REPORT ON "THE TREE PROPERTY AND THE CONTINUUM FUNCTION"

The thesis under review is concerned with two classical problems in set theory, the behaviour of the continuum function and the extent of the tree property, together with the relationship between them. This has been a very active area in recent years: major progress has been made by researchers including Itay Neeman, Moti Gitik, Dima Sinapova, Spencer Unger and Laura Fontanella. The results of the thesis are interesting and original, and fully merit the award of a doctoral degree.

The continuum problem is one of the oldest problems in set theory, and asks about the possible behaviours of the *continuum function* $\kappa \mapsto 2^\kappa$. Much is known about this problem, which has been the stimulus for many important developments. For the purposes of this review the main points are that:

- The behaviour of the continuum function at regular cardinals is subject only to the "obvious" constraints imposed by monotonicity and König's theorem, and Easton showed that all possibilities can be realised by forcing over a model of GCH with an appropriate "Easton product".
- The behaviour of the continuum function at singular cardinals is subject to many subtle constraints, some of which also involve the continuum function on the regular cardinals, and is not completely understood. Some possible behaviours have high consistency strength and are only realised by elaborate forcing constructions over models with large cardinals.

Interest in the *tree property* is more recent. The tree property holds at a regular cardinal κ if every κ-tree has a cofinal branch, that is to say there are no κ-Aronszajn trees. A related property concerns *special λ^+-trees*, that is to say λ^+-trees which are the union of λ antichains: such trees are λ^+-Aronszajn trees in any cardinal preserving extension of V. λ^+ has the *weak tree property* if there are no special λ^+-trees. Some key facts are:

- (Specker) If $\lambda^{<\lambda} = \lambda$, then there is a special λ^+-tree. In particular $\aleph_1$ does not have the weak tree property, and if κ^{++} has the weak tree property then $2^\kappa > \kappa^+$.
- (Mitchell) If $\kappa < \lambda$ with κ regular and λ weakly compact, then there is a forcing extension where $2^\kappa = \lambda = \kappa^{++}$ with κ^{++} having the tree property.

Two results from my joint work with Foreman are also relevant:

- Starting with infinitely many supercompact cardinals, we may force that $2^{\aleph_n} = \aleph_{n+2}$ and $\aleph_{n+2}$ has the tree property for all $n < \omega$.
- If $\kappa < \lambda$ with κ supercompact and λ weakly compact then we may force to obtain an extension where κ is singular strong limit of cofinality ω, $2^\kappa = \lambda = \kappa^{++}$ and κ^{++} has the tree property.

The thesis contain four main results, two of them concerned with versions of the tree property at many cardinals below $\aleph_\omega$ and two concerned with the tree property at the double successor of a singular cardinal. The main point in each case is that we can control the continuum function with considerable freedom, or to put it another way the tree property does not exert much influence on the behaviour of the continuum function.

Theorem 5.7 shows (from the optimal assumption) that the hypothesis "$\aleph_{2n}$ has the tree property for every $n > 0$ and $\aleph_\omega$ is strong limit" only influences the continuum function below $\aleph_\omega$ in the "obvious" way demanded by Specker's theorem, that is $2^{\aleph_{2n}} \geq 2^{\aleph_{2n+2}}$ for all n. The argument is ingenious, and shows the author's mastery of the relevant body of forcing and large cardinals techniques.

In a similar spirit, Theorem 5.14 shows (again from the optimal assumption) that the hypothesis "$\aleph_n$ has the weak tree property for every $n > 1$ and $\aleph_\omega$ is strong limit" only influences the continuum function in the obvious way, which in this case means that $2^{\aleph_n} \geq 2^{\aleph_{n+2}}$ for all n. The argument combines a variation of Mitchell forcing due to Spencer Unger with the ideas of Theorem 5.7

Turning to the tree property at the double successor of a singular cardinal, theorem 6.25 shows that the tree property can hold at the double successor of a singular strong limit cardinal κ with 2^κ as large as desired. The proof builds on my joint work with Foreman but contains many new ideas: in particular I was struck by the idea of doing all the work on the even coordinates up to λ, so that the odd coordinates are free to "soak up" arbitrarily long initial segments of the construction between λ and λ^+.

The hardest results in the thesis are Theorem 7.13 and its generalisation Theorem 7.20, which give versions of Theorem 6.25 in which $\kappa = \aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+n}$ (where $n = 3$ in Theorem 7.13 and $3 \leq n < \omega$ in Theorem 7.20). There are many novel ideas here:

- The Prikry-type forcing which bring κ down to $\aleph_\omega$ happens after the Mitchell forcing, rather than being woven into it as was the case in my work with Foreman.
- It is necessary to do a delicate preparation argument to ensure that after the Mitchell forcing (which in this case is actually Unger's variant) we have the hypotheses necessary to build the Prikry-type forcing.
- Some very subtle analysis is needed to show that the tree property still holds after doing the Prikry-type forcing. This is in no sense automatic: in some unpublished joint work with Magidor we have constructed examples where κ is measurable, κ^{++} has the tree property but doing Prikry-type forcing destroys the tree property.

The quality of the exposition is very high throughout. I particularly appreciated the author's attention to detail in the matter of projections between forcing posets and the associated quotient forcings. These are hard to get right (and indeed were the source of an error in my work with Foreman, later fixed by Unger).

In summary, this is an excellent thesis and I strongly recommend the award of a doctoral degree.