot X + \Big[u_x^X\left(t, Y_X(t)\right)\Big] \cdot Y.} \tag{2.70}$$

Proof: The hedging portfolio is in the form

$$P(t) = \Delta^X(t) \cdot X + \Delta^Y(t) \cdot Y,$$

and has dynamics of the form

$$dP_Y(t) = \Delta^X\left(t, X_Y(t)\right) dX_Y(t).$$

We also have

$$dV_Y(t) = du^Y\left(t, X_Y(t)\right) = u_x^Y\left(t, X_Y(t)\right) dX_Y(t).$$

In order to have

$$P(t) = V(t),$$

at all times, the hedge of this contract must satisfy

$$\boxed{\Delta^X\left(t, X_Y(t)\right) = u_x^Y\left(t, X_Y(t)\right) = \frac{\partial V_Y(t)}{\partial X_Y(t)}.} \tag{2.71}$$

The hedging position $\Delta^X$ in the asset $X$ is the sensitivity of the price of the contract $V_Y(t)$ to the changes of the underlying price $X_Y(t)$. The hedge position $\Delta^Y$ in the asset $Y$ follows from

$$\Delta^Y(t) = P_Y(t) - \Delta^X(t) \cdot X_Y(t) = u^Y(t, X_Y(t)) - u_x^Y(t, X_Y(t)) \cdot X_Y(t).$$

When $X$ is chosen as a reference asset, the price dynamics of the hedging portfolio $P$ are given by

$$dP_X(t) = \Delta^Y\left(t, Y_X(t)\right) dY_X(t).$$

We also have

$$dV_X(t) = du^X\left(t, Y_X(t)\right) = u_x^X\left(t, Y_X(t)\right) dY_X(t),$$

and thus in order to have

$$P(t) = V(t),$$

the hedging position $\Delta^Y$ must satisfy

$$\boxed{\Delta^Y\left(t, Y_X(t)\right) = u_x^X\left(t, Y_X(t)\right) = \frac{\partial V_X(t)}{\partial Y_X(t)}.} \tag{2.72}$$

The hedging position $\Delta^X(t)$ in the asset $X$ follows from

$$\Delta^X(t) = P_X(t) - \Delta^Y(t) \cdot Y_X(t) = u^X(t, Y_X(t)) - u_x^X(t, Y_X(t)) \cdot Y_X(t).$$

Recall that the prices in terms of the functions $u^Y$ and $u^X$ are related by the following symmetric relationship known as a perspective mapping:

$$\boxed{u^Y(t,x) = u^X(t, \tfrac{1}{x}) \cdot x,} \quad \text{or} \quad \boxed{u^X(t,x) = u^Y(t, \tfrac{1}{x}) \cdot x.}$$

We can connect the pricing partial differential equations for $u^Y$ and $u^X$ through the above relationship. The function $u^Y$ solves Equation (2.49):

$$u_t^Y(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^Y(t,x) = 0.$$

We can rewrite this partial differential equation in terms of $u^X$ using the following identities:

$$\begin{aligned}
u_t^Y(t,x) &= u_t^X(t, \tfrac{1}{x}) \cdot x, \\
u_x^Y(t,x) &= u^X(t, \tfrac{1}{x}) - \tfrac{1}{x} \cdot u_x^X(t, \tfrac{1}{x}), \\
u_{xx}^Y(t,x) &= -\tfrac{1}{x^2} \cdot u_x^X(t, \tfrac{1}{x}) + \tfrac{1}{x^2} \cdot u_x^X(t, \tfrac{1}{x}) + \tfrac{1}{x^3} \cdot u_{xx}^X(t, \tfrac{1}{x}) \\
&= \tfrac{1}{x^3} \cdot u_{xx}^X(t, \tfrac{1}{x}).
\end{aligned}$$

Substituting for $u_t^Y(t,x)$ and $u_{xx}^Y(t,x)$ in (2.49) we get

$$u_t^Y(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^Y(t,x) = u_t^X(t, \tfrac{1}{x}) \cdot x + \tfrac{1}{2}\sigma^2 x^2 \tfrac{1}{x^3} \cdot u_{xx}^X(t, \tfrac{1}{x}) = 0,$$

which leads to

$$u_t^X(t, \tfrac{1}{x}) + \tfrac{1}{2}\sigma^2 \tfrac{1}{x^2} \cdot u_{xx}^X(t, \tfrac{1}{x}) = 0.$$

After making the substitution $\frac{1}{x} \to x$, we can rewrite the above partial differential equation as

$$u_t^X(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}^X(t,x) = 0,$$

which is Equation (2.52). This is an independent derivation of this partial differential equation using the relationship between $u^Y$ and $u^X$. Note that the partial differential equation for $u^Y$ and $u^X$ takes the same form, so it is completely symmetric with respect to the choice of the reference asset. This is not the case for more complex products, such as for Asian options.

We have previously seen that the hedging portfolio is given by

$$P(t) = \Delta^X(t) \cdot X + \Delta^Y(t) \cdot Y = \left[u_x^Y\left(t, X_Y(t)\right)\right] \cdot X + \left[u_x^X\left(t, Y_X(t)\right)\right] \cdot Y,$$

when using both price functions $u^Y$ and $u^X$, or in other words,

$$\boxed{P(t) = \left[\frac{\partial V_Y(t)}{\partial X_Y(t)}\right] \cdot X + \left[\frac{\partial V_X(t)}{\partial Y_X(t)}\right] \cdot Y.} \tag{2.73}$$

Using the relationship between $u^Y$ and $u^X$:

$$u_x^Y(t, x) = u^X(t, \tfrac{1}{x}) - \tfrac{1}{x} \cdot u_x^X(t, \tfrac{1}{x}),$$

or

$$u_x^X(t, x) = u^Y(t, \tfrac{1}{x}) - \tfrac{1}{x} \cdot u_x^Y(t, \tfrac{1}{x}),$$

we can also write

$$P(t) = \left[u_x^Y\left(t, X_Y(t)\right)\right] \cdot X + \left[u^Y(t, X_Y(t)) - u_x^Y(t, X_Y(t)) \cdot X_Y(t)\right] \cdot Y,$$

or equivalently

$$P(t) = \left[u^X(t, Y_X(t)) - u_x^X(t, Y_X(t)) \cdot Y_X(t)\right] \cdot X + \left[u_x^X\left(t, Y_X(t)\right)\right] \cdot Y.$$

This confirms Theorem 2.18.

**Example 2.19 (Hedging of the forward contract)** *The forward contract pays off $X(T) - K \cdot Y(T)$, which corresponds to the payoff functions $f^Y(x) = x - K$, and $f^X(x) = 1 - K \cdot x$. The price of the forward contract is trivially given by $u^Y(t, x) = x - K$, and $u^X(t, x) = 1 - K \cdot x$. Therefore the hedging portfolio is given by*

$$P(t) = \left[u_x^Y(t, x)\right] \cdot X(t) + \left[u_x^X(t, x)\right] \cdot Y(t) = X(t) - K \cdot Y(t).$$

*The hedge is static; one buys one unit of the asset $X$ and sells $K$ units of $Y$. The forward contract can be thought of as a combination of two contracts to deliver: one that delivers a unit of an asset $X$ and one that delivers $-K$ units (or in other words shorts $K$ units) of an asset $Y$. A contract to deliver an asset $X$ at time $T$ is trivial: it is the asset $X$ itself. One simply buys the asset and holds it until expiration. A similar argument applies to the asset $Y$. Note that the hedge of the forward contract is model independent; it does not depend on the evolution of the price $X_Y(t)$.*

**Example 2.20 (Hedging of the European call option)**
*We have seen that the hedging position of a general European option in the asset $X$ is given by*

$$\Delta^X\left(t, X_Y(t)\right) = u_x^Y\left(t, X_Y(t)\right).$$

*This further simplifies when the payoff function is given by $f^Y(x) = (x - K)^+$. We have that*

$$\begin{aligned} u_x^Y(t,x) &= \tfrac{d}{dx}\mathbb{E}_t^Y\left(X_Y(T) - K\right)^+ \\ &= \tfrac{d}{dx}\mathbb{E}^Y\left[\left(x \cdot \tfrac{X_Y(T)}{X_Y(t)} - K\right)^+ |X_Y(t) = x\right] \\ &= \tfrac{d}{dx}\mathbb{E}^X\left[\left(x - K \cdot \tfrac{Y_X(T)}{Y_X(t)}\right)^+ |X_Y(t) = x\right] \\ &= \mathbb{P}_t^X(X_Y(T) \geq K). \end{aligned}$$

*Thus we have*

$$\Delta^X(t) = \mathbb{P}_t^X(X_Y(T) \geq K) = N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{1}{K} \cdot X_Y(t)\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right). \tag{2.74}$$

*Similarly we get*

$$\begin{aligned} \Delta^Y(t) &= -K \cdot \mathbb{P}_t^Y(X_Y(T) \geq K) \\ &= -K \cdot N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{1}{K} \cdot X_Y(t)\right) - \tfrac{1}{2}\sigma\sqrt{T-t}\right). \end{aligned} \tag{2.75}$$

Hedging of an option that has a dollar as an underlying asset has to be done in a stock $S$ and in the bond $B^T$ (or equivalently in the money market $M$). Thus the hedging portfolio $P(t)$ is in the form

$$P(t) = \Delta^S(t) \cdot S + \Delta^T(t) \cdot B^T.$$

We have already seen that

$$\Delta^S(t) = u_x^T\left(t, S_{B^T}(t)\right), \qquad \text{and} \qquad \Delta^T(t) = u_x^S\left(t, B_S^T(t)\right).$$

We can also express the hedging positions in terms of the price functions $v^\$$ and $v^S$. Since $u^T(t,x) = e^{r(T-t)} \cdot v^\$(t, e^{-r(T-t)}x)$, we have

$$u_x^T(t,x) = v_x^\$(t, e^{-r(T-t)}x),$$

and thus

$$\begin{aligned} \Delta^S(t) = u_x^T\left(t, S_{B^T}(t)\right) &= u_x^T\left(t, S_\$(t) \cdot \$_{B^T}(t)\right) \\ &= u_x^T\left(t, e^{r(T-t)} \cdot S_\$(t)\right) = v_x^\$\left(t, S_\$(t)\right). \end{aligned}$$

The hedging position in the bond $B^T$ can be obtained from the dollar price of the hedging portfolio

$$P_\$(t) = \Delta^S(t) \cdot S_\$(t) + \Delta^T(t) \cdot B_\$^T(t),$$

or in other words,

$$v^\$\left(t, S_\$(t)\right) = v_x^\$\left(t, S_\$(t)\right) \cdot S_\$(t) + \Delta^T(t) \cdot e^{-r(T-t)}.$$

Thus we have

$$\Delta^T(t) = e^{r(T-t)} \cdot \left[v^\$\left(t, S_\$(t)\right) - v_x^\$\left(t, S_\$(t)\right) \cdot S_\$(t)\right].$$

Similarly, we can express the hedging portfolio in terms of the price function $v^S$. Since $u^S(t,x) = v^S(t, e^{r(T-t)}x)$, we have

$$u_x^S(t,x) = v_x^S(t, e^{r(T-t)}x) \cdot e^{r(T-t)},$$

and thus

$$\begin{aligned}\Delta^T(t) = u_x^S\left(t, B_S^T(t)\right) &= u_x^S\left(t, \$_S(t) \cdot B_\$^T(t)\right) \\ &= u_x^S\left(t, e^{-r(T-t)} \cdot \$_S(t)\right) = e^{r(T-t)} \cdot v_x^S\left(t, \$_S(t)\right).\end{aligned}$$

The hedging position $\Delta^S(t)$ can be obtained from

$$\begin{aligned}\Delta^S(t) &= P_S(t) - \Delta^T(t) \cdot B_S^T(t) \\ &= v^S\left(t, \$_S(t)\right) - e^{r(T-t)} \cdot v_x^S\left(t, \$_S(t)\right) \cdot \$_S(t) \cdot B_\$^T(t) \\ &= v^S\left(t, \$_S(t)\right) - v_x^S\left(t, \$_S(t)\right) \cdot \$_S(t).\end{aligned}$$

**Corollary 2.21** *The hedging portfolio is given by*

$$\begin{aligned}P(t) = &\left[v_x^\$\left(t, S_\$(t)\right)\right] \cdot S \\ &+ \left[e^{r(T-t)} \cdot \left[v^\$\left(t, S_\$(t)\right) - v_x^\$\left(t, S_\$(t)\right) \cdot S_\$(t)\right]\right] \cdot B^T, \end{aligned} \tag{2.76}$$

*or*

$$\begin{aligned}P(t) = &\left[\left[v^S\left(t, \$_S(t)\right) - v_x^S\left(t, \$_S(t)\right) \cdot \$_S(t)\right]\right] \cdot S \\ &+ \left[e^{r(T-t)} \cdot v_x^S\left(t, \$_S(t)\right)\right] \cdot B^T. \end{aligned} \tag{2.77}$$

*Assuming that $M_\$(t) = 1$, or equivalently stated, $M(t) = e^{r(T-t)} \cdot B^T(t)$, we can also express the hedging portfolio in term of the stock $S$ and the money market $M$ as*

$$P(t) = \left[v_x^\$\left(t, S_\$(t)\right)\right] \cdot S + \left[v^\$\left(t, S_\$(t)\right) - v_x^\$\left(t, S_\$(t)\right) \cdot S_\$(t)\right] \cdot M, \tag{2.78}$$

*or*

$$P(t) = \left[\left[v^S\left(t, \$_S(t)\right) - v_x^S\left(t, \$_S(t)\right) \cdot \$_S(t)\right]\right] \cdot S + \left[v_x^S\left(t, \$_S(t)\right)\right] \cdot M. \tag{2.79}$$

## 2.7 Properties of European Call and Put Options

An option is **in the money** at time $t$ if $f^Y(X_Y(t)) > 0$. If the option were to expire immediately at time $t$, its holder would collect a positive payoff. An option is **deep in the money** if it is in the money and $f^Y(X_Y(T)) > 0$ with high probability, meaning that the option is likely to expire with a positive payoff. An option is **out of the money** at time $t$ if $f^Y(X_Y(t)) = 0$. An option is **deep out of the money** if it is out of the money, and $f^Y(X_Y(T)) = 0$ with high probability, meaning that the option is likely to expire worthless. An option is **at the money** if $f^Y(X_Y(t)+\epsilon) > 0$

and $f^Y(X_Y(t)-\epsilon)=0$ for $\epsilon>0$. An at the money option is a boundary case between in the money and out of the money option.

Given the hedge representation for a European call option

$$\Delta^X(t)=\mathbb{P}_t^X(X_Y(T)\geq K),$$

and

$$\Delta^Y(t)=-K\cdot\mathbb{P}_t^Y(X_Y(T)\geq K),$$

we can see that

$$0\leq\Delta^X(t)\leq 1,\quad \text{and}\quad -K\leq\Delta^Y(t)\leq 0.$$

Moreover, if the option is deep out of the money, the option is almost worthless, and the corresponding hedge is $\Delta^X(t)\approx 0$, and $\Delta^Y(t)\approx 0$. On the other hand, if the option is deep in the money, $\Delta^X(t)\approx 1$, $\Delta^Y(t)\approx -K$, and the European option contract is close to a forward $X(t)-K\cdot Y(t)$.

Another interesting observation is to see what happens when the maturity of the option approaches infinity, or equivalently, when the volatility approaches infinity. Recall that the price of a European call option is given by

$$V^{EC}(0,X,K\cdot Y,T)=N\left(d_+\right)\cdot X-K\cdot N\left(d_-\right)\cdot Y,$$

where

$$d_\pm=\tfrac{1}{\sigma\sqrt{T}}\cdot\log\left(\tfrac{1}{K}\cdot X_Y(0)\right)\pm\tfrac{1}{2}\sigma\sqrt{T},$$

and so the price is a function of a factor $\sigma\sqrt{T}$. For instance, doubling the volatility has the same effect on the option price as quadrupling time. When maturity $T\to\infty$, we simply have

$$\lim_{T\to\infty}V_Y^{EC}(0,X,K\cdot Y,T)=X_Y(0)$$

since $d_+\to\infty$, and $d_-\to-\infty$. Therefore for large $T$, $V_Y(0)\approx X_Y(0)$, and the hedge is to hold a unit of an asset $X$ and have no position in the asset $Y$. Figure 2.1 shows the price $V_Y$ of a European call option with a payoff $(X(T)-\frac{1}{2}Y(T))^+$ as a function of the price $X_Y(t)$ of the underlying asset $X$, and time to maturity $T-t$. Note that when $t=T$, the price of the contract is simply the payoff $(x-\frac{1}{2})^+$. On the other hand, for large maturities the price of the contract is approximately $X_Y$, so the price of $V_Y$ becomes approximately linear in $X_Y$.

Figures 2.2 and 2.3 show the corresponding hedging positions in the underlying assets $X$ and $Y$ as a function of the price $X_Y(t)$ and time to maturity $T-t$. Note that the hedging position in the asset $X$ is between 0 and 1, and the hedging position in the asset $Y$ is between $-\frac{1}{2}$ and 0. For short maturities, the hedging position in the asset $X$ should be close to 1 when the option is in the money, but it should be close to 0 when the option is out of the money. There is a jump in the hedging position at the strike price at the time of maturity. For large maturities, the hedging

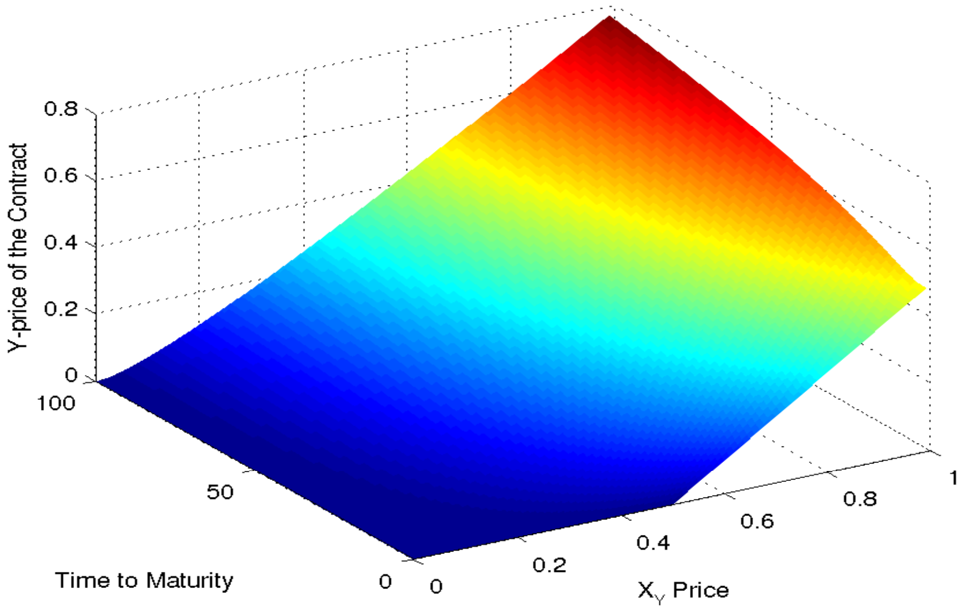



Figure 2.1: The price $V_Y(t)$ of a European option contract with a payoff $(X(T) - K \cdot Y(T))^+$ with parameters $K = \frac{1}{2}$, $\sigma = 0.2$, as a function of price $X_Y(t)$, and time to maturity $T$. We have considered unrealistically large maturities in order to show the limiting behavior of the option price.

position in the asset $X$ should be close to 1.

Similarly, for short maturities, the hedging position in the asset $Y$ should be close to $-\frac{1}{2}$ when the option is in the money, and it should be close to 0 when the option is out of the money. For long maturities, the hedging position in the asset $Y$ should be close to 0.

Figure 2.4 shows a sample path of $X_Y$ in a geometric Brownian motion model, and the corresponding price of the European option $V_Y$. Figure 2.5 shows the corresponding hedging position in the underlying assets $X$ and $Y$. Note that the hedging positions start to change dramatically when the time is close to maturity. The reason is that the price of the underlying asset happens to be near the strike price when the option is close to maturity, and the corresponding hedging position in the asset $X$ takes the values close to 0 or 1 depending whether the option is out of the money or in the money. We observe a similar behavior for the hedging position in the asset $Y$.

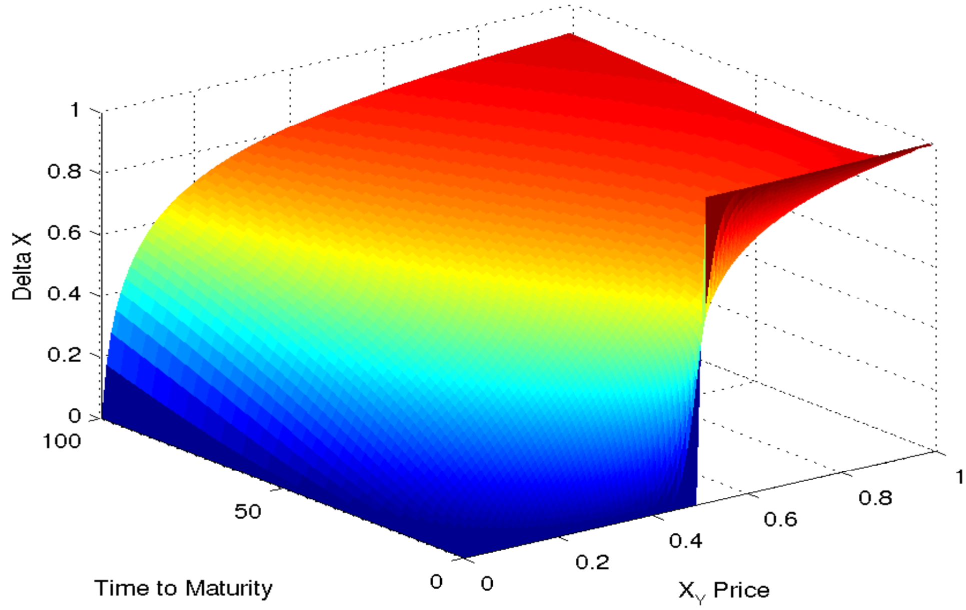



Figure 2.2: The hedging position in the asset $X$ for the European option contract $(X(T) - K \cdot Y(T))^+$ with parameters $K = \frac{1}{2}$, $\sigma = 0.2$, as a function of the price $X_Y(t)$ and time to maturity $T - t$. Note that the hedging position in the asset $X$ is between 0 and 1.

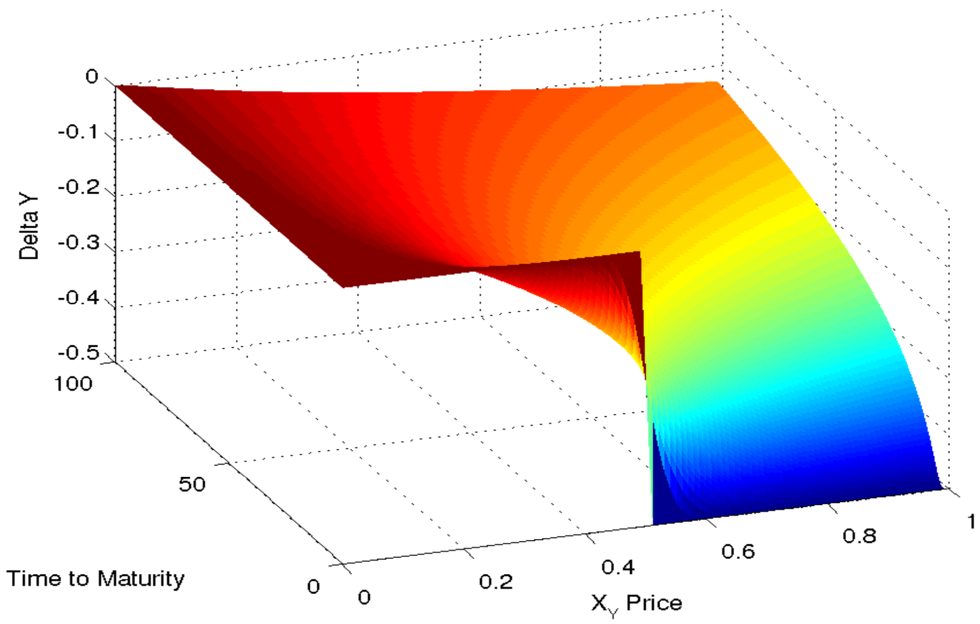



Figure 2.3: The hedging position in the asset $Y$ for the European option contract $(X(T) - K \cdot Y(T))^+$ with parameters $K = \frac{1}{2}$, $\sigma = 0.2$, as a function of the price $X_Y(t)$ and time to maturity $T - t$. Note that the hedging position in the asset $Y$ is between $-\frac{1}{2}$ and 0.

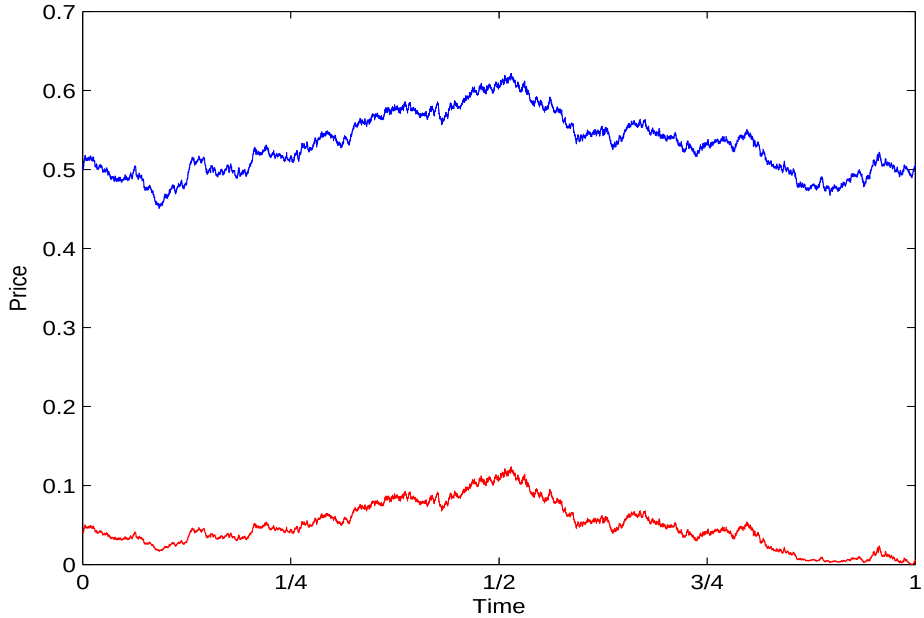



Figure 2.4: The price $X_Y(t)$ of an asset $X$ in terms of the reference asset $Y$ (top), and the price $V_Y(t)$ of a European option contract with a payoff $(X(T) - K \cdot Y(T))^+$ with parameters $X_Y(0) = \frac{1}{2}$, $K = \frac{1}{2}$, $\sigma = 0.2$, $T = 1$ (bottom).

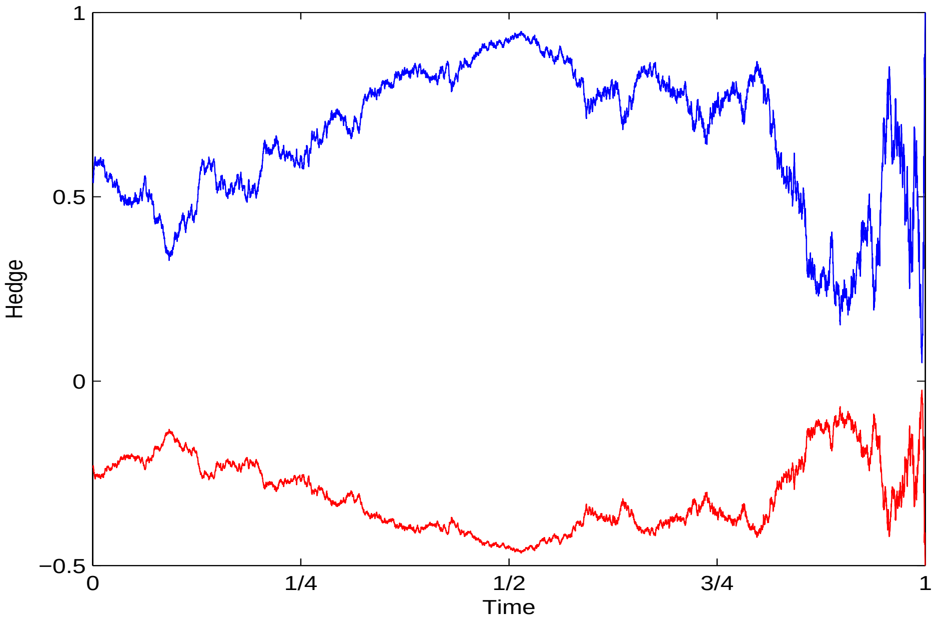



Figure 2.5: The hedging position in the asset $X$ (top) and $Y$ (bottom) for the European option contract $(X(T) - K \cdot Y(T))^+$ with parameters $X_Y(0) = \frac{1}{2}$, $K = \frac{1}{2}$, $\sigma = 0.2$, $T = 1$. Note that the hedging position in the asset $X$ is between 0 and 1, and the hedging position in the asset $Y$ is between $-\frac{1}{2}$ and 0.

**Remark 2.22 (Greeks)** **Greeks** *measure sensitivities of the prices of the portfolio (or in particular a single financial contract) to the changes of the parameters of the model. They describe how the price of the portfolio would change if the parameters change. Note that the price of the portfolio is given relative to the reference asset, so one can define portfolio sensitivities for any price function. The traditional definition of greeks applies to the price function $v^{\$}$, but it would make even better sense to apply it to the price function $u^Y$, or $u^X$. The assets $Y$ and $X$ have no time value (in contrast to a dollar \$), and thus the corresponding greeks would not be influenced by the time decay of the reference asset.*

**Delta** *is the sensitivity of the price $u^Y$ with respect to the price of the underlying $X_Y$:*

$$\Delta(t) = u^Y_x(t, X_Y(t)) = \frac{\partial u^Y(t, X_Y(t))}{\partial X_Y(t)}. \tag{2.80}$$

**Gamma** *is the sensitivity of $\Delta$ with respect to the price of the underlying $X_Y$, which is the same as the second derivative of $u^Y$ with respect to $X_Y$:*

$$\Gamma(t) = u^Y_{xx}(t, X_Y(t)) = \frac{\partial^2 u^Y(t, X_Y(t))}{\partial X^2_Y(t)}. \tag{2.81}$$

**Theta** *is the sensitivity of the price $u^Y$ with respect to time $t$:*

$$\Theta(t) = u^Y_t(t, X_Y(t)) = \frac{\partial u^Y(t, X_Y(t))}{\partial t}. \tag{2.82}$$

**Vega** *is the sensitivity of the price $u^Y$ with respect to the volatility $\sigma$:*

$$\nu(t) = u^Y_\sigma(t, X_Y(t)) = \frac{\partial u^Y(t, X_Y(t))}{\partial \sigma}. \tag{2.83}$$

**Rho** *is the sensitivity of the price $u^Y$ with respect to the interest rate $r$:*

$$\rho(t) = u^Y_r(t, X_Y(t)) = \frac{\partial u^Y(t, X_Y(t))}{\partial r}. \tag{2.84}$$

**Example 2.23** *Consider an option with a payoff $(X(T) - K \cdot Y(T))^+$. Its price is given by the Black–Scholes formula*

$$\begin{aligned} u^Y(t, X_Y(t)) = xN &\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right) \\ &- KN\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) - \tfrac{1}{2}\sigma\sqrt{T-t}\right). \end{aligned}$$

*The corresponding greeks are given by*

$$\begin{aligned} \Delta(t) &= N\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right), \\ \Gamma(t) &= \frac{1}{X_Y(t)\sigma\sqrt{T-t}} \cdot \phi\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right), \\ \theta(t) &= -\tfrac{1}{2} \cdot \frac{\sigma X_Y(t)}{\sqrt{T-t}} \cdot \phi\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right), \\ \nu(t) &= X_Y(t)\sqrt{T-t} \cdot \phi\left(\tfrac{1}{\sigma\sqrt{T-t}} \cdot \log\left(\tfrac{X_Y(t)}{K}\right) + \tfrac{1}{2}\sigma\sqrt{T-t}\right), \\ \rho(t) &= 0. \end{aligned}$$

*The sensitivity $\rho$ turns out to be zero since the price evolution $X_Y$ is not influenced by the changes of the interest rate (assets $X$ and $Y$ have no time value). The changes of the interest rate would influence contracts that depend on the assets with time value, such as a dollar \$.*

## 2.8 Stochastic Volatility Models

When we have a contingent claim $V$ whose payoff depends on the assets $X$ and $Y$, its price $V_Y(t)$ can depend on the entire price evolution $X_Y(s)$ up to time $t$. It can also depend on several additional external processes $\xi^i(s)$, such as a random process that represents stochastic volatility. In this case we can write

$$V_Y(t) = u^Y(t, \{X_Y(s)\}_{s=0}^t, \{\xi^i(s)\}_{s=0}^t).$$

While this expression would explain the price process $V_Y(t)$ in full, it would be prohibitively complicated to model the price of $V_Y(t)$ using infinitely many possible values from $\{X_Y(s)\}_{s=0}^t$ and $\{\xi^i(s)\}_{s=0}^t$. Thus it is desirable to express such dependence using only a small number of factors that would explain the price evolution $V_Y(t)$ sufficiently well.

A common approach to price modeling is to use the Markov property:

$$V_Y(t) = u^Y(t, \{X_Y(s)\}_{s=0}^t, \{\xi^i(s)\}_{s=0}^t) = u^Y(t, X_Y(t), \xi^i(t)),$$

which says that the only relevant information about the future evolution of the process $V_Y(t)$ is given by the present values of the underlying processes $X_Y(t)$ and $\xi^i(t)$.

The simplest models that we considered in the previous text assume no external processes $\xi^i(t)$, and the price of the contract $V$ can be written as

$$V(t) = u^Y(t, X_Y(t)) \cdot Y = u^X(t, Y_X(t)) \cdot X.$$

More general models of the asset prices consider a stochastic evolution of volatility. The price of a contract $V$ depends on the price of the underlying asset $X_Y(t)$, and on a process $\xi(t)$ that represents the volatility

$$V(t) = u^Y(t, X_Y(t), \xi(t)) \cdot Y = u^X(t, Y_X(t), \xi(t)) \cdot X.$$

This model has two sources of uncertainty, and it is not possible in general to hedge such contracts perfectly with only two assets $X$ and $Y$. A general rule for a complete market is to have $n+1$ assets for $n$ sources of noise, which is not the case here. Thus stochastic volatility models are not complete in general and a perfect replication of an arbitrary contingent claim may no longer be possible. As mentioned earlier, the volatility is the same for both $X_Y(t)$ and $Y_X(t)$.

Let us assume that the price process follows

$$dX_Y(t) = g(t, \xi(t)) X_Y(t) dW^Y(t), \tag{2.85}$$

where $\xi(t)$ is a stochastic process in the form

$$d\xi(t) = \alpha(t, \xi(t))dt + \beta(t, \xi(t))dW^{\xi}(t). \tag{2.86}$$

We assume that the two Brownian motions $W^Y$ and $W^{\xi}$ are correlated:

$$dW^Y(t) \cdot dW^{\xi}(t) = \rho dt.$$

Note that the price process $X_Y(t)$ is a $\mathbb{P}^Y$ martingale. The process $\xi(t)$ is a parameter of the model, and as such it can have an arbitrary evolution. In particular, it does not need to be a martingale.

**Example 2.24** *A popular stochastic volatility model is the* **Heston model***, which is given by the following choice of the functions $g$, $\alpha$, and $\beta$:*

$$g(t, \xi) = \sqrt{\xi}, \quad \alpha(t, \xi) = a - b \cdot \xi, \quad \beta(t, \xi) = \sigma\sqrt{\xi}.$$

*In this case we can write*

$$dX_Y(t) = \sqrt{\xi(t)} \cdot X_Y(t)dW^Y(t),$$

*and*

$$d\xi(t) = (a - b \cdot \xi(t))dt + \sigma\sqrt{\xi(t)}dW^{\xi}(t).$$

Let $V$ be a contingent claim whose price $V_Y(t)$ depends only on $X_Y(t)$ and on $\xi(t)$. We can write

$$V_Y(t) = u^Y(t, X_Y(t), \xi(t)).$$

Since $V_Y(t)$ is a $\mathbb{P}^Y$ martingale, we can obtain a partial differential equation for the price function $u^Y$. We have

$$\begin{aligned}
du^Y(t, X_Y(t), \xi(t)) &= u_t^Y dt + u_x^Y dX_Y(t) + u_\xi^Y d\xi(t) \\
&\quad + \tfrac{1}{2}u_{xx}^Y d^2X_Y(t) + u_{x\xi}^Y dX_Y(t)d\xi(t) + \tfrac{1}{2}u_{\xi\xi}^Y d^2\xi(t) \\
&= \Big[u_t^Y + \alpha(x, \xi)u_\xi^Y + \tfrac{1}{2}g^2X_Y(t)^2u_{xx}^Y \\
&\quad + \rho\beta g X_Y(t)u_{x\xi}^Y + \tfrac{1}{2}\beta^2u_{\xi\xi}^Y\Big]dt \\
&\quad + gX_Y(t)u_x^Y + \beta u_\xi^Y dW^{\xi}(t).
\end{aligned}$$

Since the $dt$ term must be zero, we get a partial differential equation for $u^Y$:

$$\begin{aligned}
u_t^Y(t, x, \xi) + \alpha(t, \xi)u_\xi^Y(t, x, \xi) + \tfrac{1}{2}g(t, \xi)^2x^2u_{xx}^Y(t, x, \xi)& \\
+ \rho\beta(t, \xi)g(t, \xi)xu_{x\xi}^Y(t, x, \xi) + \tfrac{1}{2}\beta(t, \xi)^2u_{\xi\xi}^Y(t, x, \xi) &= 0.
\end{aligned} \tag{2.87}$$

Similarly, we can study the evolution of the inverse price that takes the same form

$$dY_X(t) = g(t, \xi(t)) \cdot Y_X(t)dW^X(t),$$

where

$$dW^X(t) = -dW^Y(t) + g(t, \xi(t))dt.$$

This follows from Ito's formula

$$
\begin{aligned}
dY_X(t) = dX_Y(t)^{-1} &= -Y_X(t)^2 dX_Y(t) + \frac{1}{2}\cdot 2Y_X(t)^3 d^2X_Y(t)\\
&= -g(t,\xi(t))\cdot Y_X(t)dW^Y(t) + g(t,\xi(t))^2\cdot Y_X(t)dt\\
&= g(t,\xi(t))\cdot Y_X(t)dW^X(t).
\end{aligned}
$$

The correlation between $W^X(t)$ and $W^\xi(t)$ is given by

$$dW^X(t)\cdot dW^\xi(t) = (-dW^Y(t) + g(t,\xi(t))dt)\cdot dW^\xi(t) = -\rho dt.$$

The only difference is that the correlation coefficient takes an opposite sign. Thus if we have

$$V_X(t) = u^X(t, Y_X(t), \xi(t)),$$

the partial differential equation for $u^X$ differs only in the sign that corresponds to the correlation coefficient. Therefore $u^X$ satisfies

$$
\begin{aligned}
u_t^X(t,x,\xi) + \alpha(x,\xi)u_\xi^X(t,x,\xi) + \tfrac{1}{2}g(t,\xi)^2x^2u_{xx}^X(t,x,\xi)\\
- \rho\beta(t,\xi)g(t,\xi)xu_{x\xi}^X(t,x,\xi) + \tfrac{1}{2}\beta(t,\xi)^2u_{\xi\xi}^X(t,x,\xi) = 0. \quad (2.88)
\end{aligned}
$$

## 2.9 Foreign Exchange Market

This section studies contracts traded on foreign exchange markets. Let an asset $X$ be the domestic currency \$, and an asset $Y$ be the foreign currency €. Let

$$€_\$(t)$$

denote the amount of a domestic currency that is needed to acquire a unit of a foreign currency at time $t$. The quantity $€_\$(t)$ is known as an **exchange rate,** but in fact this is just a special case of the price $X_Y(t)$, where $X = €$, and $Y = \$$. Thus we can apply the results we have obtained in the previous sections for the case of the foreign exchange market.

The foreign exchange market is an excellent example to illustrate the relative concept of prices since both the domestic and the foreign currencies are legitimate choices for the reference asset. Whether a currency is domestic or foreign depends on which country one lives in. For some people \$ is the domestic currency and € is the foreign currency, but for other people € is the domestic currency and \$ is the foreign currency. Thus it makes perfect sense to study the inverse exchange rate

$$\frac{1}{€_\$(t)} = \$_€(t).$$

Note that $€_\$(t)$ and $\$_€(t)$ are prices, not assets that could be bought or sold. Moreover, the currencies themselves are arbitrage assets, and thus one needs to immediately acquire a suitable no-arbitrage asset for it in order not to lose value, such as a bond that is denominated in the corresponding currency.

### 2.9.1 Forwards

Let us consider first a **forward** contract on the foreign exchange with a payoff

$$€(T) - K \cdot \$(T) = (€_{\$}(T) - K) \cdot \$(T) = (1 - \$_{€}(T)) \cdot €(T).$$

at time $T$. Let us write this contract in terms of no-arbitrage assets. There is a corresponding foreign bond that delivers one € at time $T$. We will denote this no-arbitrage asset by $B^{€,T}$. Similarly, there is a domestic bond $B^T$ that delivers one \$ at time $T$. Therefore the forward contract is equivalent to the contract with a payoff

$$B^{€,T}(T) - K \cdot B^T(T).$$

Let us denote the forward contract by $F(B^{€,T}, K \cdot B^T, T)$, and let us compute its price. The contract depends on two no-arbitrage assets, namely on $B^{€,T}$, and $B^T$. A possible numeraire for pricing is $B^T$, a domestic bond maturing at time $T$. We have

$$F_{B^T}(t) = \mathbb{E}_t^T \left[ \left( B^{€,T} - K \cdot B^T \right)_{B^T} (T) \right] = B^{€,T}_{B^T}(t) - K. \tag{2.89}$$

The last identity follows from the fact that the price of the bond $B^{€,T}$ in terms of the bond $B^T$ is a martingale under the T-forward measure that corresponds to $B^T$ as a reference asset. Thus we conclude that the forward contract is equal to

$$F(t, B^{€,T}, K \cdot B^T, T) = B^{€,T}(t) - K \cdot B^T(t). \tag{2.90}$$

Note that this is a model-independent formula (we have not assumed any particular dynamics). A **forward exchange rate** is a choice of $\bar{K}$ that makes the forward contract equal to zero:

$$F(0, B^{€,T}, \bar{K} \cdot B^T, T) = 0. \tag{2.91}$$

Solving this equation, we get

$$\boxed{\bar{K} = B^{€,T}_{B^T}(0).} \tag{2.92}$$

If we assume constant interest rates for both the domestic and the foreign zero coupon bond, we can express the above relationship in terms of the exchange rate $€_{\$}(0)$. The domestic bond price in terms of the domestic currency is

$$B^T(t) = e^{-r(T-t)} \cdot \$(t),$$

and the foreign bond price in terms of the foreign currency is

$$B^{€,T}(t) = e^{-r^F(T-t)} \cdot €(t).$$

Thus we can write

$$\begin{aligned} B^{€,T}_{B^T}(t) &= B^{€,T}_{€}(t) \cdot €_{\$}(t) \cdot \$_{B^T}(t) \\ &= e^{-r^F(T-t)} \cdot €_{\$}(t) \cdot e^{r(T-t)} = e^{(r-r^F)(T-t)} \cdot €_{\$}(t). \end{aligned}$$

In other words,

$$\boxed{\bar{K} = B^{€,T}_{B^T}(0) = e^{(r-r^F)T} \cdot €_{\$}(0).} \tag{2.93}$$

We can consider a similar contract on the inverse exchange rate $\$_{€}(T)$

$$\$(T) - K^f \cdot €(T) = \left(\$_{€}(T) - K^f\right) \cdot €(T) = (1 - K^f \cdot €_{\$}(T)) \cdot \$(T).$$

The corresponding no-arbitrage assets are $B^T$ and $B^{€,T}$. We can rewrite the payoff of the contract as

$$B^T(T) - K^f \cdot B^{€,T}(T).$$

If we denote this contract by $F(B^T, K^f \cdot B^{€,T}, T)$, and choose the corresponding foreign bond $B^{€,T}$ as a numeraire, we get

$$F_{B^{€,T}}(t) = \mathbb{E}_t^{€,T}\left[\left(B^T - K^f \cdot B^{€,T}\right)_{B^{€,T}}(T)\right] = B^T_{B^{€,T}}(t) - K^f. \tag{2.94}$$

The last equation follows from the fact that the price of $B^T$ in terms of $B^{€,T}$ is a martingale under the measure that corresponds to $B^{€,T}$ as a numeraire. The forward contract is equal to

$$F(t, B^T, K^f \cdot B^{€,T}, T) = B^T(t) - K^f \cdot B^{€,T}(t). \tag{2.95}$$

The corresponding forward exchange rate from the point of view of the foreign currency is a choice of $\bar{K}^f$ that makes the value of the forward contract zero:

$$F(0, B^T, \bar{K}^f \cdot B^{€,T}, T) = 0. \tag{2.96}$$

Solving for $\bar{K}^f$, we get

$$\boxed{\bar{K}^f = B^T_{B^{€,T}}(0) = e^{(r^F - r)T} \cdot \$_{€}(0) = \frac{1}{\bar{K}}.} \tag{2.97}$$

Note that the forward exchange rates as seen from the domestic currency and from the foreign currency point of view are linked through $\bar{K} = \frac{1}{\bar{K}^f}$.

### 2.9.2 Options

European-type contracts on foreign exchange are special cases of general European contracts where the roles of the no-arbitrage assets $X$ and $Y$ are played by no-arbitrage assets $B^{€,T}$ and $B^T$. For instance, a call option with payoff

$$(€(T) - K \cdot \$(T))^+$$

can be rewritten in terms of the no-arbitrage assets as

$$(B^{€,T}(T) - K \cdot B^T(T))^+.$$

We have a special case of the Black–Scholes formula that is also known as Garman–Kohlhagen formula.

**Remark 2.25 (Garman–Kohlhagen formula)**
*The value* $V^{EC}(t, B^{€,T}, KB^T, T)$ *of a European option contract with a payoff* $\left(B^{€,T}(T) - K \cdot B^T(T)\right)^+$ *is given by*

$$\begin{aligned} V^{EC}(t, B^{€,T}, KB^T, T) &= \mathbb{P}_t^{€,T}(B_{B^T}^{€,T}(T) \geq K) \cdot B^{€,T}(t) \\ &\qquad - K \cdot \mathbb{P}_t^T(B_{B^T}^{€,T}(T) \geq K) \cdot B^T(t) \qquad (2.98) \\ &= \mathbb{P}_t^{€,T}(€_{\$}(T) \geq K) \cdot e^{-r^f(T-t)} \cdot €(T) \\ &\qquad - K \cdot \mathbb{P}_t^T(€_{\$}(T) \geq K) \cdot e^{-r(T-t)} \cdot \$(T). \end{aligned}$$

*Moreover, the corresponding deltas are given in the geometric Brownian motion model by*

$$\Delta^{€,T}(t) = \mathbb{P}_t^{€,T}(B_{B^T}^{€,T}(T) \geq K) = \mathbb{P}_t^{€,T}(€_{\$}(T) \geq K), \tag{2.99}$$

*and*

$$\Delta^T(t) = -K \cdot \mathbb{P}_t^T(B_{B^T}^{€,T}(T) \geq K) = -K \cdot \mathbb{P}_t^T(€_{\$}(T) \geq K). \tag{2.100}$$

# References and Further Reading

The first introduction of Brownian motion to finance is by Bachelier [3]. He used it as a model for stock prices, although Brownian motion can take negative values. The general theory of stochastic calculus was developed by Ito [46]. Merton [61] was the first person who used it in finance. Samuelson [72, 73] argued that geometric Brownian motion is a good model for stock prices. The first derivation of the Black–Scholes formula appears in Black and Scholes [8]. A similar result appears in the independent work of Merton [60]. The Black–Scholes formula is quite robust to model misspecifications as shown in El Karoui et al. [25]. Garman and Kohlhagen [29] found the analogous formula to the Black–Scholes for currency options.

Girsanov's theorem is due to Girsanov [33], although the result for the constant $\sigma$ appeared already in Cameron and Martin [13]. The principle of exchangeability of the reference assets appears implicitly already in Carr and Bowie [14], and more recently the symmetries between the pricing martingale measures are explored in detail in Carr and Lee [15]. Hoogland and Neumann [42, 43] explored the symmetries in pricing with respect to different reference assets using the partial differential equation approach. They noticed the advantages of using no-arbitrage assets for the numeraire. Wystup [86] applied symmetry analysis in the foreign exchange market and showed various relationships of greeks for specific options. Preservation of convexity by the perspective mapping is shown for instance in Hiriart-Urruty and Lemarechal [40].

Books that explain the numerical implementation of partial differential equations in detail are, for instance, Tavella and Randall [78] or Duffy [23]. Monte Carlo methods for pricing financial derivatives are employed for instance in the papers of Boyle [10], Boyle et al. [9], Brodie and Glasserman [12], or in the monographs Glasserman [34], Jaeckel [47], or Korn et al. [54]. Books on pricing derivative contracts under stochastic volatility include Lewis [55], Gatheral [30], Fouque et al. [28], or Rebonato [70]. The Heston stochastic volatility model was introduced in Heston [39].

# Chapter 3

# Asian Options

Asian options are contracts that depend on underlying assets $X$ and $Y$ and upon the average of the price process $X_Y(t)$. The average price process is captured by a no-arbitrage contract $A$ called the **average asset.** The payoff of the average asset is defined as

$$A(T) = \left[\int_0^T X_Y(t)\mu(dt)\right] \cdot Y(T). \tag{3.1}$$

The average asset is a contract that pays off a number of units of an asset $Y$, where the number of units is the weighted average price of an asset $X$ with respect to the asset $Y$. The weights are determined by the weighting measure $\mu$ which can represent both continuous or discrete averaging. Our definition of the average asset guarantees that its price is always positive, and thus the average asset can be used as a numeraire. The average asset is analogous to the maximal asset $M^*$ that appears in pricing of lookback options. The important difference is that the average asset $A$ turns out to be a no-arbitrage asset in contrast to the maximal asset $M^*$.

The average asset is typically not traded, but we can still use it as a numeraire in order to derive the pricing equations for Asian options. The pricing techniques for Asian options do not require the existence of the average asset as a traded contract. We will express all hedging positions in terms of assets $X$ and $Y$ only. Moreover as we will show in the following text, the Asian forward can be perfectly replicated by trading in the underlying assets $X$ and $Y$, and the hedge is model independent. Therefore $A$ itself is a no-arbitrage asset.

We can apply the First Fundamental Theorem of Asset Pricing as long as the assets $X$ and $Y$ are no-arbitrage assets. This is not the case when $X$ is a stock $S$ and $Y$ is dollars \$, when the average asset contract becomes

$$A(T) = \left[\int_0^T S_\$(t)\mu(dt)\right] \cdot \$(T).$$

However, we can still rewrite this contract in terms of no-arbitrage assets when the bond price follows a deterministic term structure $B^T(t) = e^{-r(T-t)}\$(t)$ as

$$A(T) = \left[\int_0^T S_{B^T}(t)e^{-r(T-t)}\mu(dt)\right] \cdot B^T(T),$$

which is of the form of (3.1), with the underlying two no-arbitrage assets $S$ and $B^T$. Note that hedging must be done in no-arbitrage assets exclusively as opposed to arbitrage assets such as currencies. A typical Asian option contract uses equal weights. A continuously sampled average asset pays off

$$A(T) = \tfrac{1}{T}\left[\int_0^T S_\$(t)dt\right] \cdot \$(T),$$

which corresponds to an averaging of the form

$$\mu(dt) = \tfrac{1}{T}e^{-r(T-t)}dt,$$

when expressed in terms of $S$ and $B^T$. A discretely sampled average asset pays off

$$A(T) = \tfrac{1}{n}\sum_{k=1}^{n} S_\$(\tfrac{k}{n}T),$$

which corresponds to an averaging of the form

$$\mu(dt) = \tfrac{1}{n}\sum_{k=1}^{n} \delta_{(\frac{k}{n}T)}(t)e^{-r(T-t)}dt$$

when expressed in terms of $S$ and $B^T$.

Let us define the most general form of an **Asian option.**

**Definition 3.1** *An Asian option is a contract that pays off one of the following:*

- $f^Y\left(X_Y(T), A_Y(T)\right)$ *units of an asset* $Y$*,*
- $f^X\left(Y_X(T), A_X(T)\right)$ *units of an asset* $X$*,*
- $f^A\left(X_A(T), Y_A(T)\right)$ *units of an asset* $A$*.*

When the payoff functions are linked by the perspective mapping $f^Y(x,y) = f^X(\frac{1}{x}, \frac{y}{x}) \cdot x = f^A(\frac{x}{y}, \frac{1}{y}) \cdot y$, the three payoffs represent the same contract.

**Example 3.2** *The Asian call option with a* **fixed strike** *pays off*

$$(A(T) - K \cdot Y(T))^+. \tag{3.2}$$

*This corresponds to the payoff functions* $f^Y(x,y) = (y-K)^+$, $f^X(x,y) = (y-K\cdot x)^+$*, or* $f^A(x,y) = (1 - K \cdot y)^+$ *in the above definition of the Asian option. This means that the payoff can be settled in three equivalent ways:*

$$(A_Y(T) - K)^+ \cdot Y = (A_X(T) - K \cdot Y_X(T))^+ \cdot X = (1 - K \cdot Y_A(T))^+ \cdot A.$$

*The Asian call option with a* **floating strike** *pays off*

$$\left(A(T) - K \cdot X(T)\right)^+, \tag{3.3}$$

*which corresponds to the payoff functions* $f^Y(x,y) = (y - K \cdot x)^+$, $f^X(x,y) = (y - K)^+$, *or* $f^A(x,y) = (1 - K \cdot x)^+$. *The payoff can be settled in the following three ways:*

$$(A_Y(T) - KX_Y(T))^+ \cdot Y = (A_X(T) - K)^+ \cdot X = (1 - K \cdot X_A(T))^+ \cdot A.$$

*Asian options with the fixed or the floating strike are the two most typical Asian option contracts.*

*It is interesting to note that the prices of the Asian fixed strike and the Asian floating strike options can be written as a Black–Scholes formula. The price of the fixed strike option is simply*

$$\mathbb{P}_t^A(A_Y(T) \geq K) - K \cdot \mathbb{P}_t^Y(A_Y(T) \geq K), \tag{3.4}$$

*and the price of the floating strike option is*

$$\mathbb{P}_t^A(A_X(T) \geq K) - K \cdot \mathbb{P}_t^X(A_X(T) \geq K). \tag{3.5}$$

*This follows from the fact that the Asian option can be written as a combination of two digital options whose price is given by the above expressions. However, the hard part is that the prices* $A_Y(T)$ *and* $A_X(T)$ *do not have a simple analytical distribution as opposed to the case of* $X_Y(T)$ *which has a known density, and thus determination of the corresponding probabilities is a nontrivial task. Semianalytical representations of these probabilities exist for continuous averaging, but they still require significant computational effort to obtain any numerical result. In our text we present the partial differential equations that correspond to the Asian option pricing problem which applies to both discrete and continuous averaging. These partial differential equations can be solved numerically in a straightforward way.*

*The foreign exchange market also trades contracts written on the* **harmonic average** *of the price. The harmonic average is defined as the reciprocal of the arithmetic average of the reciprocals:*

$$\frac{1}{\int_0^T \frac{1}{X_Y(t)}\mu(dt)} = \frac{1}{\int_0^T Y_X(t)\mu(dt)}.$$

*If we denote by*

$$\tilde{A}(T) = \left[\int_0^T Y_X(t)\mu(dt)\right] \cdot X(T)$$

*the average asset where the roles of the assets* $X$ *and* $Y$ *are flipped, we can define the harmonic average asset as*

$$H(T) = \left[\frac{1}{\int_0^T Y_X(t)\mu(dt)}\right] \cdot Y(T) = \frac{1}{\tilde{A}_X(T)} \cdot Y(T).$$

*Natural contracts to consider are the harmonic Asian option with a fixed strike with payoff*

$$(H(T) - K \cdot Y(T))^+ = \left(\frac{1}{\tilde{A}_X(T)} \cdot Y(T) - K \cdot Y(T)\right)^+$$

*and the harmonic Asian option with a floating strike with payoff*

$$(H(T) - K \cdot X(T))^+ = \left( \frac{1}{\tilde{A}_X(T)} \cdot Y(T) - K \cdot X(T) \right)^+ .$$

*We can also write the payoffs in terms of the original average asset $A(T)$ if we flip the roles of the assets $Y$ and $X$ (it is just a matter of naming the assets). In this case the harmonic Asian option with a fixed strike has payoff*

$$\left( \tfrac{1}{A_Y(T)} \cdot X(T) - K \cdot X(T) \right)^+ , \tag{3.6}$$

*which corresponds to the payoff functions $f^Y(x, y) = (\frac{x}{y} - K \cdot x)^+$, $f^X(x, y) = (\frac{x}{y} - K)^+$, and $f^A(x, y) = (x \cdot y - K \cdot x)^+$. The harmonic Asian option with a floating strike has payoff*

$$\left( \tfrac{1}{A_Y(T)} \cdot X(T) - K \cdot Y(T) \right)^+ , \tag{3.7}$$

*which corresponds to the payoff functions $f^Y(x, y) = (\frac{x}{y} - K)^+$, $f^X(x, y) = (\frac{x}{y} - K \cdot x)^+$, and $f^A(x, y) = (x \cdot y - K \cdot y)^+$.*

*We can also consider more exotic payoffs, such as Asian powers $f^X(x, y) = y^\alpha$. The advantage of this contract is that it admits a closed form solution for integer valued $\alpha$, and thus it can be used for calibrating numerical schemes. This payoff corresponds to $f^Y(x, y) = y^\alpha \cdot x^{1-\alpha}$, or equivalently to $f^A(x, y) = x^{1-\alpha}$. We can write the payoff as*

$$A_X(T)^\alpha \cdot X = A_Y(T)^\alpha \cdot (X_Y(T))^{1-\alpha} \cdot Y = (X_A(T))^{1-\alpha} \cdot A.$$

Let $V$ denote an Asian option contract. The price of this contract can be expressed in the following ways:

$$V = V_Y(t) \cdot Y = V_X(t) \cdot X = V_A(t) \cdot A.$$

In the Markovian model, we can also write

$$\begin{aligned} V(t) = u^Y\left(t, X_Y(t), A_Y(t)\right) \cdot Y = u^X\left(t, Y_X(T), A_X(T)\right) \cdot X \\ = u^A\left(t, X_A(T), Y_A(T)\right) \cdot A, \end{aligned}$$

giving us the following relationships between $u^Y$, $u^X$, and $u^A$ via the perspective mapping:

$$u^Y(t, x, y) = u^X(t, \tfrac{1}{x}, \tfrac{y}{x}) \cdot x, \quad u^X(t, x, y) = u^Y(t, \tfrac{1}{x}, \tfrac{y}{x}) \cdot x, \tag{3.8}$$

$$u^Y(t, x, y) = u^A(t, \tfrac{x}{y}, \tfrac{1}{y}) \cdot y, \quad u^A(t, x, y) = u^Y(t, \tfrac{x}{y}, \tfrac{1}{y}) \cdot y, \tag{3.9}$$

$$u^X(t, x, y) = u^A(t, \tfrac{1}{y}, \tfrac{x}{y}) \cdot y, \quad u^A(t, x, y) = u^X(t, \tfrac{y}{x}, \tfrac{1}{x}) \cdot x. \tag{3.10}$$

When $X$ and $Y$ are no-arbitrage assets, then $A$ is a no-arbitrage asset (shown below), and from the First Fundamental Theorem of the Asset Pricing we have the following stochastic representations:

$$\begin{aligned} u^Y(t,x,y) &= \mathbb{E}^Y\left[V_Y(T)|X_Y(t)=x, A_Y(t)=y\right] \\ &= \mathbb{E}^Y\left[f^Y\left(X_Y(T), A_Y(T)\right)|X_Y(t)=x, A_Y(t)=y\right], \end{aligned} \tag{3.11}$$

$$\begin{aligned} u^X(t,x,y) &= \mathbb{E}^X\left[V_X(T)|Y_X(t)=x, A_X(t)=y\right] \\ &= \mathbb{E}^X\left[f^X\left(Y_X(T), A_X(T)\right)|Y_X(t)=x, A_X(t)=y\right], \end{aligned} \tag{3.12}$$

$$\begin{aligned} u^A(t,x,y) &= \mathbb{E}^A\left[V_A(T)|X_A(t)=x, Y_A(t)=y\right] \\ &= \mathbb{E}^A\left[f^A\left(X_A(T), Y_A(T)\right)|X_A(t)=x, Y_A(t)=y\right]. \end{aligned} \tag{3.13}$$

Let us show that the average asset $A$ is indeed a no-arbitrage asset.

**Theorem 3.3** *Let $X$ and $Y$ be two no-arbitrage assets. Then the replicating portfolio for the average asset contract that pays off*

$$A(T) = \left[\int_0^T X_Y(t)\mu(dt)\right] \cdot Y(T) \tag{3.14}$$

*is given by*

$$A(t) = \left[\int_t^T \mu(ds)\right] \cdot X + \left[\int_0^t X_Y(s)\mu(ds)\right] \cdot Y. \tag{3.15}$$

*This result does not depend on the dynamics of the price $X_Y(t)$. In particular,*

$$dA_Y(t) = \left[\int_t^T \mu(ds)\right] dX_Y(t). \tag{3.16}$$

Proof: Let $A(t) = \bar{\Delta}^X(t)X(t) + \bar{\Delta}^Y(t)Y(t)$ be the replicating portfolio of the average asset. Then

$$dA_Y(t) = \bar{\Delta}^X(t)dX_Y(t).$$

Using the product rule, this can be rewritten as

$$dA_Y(t) = \bar{\Delta}^X(t)dX_Y(t) = d\left(\bar{\Delta}^X(t)\cdot X_Y(t)\right) - X_Y(t)d\bar{\Delta}^X(t).$$

Integrating this equation, we get

$$A_Y(T) = A_Y(0) + \bar{\Delta}^X(T)\cdot X_Y(T) - \bar{\Delta}^X(0)\cdot X_Y(0) - \int_0^T X_Y(t)d\bar{\Delta}^X(t).$$

Since the terminal position of the average asset is completely invested in the asset $Y$, and has a zero position in the asset $X$, we have $\bar{\Delta}^X(T) = 0$. We thus have the following identity:

$$\left(\int_0^T X_Y(t)\mu(dt)\right) = A_Y(0) - \bar{\Delta}^X(0)\cdot X_Y(0) - \int_0^T X_Y(t)d\bar{\Delta}^X(t).$$

The only way to match the payoff is when

$$0 = A_Y(0) - \bar{\Delta}^X(0) \cdot X_Y(0),$$

which is equivalent to

$$A(0) = \bar{\Delta}^X(0)X(0),$$

and

$$\int_0^T X_Y(t)\mu(dt) = -\int_0^T X_Y(t)d\bar{\Delta}^X(t).$$

This implies

$$-d\bar{\Delta}^X(t) = \mu(dt),$$

which is the same as

$$\bar{\Delta}^X(t) = -\int_t^T d\bar{\Delta}^X(s) = \int_t^T \mu(ds).$$

The hedging position $\bar{\Delta}^Y(t)$ in the asset $Y$ follows from the identity

$$A_Y(t) \quad = \quad \bar{\Delta}^X(t)X_Y(t) + \int_0^t X_Y(s)\mu(ds)$$

which concludes the proof.

**Remark 3.4** *Note that the hedging position $\bar{\Delta}^X(t)$ in the asset $X$ is deterministic:*

$$\boxed{\bar{\Delta}^X(t) = \int_t^T \mu(ds).} \tag{3.17}$$

*For instance, when $\mu(dt) = \frac{1}{T}e^{-r(T-t)}dt$, we get*

$$\bar{\Delta}^X(t) = \int_t^T \mu(ds) = \int_t^T \tfrac{1}{T}e^{-r(T-s)}ds = \tfrac{1}{rT}\left(1 - e^{-r(T-t)}\right).$$

*In the case of uniform weighting $\mu(dt) = \frac{1}{T}dt$, the hedge of the average asset simplifies to*

$$\bar{\Delta}^X(t) = \int_t^T \mu(ds) = \int_t^T \tfrac{1}{T}ds = \left(1 - \tfrac{t}{T}\right).$$

*For discrete averaging when $\mu(dt) = \frac{1}{n}\sum_{k=1}^n \delta_{(\frac{k}{n}T)}(t)e^{-r(T-t)}dt$, we get*

$$\begin{aligned}\bar{\Delta}^X(t) = \int_t^T \mu(ds) = \tfrac{1}{n}\int_t^T \sum_{k=1}^n \delta_{(\frac{k}{n}T)}(s)e^{-r(T-s)}ds& \\ = \tfrac{1}{n}\sum_{k=\left[\frac{nt}{T}\right]+1}^n \exp\left(-r(\tfrac{n-k}{n})T\right),&\end{aligned}$$

*where $[\cdot]$ denotes the integer part function. This simplifies to*

$$\bar{\Delta}^X(t) = 1 - \tfrac{1}{n}\left[n\tfrac{t}{T}\right] \tag{3.18}$$

*when the averaging is uniform, i.e. when $\mu(dt) = \frac{1}{n}\sum_{k=1}^n \delta_{(\frac{k}{n}T)}(t)dt$.*

The insight of this result is the following: the trader who is replicating the average asset contract starts with a hedging portfolio of $\int_0^T \mu(dt)$ units of $X$ and no units of $Y$:

$$\bar{\Delta}(0) = (\bar{\Delta}^X(0), \bar{\Delta}^Y(0)) = \left( \int_0^T \mu(dt), 0 \right).$$

The amount of $\int_0^T \mu(dt)$ units of the asset $X$ is used for replicating the average of the price. The trader then gradually liquidates his position in the asset $X$, keeping just $\int_t^T \mu(dt)$ fraction of it at time $t$, and the rest of the portfolio is invested in the asset $Y$. The position in the asset $Y$ corresponds to the running average $\int_0^t X_Y(s)\mu(ds)$. At the final time $T$, the hedge becomes

$$\bar{\Delta}(T) = (\bar{\Delta}^X(T), \bar{\Delta}^Y(T)) = \left( 0, \int_0^T X_Y(t)\mu(dt) \right),$$

so the asset $X$ is completely unloaded, and the position in the asset $Y$ is the number that corresponds to the average price.

## 3.1 Pricing in the Geometric Brownian Motion Model

The prices of assets should be martingales under their corresponding numeraire measures. Since we have three underlying assets $X$, $Y$ and $A$, we have six price processes to consider: $X_Y(t)$, $A_Y(t)$, $Y_X(t)$, $A_X(t)$, $X_A(t)$, and $Y_A(t)$. The price processes $X_Y(t)$ and $A_Y(t)$ are $\mathbb{P}^Y$ martingales, the price processes $Y_X(t)$ and $A_X(t)$ are $\mathbb{P}^X$ martingales, and the price processes $X_A(t)$ and $Y_A(t)$ are $\mathbb{P}^A$ martingales.

In the geometric Brownian motion model we assume the following price dynamics:

$$\boxed{dX_Y(t) = \sigma X_Y(t) dW^Y(t),} \tag{3.19}$$

and a similar evolution for the inverse price

$$\boxed{dY_X(t) = \sigma Y_X(t) dW^X(t).} \tag{3.20}$$

The evolution of $A_Y(t)$ follows from the hedging formula for the average asset:

$$\boxed{dA_Y(t) = \bar{\Delta}^X(t) dX_Y(t) = \sigma \bar{\Delta}^X(t) X_Y(t) dW^Y(t).} \tag{3.21}$$

Note that this evolution is not Markovian in $A_Y(t)$ since it depends on another process $X_Y(t)$, but it is Markovian in the pair $(X_Y(t), A_Y(t))$. Thus even when the Asian option contract payoff depends only on $A_Y(t)$, the corresponding pricing partial differential equation would depend on both prices.

The evolution of the average asset price under the reference asset $X$ can be expressed as

$$\begin{aligned} dA_X(t) &= \bar{\Delta}^Y(t)dY_X(t) \\ &= \left[A_Y(t) - \bar{\Delta}^X(t) \cdot X_Y(t)\right] dY_X(t) \\ &= \left[A_Y(t) - \bar{\Delta}^X(t) \cdot X_Y(t)\right] \sigma Y_X(t) dW^X(t) \\ &= \sigma \left[A_X(t) - \bar{\Delta}^X(t)\right] dW^X(t). \end{aligned}$$

The second equality $\bar{\Delta}^Y(t) = A_Y(t) - \bar{\Delta}^X(t) \cdot X_Y(t)$ follows from the relationship $A(t) = \bar{\Delta}^X(t) \cdot X + \bar{\Delta}^Y(t) \cdot Y$. The reason to write the evolution of $A_X(t)$ in terms of $\bar{\Delta}^X(t)$ rather than in terms of $\bar{\Delta}^Y(t)$ is that $\bar{\Delta}^X(t)$ is deterministic, while $\bar{\Delta}^Y(t)$ is stochastic. This means that unlike the price evolution of $A_Y(t)$, the price evolution of $A_X(t)$ is Markovian in just one variable, and thus contracts whose payoff depends only on $A_X(T)$ admit a simpler partial differential equation with one spatial variable. Thus

$$\boxed{dA_X(t) = \sigma \left[A_X(t) - \bar{\Delta}^X(t)\right] dW^X(t).} \tag{3.22}$$

Let us determine the evolution of the remaining prices: $Y_A(t)$, and $X_A(t)$. From Ito's formula we have

$$\begin{aligned} dY_A(t) = dA_Y(t)^{-1} &= -A_Y(t)^{-2} dA_Y(t) + A_Y(t)^{-3} d^2 A_Y(t) \\ &= -Y_A(t)^2 \sigma \bar{\Delta}^X(t) X_Y(t) dW^Y(t) \\ &\quad + Y_A(t)^3 \sigma^2 \bar{\Delta}^X(t)^2 X_Y(t)^2 dt \\ &= \sigma \bar{\Delta}^X(t) Y_A(t) X_A(t) \Big[ - dW^Y(t) + \sigma \bar{\Delta}^X(t) X_A(t) dt \Big]. \end{aligned}$$

According to the First Fundamental Theorem of Asset Pricing, the evolution of $Y_A(t)$ has to be a martingale under the corresponding $\mathbb{P}^A$ measure. Thus we have

$$\boxed{dY_A(t) = \sigma \bar{\Delta}^X(t) Y_A(t) X_A(t) dW^A(t),} \tag{3.23}$$

where $W^A(t)$ is a Brownian motion under $\mathbb{P}^A$ measure. Similarly,

$$\begin{aligned} dX_A(t) &= dA_X(t)^{-1} = -A_X(t)^{-2} dA_X(t) + A_X(t)^{-3} d^2 A_X(t) \\ &= -X_A(t)^2 \sigma \left[A_X(t) - \bar{\Delta}^X(t)\right] dW^X(t) \\ &\qquad + X_A(t)^3 \sigma^2 \left[A_X(t) - \bar{\Delta}^X(t)\right]^2 dt \\ &= \sigma X_A(t) \cdot \left[\bar{\Delta}^X(t) X_A(t) - 1\right] \cdot \Big[ dW^X(t) - \sigma \left[1 - \bar{\Delta}^X(t) X_A(t)\right] dt \Big]. \end{aligned}$$

Therefore

$$\boxed{dX_A(t) = \sigma X_A(t) \left[\bar{\Delta}^X(t) X_A(t) - 1\right] dW^A(t),} \tag{3.24}$$

which is a martingale under the $\mathbb{P}^A$ measure.

The price of the Asian option is determined in the next theorem.

**Theorem 3.5** *The price function*

$$u^Y(t,x,y) = \mathbb{E}^Y\left[f^Y\left(X_Y(T), A_Y(T)\right)|X_Y(t)=x, A_Y(t)=y\right],$$

*satisfies partial differential equation*

$$u_t^Y(t,x,y) + \tfrac{1}{2}\sigma^2 x^2\Big[u_{xx}^Y(t,x,y) + 2\bar{\Delta}^X(t)u_{xy}^Y(t,x,y) + \bar{\Delta}^X(t)^2 u_{yy}^Y(t,x,y)\Big] = 0 \quad (3.25)$$

*with the terminal condition*

$$u^Y(T,x,y) = f^Y(x,y). \quad (3.26)$$

*The price function*

$$u^X(t,x,y) = \mathbb{E}^X\left[f^X\left(Y_X(T), A_X(T)\right)|Y_X(t)=x, A_X(t)=y\right],$$

*satisfies partial differential equation*

$$u_t^X(t,x,y) + \tfrac{1}{2}\sigma^2\Big[x^2 u_{xx}^X(t,x,y) + 2x(y-\bar{\Delta}^X(t))u_{xy}^X(t,x,y) + (y-\bar{\Delta}^X(t))^2 u_{yy}^X(t,x,y)\Big] = 0, \quad (3.27)$$

*with the terminal condition*

$$u^X(T,x,y) = f^X(x,y). \quad (3.28)$$

*The price function*

$$u^A(t,x,y) = \mathbb{E}^A\left[f^A\left(X_A(T), Y_A(T)\right)|X_A(t)=x, Y_A(t)=y\right]$$

*satisfies partial differential equation*

$$u_t^A(t,x,y) + \tfrac{1}{2}\sigma^2 x^2\Big([x\bar{\Delta}^X(t)-1]^2\cdot u_{xx}^A(t,x,y) + 2y\bar{\Delta}^X(t)[x\bar{\Delta}^X(t)-1]\cdot u_{xy}^A(t,x,y) + y^2(\bar{\Delta}^X(t))^2\cdot u_{yy}^A(t,x,y)\Big) = 0, \quad (3.29)$$

*with the terminal condition*

$$u^A(T,x,y) = f^A(x,y). \quad (3.30)$$

Proof: The price of the Asian option with respect to the reference asset $Y$, $u^Y(t, X_Y(t), A_Y(t))$, is a $\mathbb{P}^Y$ martingale, and thus $du^Y$ has a zero $dt$ term. Using Ito's formula, we get

$$\begin{aligned} du^Y &= u_t^Y dt + u_x^Y dX_Y(t) + u_y^Y dA_Y(t) \\ &\quad + \tfrac{1}{2}\left[u_{xx}^Y d^2X_Y(t) + 2u_{xy}^Y dX_Y(t)dA_Y(t) + u_{yy}^Y d^2A_Y(t)\right] \\ &= [u_t^Y + \tfrac{1}{2}\sigma^2x^2(u_{xx}^Y + 2\bar{\Delta}^X(t)u_{xy}^Y + \bar{\Delta}^X(t)^2u_{yy}^Y)]dt \\ &\quad + u_x^Y dX_Y(t) + u_y^Y dA_Y(t). \end{aligned}$$

Since the $dt$ term is zero, we obtain the following partial differential equation:

$$u_t^Y(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 \left[u_{xx}^Y(t,x,y) + 2\bar{\Delta}^X(t) u_{xy}^Y(t,x,y) + \bar{\Delta}^X(t)^2 u_{yy}^Y(t,x,y)\right] = 0.$$

The terminal condition is given by

$$u^Y(T,x,y) = f^Y(x,y).$$

Similarly, the price of the Asian option with respect to the reference asset $X$, $u^X(t, Y_X(t), A_X(t))$, is a $\mathbb{P}^X$ martingale, and thus the $dt$ term of $du^X$ is zero. Using the evolution of the price of the average asset under the reference asset $X$, we get

$$\begin{aligned} du^X &= u_t^X dt + u_x^X dY_X(t) + u_y^X dA_X(t) \\ &\quad + \tfrac{1}{2}\left[u_{xx}^X d^2 Y_X(t) + 2u_{xy}^X dY_X(t) dA_X(t) + u_{yy}^X d^2 A_X(t)\right] \\ &= [u_t^X + \tfrac{1}{2}\sigma^2 (x^2 u_{xx}^X + 2x(y - \bar{\Delta}^X(t)) u_{xy}^X + (y - \bar{\Delta}^X(t))^2 u_{yy}^X)] dt \\ &\quad + u_x^X dY_X(t) + u_y^X dA_X(t). \end{aligned}$$

Since the $dt$ term is zero, we have the following partial differential equation

$$u_t^X(t,x,y) + \tfrac{1}{2}\sigma^2 \Big[x^2 u_{xx}^X(t,x,y) + 2x(y - \bar{\Delta}^X(t)) u_{xy}^X(t,x,y) + (y - \bar{\Delta}^X(t))^2 u_{yy}^X(t,x,y)\Big] = 0,$$

with the terminal condition

$$u^X(T,x,y) = f^X(x,y).$$

Finally, the price of the Asian option with respect to the reference asset $A$, $u^A(t, X_A(t), Y_A(t))$, is a $\mathbb{P}^A$ martingale, and thus the $dt$ term of $du^A$ is zero. Using the evolution of the prices of $X$ and $Y$ under the reference asset $A$, we get

$$\begin{aligned} du^A &= u_t^A dt + u_x^A dX_A(t) + u_y^A dY_A(t) \\ &\quad + \tfrac{1}{2}\left[u_{xx}^A d^2 X_A(t) + 2u_{xy}^A dX_A(t) dY_A(t) + u_{yy}^A d^2 Y_A(t)\right] \\ &= \Big[u_t^A + \tfrac{1}{2}\sigma^2 x^2 [[x\bar{\Delta}^X(t) - 1]^2 \cdot u_{xx}^A \\ &\quad + 2y\bar{\Delta}^X(t)[x\bar{\Delta}^X(t) - 1] \cdot u_{xy}^A + y^2 \bar{\Delta}^X(t)^2 \cdot u_{yy}^A]\Big] dt \\ &\quad + u_x^A dX_A(t) + u_y^A dY_A(t). \end{aligned}$$

Since the $dt$ term is zero, we have the following partial differential equation

$$u_t^A(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 \Big([x\bar{\Delta}^X(t) - 1]^2 \cdot u_{xx}^A(t,x,y) + 2y\bar{\Delta}^X(t)[x\bar{\Delta}^X(t) - 1] \cdot u_{xy}^A(t,x,y) + y^2 \bar{\Delta}^X(t)^2 \cdot u_{yy}^A(t,x,y)\Big) = 0,$$

with the terminal condition

$$u^A(T,x,y) = f^A(x,y). \tag{3.31}$$

## 3.2 Hedging of Asian Options

Since Asian options depend on three assets: $X$, $Y$, and the Asian forward $A$, the hedge should take positions in all these assets. The hedging portfolio should be of the form

$$P(t) = \Delta^X(t) \cdot X + \Delta^Y(t) \cdot Y + \Delta^A(t) \cdot A. \tag{3.32}$$

However, the average asset itself can be hedged by assets $X$ and $Y$:

$$A(t) = \bar{\Delta}^X(t) \cdot X + \bar{\Delta}^Y(t) \cdot Y, \tag{3.33}$$

and thus the Asian option hedge can be reduced to positions in just two assets, $X$ and $Y$:

$$P(t) = [\Delta^X(t) + \Delta^A(t) \cdot \bar{\Delta}^X(t)] \cdot X + [\Delta^Y(t) + \Delta^A(t) \cdot \bar{\Delta}^Y(t)] \cdot Y. \tag{3.34}$$

The hedging position in the underlying assets $X$ and $Y$ has two components: one part ($\Delta^X(t)$ or $\Delta^Y(t)$) represents the usual delta sensitivity of the Asian option price with respect to the price of the underlying asset, and the other part represents the delta sensitivity of the Asian option price with respect to the average asset price ($\Delta^A(t)$), multiplied by the hedge of the average asset in terms of the assets $X$ and $Y$ ($\bar{\Delta}^X(t)$, or $\bar{\Delta}^Y(t)$). This feature is rather unique among contingent claims. The exact forms of the hedging portfolio are given in the following theorem. Recall that

$$\bar{\Delta}^X(t) = \int_t^T \mu(ds),$$

and

$$\bar{\Delta}^Y(t) = \int_0^t X_Y(s)\mu(ds).$$

**Theorem 3.6** *The hedging portfolio $P(t)$ of the Asian option admits each of the following equivalent represenations:*

$$\begin{aligned} P(t) = &\Big[u_x^Y(t, X_Y(t), A_Y(t)) + \bar{\Delta}^X(t) \cdot u_y^Y(t, X_Y(t), A_Y(t))\Big] \cdot X \\ &+ \Big[u^Y(t, X_Y(t), A_Y(t)) - X_Y(t) \cdot u_x^Y(t, X_Y(t), A_Y(t)) \\ &\qquad + \big(\bar{\Delta}^Y(t) - A_Y(t)\big) \cdot u_y^Y(t, X_Y(t), A_Y(t))\Big] \cdot Y, \end{aligned} \tag{3.35}$$

$$\begin{aligned} P(t) = &\Big[u^X(t, Y_X(t), A_X(t)) - Y_X(t) \cdot u_x^X(t, Y_X(t), A_X(t)) \\ &\qquad + \big(\bar{\Delta}^X(t) - A_X(t)\big) \cdot u_y^X(t, Y_X(t), A_X(t))\Big] \cdot X \\ &+ \Big[u_x^X(t, Y_X(t), A_X(t)) + \bar{\Delta}^Y(t) \cdot u_y^X(t, Y_X(t), A_X(t))\Big] \cdot Y, \end{aligned} \tag{3.36}$$

$$P(t) = \Bigg[\Big[u^A\left(t, X_A(t), Y_A(t)\right) - u_y^A\left(t, X_A(t), Y_A(t)\right) \cdot Y_A(t)\Big] \cdot \bar{\Delta}^X(t)$$
$$+ u_x^A\left(t, X_A(t), Y_A(t)\right) \cdot \Big[1 - \bar{\Delta}^X(t) X_A(t)\Big]\Bigg] \cdot X$$
$$+ \Bigg[\Big[u^A\left(t, X_A(t), Y_A(t)\right) - u_x^A\left(t, X_A(t), Y_A(t)\right) \cdot X_A(t)\Big] \cdot \bar{\Delta}^Y(t)$$
$$+ u_y^A\left(t, X_A(t), Y_A(t)\right) \cdot \Big[1 - \bar{\Delta}^Y(t) Y_A(t)\Big]\Bigg] \cdot Y. \quad (3.37)$$

PROOF: Let us find a hedge for the Asian option of the form

$$P(t) = \Delta^X(t) \cdot X + \Delta^Y(t) \cdot Y.$$

Using the fact that the process $u^Y\left(t, X_Y(t), A_Y(t)\right)$ has a zero $dt$ term, we get

$$\begin{aligned} du^Y &= u_x^Y \cdot dX_Y(t) + u_y^Y \cdot dA_Y(t) \\ &= \left(u_x^Y + \bar{\Delta}^X(t) u_y^Y\right) \cdot dX_Y(t). \end{aligned}$$

Thus the hedging position in the asset $X$ is given by the formula

$$\Delta^X\left(t, X_Y(t), A_Y(t)\right) = u_x^Y\left(t, X_Y(t), A_Y(t)\right) + \bar{\Delta}^X(t) u_y^Y\left(t, X_Y(t), A_Y(t)\right). \quad (3.38)$$

Similarly, using the evolution of $u^X\left(t, Y_X(t), A_X(t)\right)$

$$\begin{aligned} du^X &= u_x^X \cdot dY_X(t) + u_y^X \cdot dA_X(t) \\ &= \left(u_x^X + \bar{\Delta}^Y(t) \cdot u_y^X\right) \cdot dY_X(t), \end{aligned}$$

we get the following representation of the hedging position in the asset $Y$:

$$\Delta^Y\left(t, Y_X(t), A_X(t)\right) = u_x^X\left(t, Y_X(t), A_X(t)\right) + \bar{\Delta}^Y(t) u_y^X\left(t, Y_X(t), A_X(t)\right). \quad (3.39)$$

Therefore the hedging portfolio takes the following form

$$\begin{aligned} P(t) = &\left[u_x^Y\left(t, X_Y(t), A_Y(t)\right) + \bar{\Delta}^X(t) u_y^Y\left(t, X_Y(t), A_Y(t)\right)\right] \cdot X \\ &+ \left[u_x^X\left(t, Y_X(t), A_X(t)\right) + \bar{\Delta}^Y(t) u_y^X\left(t, Y_X(t), A_X(t)\right)\right] \cdot Y. \quad (3.40) \end{aligned}$$

We can also rewrite the above representation of the hedging portfolio using the function $u^Y$ of the function $u^X$ only. From

$$u^X(t, x, y) = u^Y(t, \tfrac{1}{x}, \tfrac{y}{x}) \cdot x,$$

we get

$$u_x^X(t, x, y) = u^Y(t, \tfrac{1}{x}, \tfrac{y}{x}) - \tfrac{1}{x} \cdot u_x^Y(t, \tfrac{1}{x}, \tfrac{y}{x}) - \tfrac{y}{x} \cdot u_y^Y(t, \tfrac{1}{x}, \tfrac{y}{x}),$$

and

$$u_y^X(t, x, y) = u_y^Y(t, \tfrac{1}{x}, \tfrac{y}{x}).$$

Substituting into (3.40), we get

$$P(t) = \Big[ u^Y_x\left(t, X_Y(t), A_Y(t)\right) + \bar{\Delta}^X(t) \cdot u^Y_y\left(t, X_Y(t), A_Y(t)\right) \Big] \cdot X$$
$$+ \Big[ u^Y\left(t, X_Y(t), A_Y(t)\right) - X_Y(t) \cdot u^Y_x\left(t, X_Y(t), A_Y(t)\right)$$
$$+ \left(\bar{\Delta}^Y(t) - A_Y(t)\right) \cdot u^Y_y\left(t, X_Y(t), A_Y(t)\right) \Big] \cdot Y.$$

Similarly, from

$$u^Y(t,x,y) = u^X(t, \tfrac{1}{x}, \tfrac{y}{x}) \cdot x,$$

we get

$$u^Y_x(t,x,y) = u^X(t, \tfrac{1}{x}, \tfrac{y}{x}) - \tfrac{1}{x} \cdot u^X_x(t, \tfrac{1}{x}, \tfrac{y}{x}) - \tfrac{y}{x} \cdot u^X_y(t, \tfrac{1}{x}, \tfrac{y}{x}),$$

and

$$u^Y_y(t,x,y) = u^X_y(t, \tfrac{1}{x}, \tfrac{y}{x}).$$

Substituting to (3.40), we get

$$P(t) = \Big[ u^X\left(t, Y_X(t), A_X(t)\right) - Y_X(t) \cdot u^X_x\left(t, Y_X(t), A_X(t)\right)$$
$$+ \left(\bar{\Delta}^X(t) - A_X(t)\right) \cdot u^X_y\left(t, Y_X(t), A_X(t)\right) \Big] \cdot X$$
$$+ \Big[ u^X_x\left(t, Y_X(t), A_X(t)\right) + \bar{\Delta}^Y(t) \cdot u^X_y\left(t, Y_X(t), A_X(t)\right) \Big] \cdot Y.$$

Finally, from

$$du^A = u^A_x dX_A(t) + u^A_y dY_A(t),$$

we get a hedging portfolio representation of the form

$$P(t) = u^A \cdot A(t) = u^A_x \cdot X(t) + u^A_y \cdot Y(t)$$
$$+ \left[u^A - X_A(t) \cdot u^A_x - Y_A(t) \cdot u^A_y\right] \cdot A(t).$$

Using the fact that $A(t) = \bar{\Delta}^X(t) \cdot X + \bar{\Delta}^Y(t) \cdot Y$, we conclude that

$$P(t) = \Big[ \Big[ u^A\left(t, X_A(t), Y_A(t)\right) - u^A_y\left(t, X_A(t), Y_A(t)\right) \cdot Y_A(t) \Big] \cdot \bar{\Delta}^X(t)$$
$$+ u^A_x\left(t, X_A(t), Y_A(t)\right) \cdot \Big[ 1 - \bar{\Delta}^X(t) X_A(t) \Big] \Big] \cdot X$$
$$+ \Big[ \Big[ u^A\left(t, X_A(t), Y_A(t)\right) - u^A_x\left(t, X_A(t), Y_A(t)\right) \cdot X_A(t) \Big] \cdot \bar{\Delta}^Y(t)$$
$$+ u^A_y\left(t, X_A(t), Y_A(t)\right) \cdot \Big[ 1 - \bar{\Delta}^Y(t) Y_A(t) \Big] \Big] \cdot Y.$$

## 3.3 Reduction of the Pricing Equations

When the Asian option contract depends only on the assets $A$ and $X$, such as in the case of an Asian call option with a floating strike that has a payoff $(A(T) - K \cdot X(T))^+$, the option pricing problem depends only on the price process $A_X(t)$, and thus the corresponding partial differential equations depend only on one spatial variable. In this case the pricing equation (3.27) does not depend on the variable $x$ that represents the price $Y_X(t)$ that is irrelevant to this problem, and thus it reduces to the partial differential equation

$$\boxed{u_t^X(t,y) + \tfrac{1}{2}\sigma^2(y - \bar{\Delta}^X(t))^2 u_{yy}^X(t,y) = 0,} \tag{3.41}$$

with the terminal condition

$$u^X(T,y) = f^X(y), \tag{3.42}$$

where

$$u^X(t,y) = \mathbb{E}^X[f^X(A_X(T))|A_X(t) = y].$$

We keep the notation $y$ (as opposed to $x$) for the only spatial variable in order to be consistent with the pricing problem (3.27). Similarly, when $A$ is a reference asset and the payoff depends only on $X_A(T)$, the pricing equation (3.29) does not depend on the variable $y$, and the partial differential equation simplifies to

$$\boxed{u_t^A(t,x) + \tfrac{1}{2}\sigma^2 x^2 [x\bar{\Delta}^X(t) - 1]^2 \cdot u_{xx}^A(t,x) = 0,} \tag{3.43}$$

with the terminal condition

$$u^A(T,x) = f^A(x), \tag{3.44}$$

where

$$u^A(t,x) = \mathbb{E}^A[f^A(X_A(T))|X_A(t) = x].$$

The formulas for the hedging portfolio given in Equations (3.36) and (3.37) also simplify to

$$\begin{aligned} P(t) = &\left[u^X\left(t, A_X(t)\right) + \left(\bar{\Delta}^X(t) - A_X(t)\right) \cdot u_y^X\left(t, A_X(t)\right)\right] \cdot X \\ &+ \left[\bar{\Delta}^Y(t) \cdot u_y^X\left(t, A_X(t)\right)\right] \cdot Y, \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} P(t) = &\left[\bar{\Delta}^X(t) \cdot u^A\left(t, X_A(t)\right) + u_x^A\left(t, X_A(t)\right) \cdot \left[1 - \bar{\Delta}^X(t) X_A(t)\right]\right] \cdot X \\ &+ \left[\bar{\Delta}^Y(t) \cdot \left[u^A\left(t, X_A(t)\right) - u_x^A\left(t, X_A(t)\right) \cdot X_A(t)\right]\right] \cdot Y. \end{aligned} \tag{3.46}$$

The pricing equation (3.25) does not reduce in this case, and it is strictly suboptimal to employ it for pricing Asian options that do not depend on the asset $Y$.

When the contract depends on the assets $A$ and $Y$ only, such as in the case of the Asian call option with a fixed strike that has a payoff $(A(T) - K \cdot Y(T))^+$, the reduction of the pricing equations is possible only in special cases, not in general. The reason is that the evolution of the price process $A_Y(t)$ depends on both prices $A_Y(t)$ and $X_Y(t)$ (in contrast to the evolution of the price $A_X(t)$ that depends only on itself), and thus the partial differential equation (3.25) cannot be reduced to only one spatial variable. However, when the payoff of the contract is only a function of the asset $F$ known as the **Asian forward** defined as

$$F(T) = A(T) - K_1 Y(T), \tag{3.47}$$

a reduction of the pricing problem similar to Equation (3.41) is possible when the asset $X$ is taken as a numeraire. Consider a contract that pays off $f^X(F_X(T))$ units of an asset $X$, where $K_1$ in (3.47) is a constant. When the payoff function is given by $f^X(x) = (x - K_2)^+$, the contract that corresponds to it is

$$\begin{aligned}[F_X(T) - K_2]^+ \cdot X = (A_X(T) - K_1 Y_X(T) - K_2)^+ \cdot X& \\ = (A(T) - K_1 Y(T) - K_2 X(T))^+&,\end{aligned}$$

which covers both the floating strike option when $K_1 = 0$, and the fixed strike option when $K_2 = 0$.

Let us define

$$u^X(t,x) = \mathbb{E}^X[f^X(F_X(T))|F_X(t) = x].$$

In order to get the partial differential equation for $u^X$, we need to determine $dF_X(t)$. Note that

$$\begin{aligned}
dF_X(t) &= d\left[A_X(t) - K_1 Y_X(t)\right] \\
&= \left[\bar{\Delta}^Y(t) - K_1\right] \cdot dY_X(t) \\
&= \left[A_Y(t) - \bar{\Delta}^X(t) \cdot X_Y(t) - K_1\right] \cdot dY_X(t) \\
&= \left[A_Y(t) - \bar{\Delta}^X(t) \cdot X_Y(t) - K_1\right] \sigma Y_X(t) dW^X(t) \\
&= \sigma\left[\left[A_X(t) - K_1 Y_X(t)\right] - \bar{\Delta}^X(t)\right] dW^X(t) \\
&= \sigma\left[F_X(t) - \bar{\Delta}^X(t)\right] dW^X(t).
\end{aligned}$$

Therefore

$$dF_X(t) = \sigma\left[F_X(t) - \bar{\Delta}^X(t)\right] dW^X(t),$$

which is identical to an evolution of the average asset $A$. Therefore the pricing partial differential equation takes the same form as (3.41):

$$u_t^X(t,x) + \tfrac{1}{2}\sigma^2(x - \bar{\Delta}^X(t))^2 u_{xx}^X(t,x) = 0 \tag{3.48}$$

with the terminal condition

$$u^X(T,x) = f^X(x). \tag{3.49}$$

Thus we can also efficiently solve the Asian call option with the fixed strike using the above partial differential equation. Note the important difference from Equation (3.41). In the previous case, the basic price process was $A_X(t)$, the price of the average asset $A$ in terms of the reference asset $X$. The partial differential equation (3.48) applies to the price process $F_X(t)$, the price of the Asian forward $F$ in terms of the reference asset $X$. The corresponding spatial variables are shifted by the factor $K_1 Y_X(t)$ as

$$A_X(t) - F_X(t) = A_X(t) - A_X(t) + K_1 Y_X(t) = K_1 Y_X(t).$$

Note that while $A_X(t)$ is always positive, $F_X(t)$ can become zero or even become negative, and thus the Asian forward $F$ cannot be used as a reference asset for the purposes of pricing. Thus in contrast to the case of the average asset $A$, there is no partial differential equation where $F$ serves as a reference asset.

The hedging portfolio agrees with (3.45), but the value of $A_X(t)$ is replaced by $F_X(t)$:

$$\begin{aligned} P(t) = &\left[ u^X\left(t, F_X(t)\right) + \left(\bar{\Delta}^X(t) - F_X(t)\right) \cdot u_x^X\left(t, F_X(t)\right) \right] \cdot X \\ &+ \left[ \bar{\Delta}^Y(t) \cdot u_x^X\left(t, F_X(t)\right) \right] \cdot Y. \end{aligned} \tag{3.50}$$

# References and Further Reading

The approach to Asian options presented in this text extends previous works of Vecer [79, 80] which used the average asset as the natural asset for pricing. The characterization of the Asian option price with partial differential equations was known even earlier; see for instance Rogers and Shi [71], but the asset that was considered for pricing was the running average which is an arbitrage asset, and the corresponding partial differential equation had extra terms that appear in the connection with the time value of the running average. Use of the average asset for pricing Asian options is not limited to the geometric Brownian motion. It is possible to generalize this approach to other martingale models of the price as shown by Fouque and Han [27] for stochastic volatility or for models with jumps as shown in Vecer and Xu [84], and later by Bayraktar and Xing [5]. Hoogland and Neumann [41] and Henderson and Wojakowski [38] pointed out the symmetries between the fixed and the floating strike Asian options. Other relevant papers include Geman and Yor [31], Curran [17], Linetsky [56], Dufresne [24], D'Halluin et al. [21], Milevsky and Posner [63], or Nielsen and Sandmann [66].

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