Charles University

Faculty of Arts
Department of Logic

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The tree property and the continuum function

Šárka Stejskalová

Program
Logic

Supervisor of the PhD thesis
Radek Honzík, PhD

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Author: Šárka Stejskalová

University: Charles University

Faculty: Faculty of Arts

Department: Department of Logic

Program: Logic

Supervisor: Radek Honzík

Abstract: The continuum function is a function which maps every infinite cardinal κ to 2^κ. We say that a regular uncountable cardinal κ has the tree property if every κ-tree has a cofinal branch, or equivalently if there are no κ-Aronszajn trees. We say that a regular uncountable cardinal κ has the weak tree property if there are no special κ-Aronszajn trees. It is known that the tree property, and the weak tree property, have the following non-trivial effect on the continuum function: $(*)$ If the (weak) tree property holds at κ^{++}, then $2^\kappa \geq \kappa^{++}$. In this thesis we show several results which suggest that $(*)$ is the only restriction which the tree property and the weak tree property put on the continuum function in addition to the usual restrictions provable in ZFC (monotonicity and the fact that the cofinality of 2^κ must be greater than κ; let us denote these conditions by $(**)$). First we show that the tree property at $\aleph_{2n}$ for every $1 \leq n < \omega$, and the weak tree property at $\aleph_n$ for $2 \leq n < \omega$, does not restrict the continuum function below $\aleph_\omega$ more than is required by $(*)$, i.e. every behaviour of the continuum function below $\aleph_\omega$ which satisfies the conditions $(*)$ and $(**)$ is realisable in some generic extension. We use infinitely many weakly compact cardinals (for the tree property) and infinitely many Mahlo cardinals (for the weak tree property) as the optimal large cardinal assumption. In the second result we show that the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality does not limit the size of 2^κ except for conditions $(*)$ and $(**)$. We use the assumption of the existence of a supercompact cardinal with a weakly compact cardinal above it for the result. In the final result we show that the tree property at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit is consistent with $2^{\aleph_\omega}$ being equal to $\aleph_{\omega+2+n}$ for any prescribed $0 \leq n < \omega$. We use the existence of a strong cardinal of a suitable degree and a weakly compact cardinal above it for this result.

Keywords: The tree property, the continuum function

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Autor: Šárka Stejskalová

Univerzita: Univerzita Karlova

Fakulta: Filozofická fakulta

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Abstrakt: Funkce kontinua je funkce, která libovolnému nekonečnému kardinálu κ přiřadí hodnotu 2^κ. Řekneme, že regulární nespočetný kardinál κ má stromovou vlastnost, jestliže každý κ-strom má kofinální větev, ekvivalentně, že neexistuje žádný κ-Aronszajnův strom. Obdobně definujeme, že regulární nespočetný kardinál κ má slabou stromovou vlastnost, jestliže neexistuje žádný speciální κ-Aronszajnův strom. Stromová vlastnost a slabá stromová vlastnost mají následující netriviální efekt na funkci kontinua: (*) Jestliže (slabá) stromová vlastnost platí na κ^{++}, pak $2^\kappa \geq \kappa^{++}$. V této práci se věnujeme několika výsledkům, které naznačují, že (*) je jediná restrikce, kterou na funkci kontinua kladou stromová vlastnost a slabá stromová vlastnost kromě obvyklých restrikcí dokazatelných v ZFC (monotonie a tvrzení, že kofinalita 2^κ musí být větší než κ; označme tyto restrikce (**)). Nejprve ukážeme, že stromová vlastnost na $\aleph_{2n}$ pro každé $1 \leq n < \omega$ a slabá stromová vlastnost na $\aleph_n$ pro $2 \leq n < \omega$ neovlivňují funkci kontinua pod $\aleph_\omega$ víc, než je dáno podmínkami (*) a (**), tedy že každé chování funkce kontinua pod $\aleph_\omega$, které splňuje podmínky (*) a (**), je realizovatelné v nějaké generické extenzi. Pro důkaz stromové vlastnosti předpokládáme existenci nekonečně mnoha slabě kompaktních kardinálů a pro důkaz slabé stromové vlastnosti předpokládáme existenci nekonečně mnoha Mahlových kardinálů, což jsou optimální předpoklady vzhledem ke konzistentní síle daných tvrzení. V další části ukážeme, že stromová vlastnost na dvojitém následníku singulárního silně limitního kardinálu κ se spočetnou kofinalitou neovlivňuje hodnotu 2^κ kromě podmínek (*) a (**). Pro tento výsledek používáme předpoklad existence superkompaktního kardinálu κ se slabě kompaktním kardinálem nad κ. Poslední výsledek ukazuje, že stromová vlastnost na $\aleph_{\omega+2}$ s $\aleph_\omega$ silně limitním je konzistentní s tvrzením $2^{\aleph_\omega} = \aleph_{\omega+2+n}$ pro libovolné n, $0 \leq n < \omega$. Pro důkaz využíváme předpoklad existence silného kardinálu κ jistého stupně se slabě kompaktním kardinálem nad κ.

Klíčová slova: Stromová vlastnost, funkce kontinua

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Contents

1 Introduction

In the thesis, we study the tree property and its interaction with the continuum function.

If $\kappa > \omega$ is a regular uncountable cardinal, we say that κ has *the tree property*, and we denote it by $\mathsf{TP}(\kappa)$, if all κ-trees have a cofinal branch.[1] The tree property of κ is a compactness property which derives its motivation from compactness of the infinitary logic $L_{\kappa,\kappa}$ for an inaccessible κ (see [48] for more details). Indeed, κ is weakly compact if and only if κ has the tree property and it is inaccessible. The notion of the tree property at κ is a priori weaker than weak compactness as it does not require κ to be inaccessible. We will give more background later on (see Section 4.1), but let us state here that the existence of κ with the tree property is equiconsistent with the existence of a weakly compact cardinal, and that the tree property can also hold at successor cardinals greater or equal to $\aleph_2$.

A κ-tree T which witnesses the failure of the tree property at κ is called a *κ-Aronszajn tree*, i.e. T is a κ-tree which has no cofinal branches. By results of Aronszajn and Specker ([50] and [69]), GCH ensures the existence of many counterexamples to the tree property:

$$(\forall \kappa \geq \omega)\ (\kappa^{<\kappa} = \kappa \rightarrow \neg\mathsf{TP}(\kappa^+)). \tag{1.1}$$

In particular, the tree property can never hold at $\aleph_1$ (or at the successor of an inaccessible cardinal). In fact, the tree constructed to witness (1.1) can be required to have the additional property that there exists a function $T \to \kappa$ which is injective on the chains in the tree ordering such trees are called *special Aronszajn trees*. It is consistent that special Aronszajn trees form a strictly smaller family than the Aronszajn trees, and we therefore introduce the notion of the *weak tree property*, and we denote it by $\mathsf{wTP}(\kappa)$: $\mathsf{wTP}(\kappa)$ says that there are no special κ-Aronszajn trees. More details can be found in Section 4.1, but let us say that the existence of κ with the weak tree property is equiconsistent with the existence of a Mahlo cardinal.

The inequality in (1.1) generalises to the weak tree property:

$$(\forall \kappa \geq \omega)\ (\kappa^{<\kappa} = \kappa \rightarrow \neg\mathsf{wTP}(\kappa^+)). \tag{1.2}$$

In fact, the antecedent of the implication in (1.2) can be weakened to the existence of the weak square sequence at κ (denoted $\square^*_\kappa$) (see Definition 3.8 and [8] for more details):

$$(\forall \kappa > \omega)\ (\square^*_\kappa \rightarrow \neg\mathsf{wTP}(\kappa^+)). \tag{1.3}$$

By results of Jensen [46], $\square^*_\kappa$ is actually equivalent to the existence of a special κ^+-Aronszajn tree, and therefore to the failure of the weak tree property.

Recall that the function which maps an infinite cardinal κ to 2^κ is called the *continuum function*. As is well known, the continuum function on regular cardinals can behave very arbitrarily (see Section 3.5). While large cardinals and the singular strong limit cardinals of uncountable cofinality do reflect the pattern of the continuum function to smaller cardinals – and therefore restrict the freedom of the continuum function –, this limits the arbitrariness of the continuum function on regular cardinals only modulo "large

[1] See Section 3.1 for definitions.

sets" (such as the stationary sets); there is no local control over the continuum function. It is of interest to note that (1.1) does provide such control: for instance $\mathsf{TP}(\aleph_2)$ implies the failure of CH.

The natural question on which we focus in this thesis is the following:

(Q) Do the tree property and the weak tree property put more restrictions on the continuum function apart from (1.1) and (1.2)?

This question can in principle be approached either locally or globally, i.e. with the tree property holding at more cardinals at the same time. That is, we may ask how $\mathsf{TP}(\kappa)$ influences the continuum function for a fixed κ, or consider a set of regular cardinals $\{\kappa_i \mid i \in I\}$ (usually an interval) and ask about the influence of $\mathsf{TP}(\kappa_i)$ for all $i \in I$.

Let us note in this context that it is highly non-trivial even to obtain a model with a large interval of regular cardinals with the tree property: while an easy modification of the original Mitchell's construction (see [55]; we call the forcing *Mitchell forcing*) yields two successive cardinals with the weak tree property, the existence of two successive cardinals with the tree property requires a major modification of the argument (see [1]). We will not go into details here, but let us mention some crucial problems which make it hard to get long intervals with the tree property: by (1.3), obtaining the weak tree property at the successor of a singular cardinal is hard since it requires the killing of weak square sequences (which exist in core models); and just from (1.2), obtaining the weak tree property at the double successor of a singular strong limit cardinal requires the failure of SCH (see Section 3.5 for more details). Importantly, dealing with these restrictions at more cardinals at the same time complicates the matters even more: it is noteworthy that obtaining $\mathsf{TP}(\aleph_2)$ and $\mathsf{TP}(\aleph_3)$ at the same time requires a much large cardinal strength than $\mathsf{TP}(\aleph_2)$ or $\mathsf{TP}(\aleph_3)$ alone (see Section 4.1 and [21] for more details).

Returning to our question (Q), we provide three original results which show that the answer to (Q) is negative in some special cases: any behaviour – consistent with (1.1) and (1.2) – of the continuum function on the cardinals considered in our results is consistent with the tree property (locally and globally).

The thesis is structured as follows:

Section 2 contains a brief description of the original results which we will prove in Sections 5, 6, and 7.

In Section 3 we review basic notions and give basic facts for concepts appearing in the thesis (trees, large cardinals, forcing conventions and the forcing notions we will use, and basic facts about the continuum function).

In Section 4 we survey the existing results related to the tree property and provide background for the tree property arguments, including a sketch of the original Mitchell argument which achieves the tree property at a single cardinal of the form κ^{++} where κ is regular.

In Section 5 we answer a limited global version of (Q) and show that the continuum function below a strong limit $\aleph_\omega$ can be anything consistent with (1.1) and (1.2) (and the usual restrictions which the continuum function needs to satisfy) while the tree property holds at every even cardinal below $\aleph_\omega$ larger than $\aleph_1$, or the weak tree property holds at every regular cardinal below $\aleph_\omega$ larger than $\aleph_1$.

In Sections 6 and 7 we focus on a local version of (Q): in Section 6 we show that the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality does not put any restrictions on the value of 2^κ apart from (1.1). In Section 7 we follow up with the result that $2^{\aleph_\omega}$ can be equal to $\aleph_{\omega+2+n}$ for any $n < \omega$ with the tree property holding at $\aleph_{\omega+2}$.

Let us end the introduction by saying that we expect that the answer to (Q) will be negative even when more cardinals with the tree property are considered. We consider further development and open question in Section 8.

2 Original results of the thesis

Let us briefly introduce the results in the thesis and discuss how they relate to existing results.

The results in Section 5, joint with Radek Honzik, were submitted as [42] and deal with the tree property and the weak tree property at cardinals $\aleph_n$, $1 < n < \omega$. We show that the tree property and the weak tree property at these cardinals do not put any restrictions on the continuum function below $\aleph_\omega$ apart from the trivial implication that $\mathsf{wTP}(\aleph_{n+2})$ implies $2^{\aleph_n} > \aleph_{n+1}$ for $0 \leq n < \omega$.

A succinct statement of the theorems is as follows:

Theorem 2.1. (GCH) *Assume there are infinitely many weakly compact cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \to f(m) \leq f(n)$.
(ii) $f(2n) \geq 2n + 2$ for all $n < \omega$.

Then there is a model where the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Theorem 2.2. (GCH) *Assume there are infinitely many Mahlo cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \to f(m) \leq f(n)$,
(ii) $f(n) > n + 1$ for all $n < \omega$.

Then there is a model where the weak tree property holds at every $\aleph_n$, $1 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Theorem 2.1 is based on the construction in [26, Section 5] – which just ensures $2^{\aleph_m} = \aleph_{m+2}$, $0 \leq m < \omega$, and the tree property at $\aleph_{2n}$, $0 < n < \omega$ –, and adds extra forcings to control the continuum function. Similarly, Theorem 2.2 builds on the proof in Unger [74] and adds extra forcings to control the continuum function while ensuring the weak tree property at every $\aleph_n$, $1 < n < \omega$. In both cases we used the product of Cohen forcings at relevant cardinals, and computed that their presence will not destroy the tree property ensured by the rest of the forcing.

The results in Section 6, joint with Sy-David Friedman and Radek Honzik, were submitted as [27] and focus on the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality.

A succinct statement of the theorem is as follows:

Theorem 2.3. *Assume* GCH *and let κ be a Laver-indestructible supercompact cardinal, λ a weakly compact cardinal and μ a cardinal of cofinality greater than κ such that $\kappa < \lambda < \mu$. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ preserves cardinals $\leq \kappa^+$ and $\geq \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \;\&\; 2^\kappa = \mu \;\&\; \mathrm{cf}(\kappa) = \omega \;\&\; \kappa$ is strong limit).
(iii) $V[\mathbb{R}] \models \mathsf{TP}(\lambda)$.

Theorem 2.3 generalises the construction in [10] in which Cummings and Foreman obtained a singular strong limit cardinal with countable cofinality with $2^\kappa = \kappa^{++}$ and

$\mathsf{TP}(\kappa^{++})$, starting with a Laver-indestructible supercompact κ. We modify their original forcing – which integrates Prikry forcing with Mitchell forcing – by adding more Cohen subsets of κ to control the continuum function at κ so that $2^\kappa = \mu$ for any $\mu \geq \lambda$ of cofinality greater than κ. This modification required substantial changes in the argument built as it is on reflecting Prikry forcing defined after adding λ-many Cohen subsets of κ: we add μ-many subsets of κ, with $\mu > \lambda$, and therefore the reflection is more complicated.

In Section 7, submitted as [28] and joint with Sy-David Friedman and Radek Honzik, we bring the cardinal κ in Theorem 2.3 down to $\aleph_\omega$. The method of the proof is different from [10] and [27]: we do not integrate Prikry forcing with collapses into Mitchell forcing, but force with Prikry forcing after Mitchell forcing.[2] Also, as the value of $2^{\aleph_\omega}$ cannot be arbitrarily high (see Section 3.5), we only achieve an arbitrary finite gap. It is an open question whether we can achieve an infinite gap.

A succinct statement of the theorem is as follows:

Theorem 2.4. *Suppose* GCH *holds in the universe. Assume n is a natural number, $2 \leq n < \omega$, $\kappa < \lambda$ are cardinals such that λ is the least weakly compact cardinal above κ, and κ is $H(\lambda^{+n-2})$-strong. Then there is a forcing extension where the following hold:*

(i) $\kappa = \aleph_\omega$ *is strong limit;*
(ii) $2^{\aleph_\omega} = \aleph_{\omega+n}$*;*
(iii) $\mathsf{TP}(\aleph_{\omega+2})$.

[2] This modification is necessary: the original method does not work since λ (the weakly compact cardinal above κ) must be first collapsed to κ^{++}, and only then Prikry forcing with collapses can be used.

3 Preliminaries

The purpose of this section is to introduce basic definitions and facts which are used later in the text. Our notation is standard (see [44] or [48]).

3.1 Trees

In this section we introduce the notion of a tree and state some related facts.

Definition 3.1. We say that a set T with an ordering $<_T$ is a *tree* if for all $t \in T$ the set $\{s \in T \mid s <_T t\}$ is well-ordered by $<_T$.

Let T be a tree. We say that $S \subseteq T$ is a *subtree* of T in the induced ordering $<_S = <_T \restriction S$ if for all $s \in S$ and all $t \in T$:

$$\text{if } t <_T s \text{ then } t \in S. \tag{3.1}$$

Definition 3.2. Let T be a tree.

(i) For $t \in T$, the *height of t in T* is defined by $\mathrm{ht}(t, T) = \mathrm{ot}(\{s \in T \mid s <_T t\})$, where $\mathrm{ot}(x)$ is the order-type of a set x (with an understood wellordering on x).
(ii) For an ordinal α, the *α-th level of T* is defined by $T_\alpha = \{t \in T \mid \mathrm{ht}(t) = \alpha\}$.
(iii) The *height of T*, denoted by $\mathrm{ht}(T)$, is the least ordinal α such that the level T_α is empty.
(iv) For an ordinal $\alpha < \mathrm{ht}(T)$, the subtree $T \restriction \alpha$ is defined by $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$.

In this thesis, we deal with trees which are thin in the sense that their levels have size smaller than the height of the tree.

Definition 3.3. For an infinite regular cardinal κ, we say that a tree T is a *κ-tree* if it has height κ and all levels of T have size less than κ.

Let T be a tree. We say that $b \subseteq T$ is a *branch* if it is a maximal chain in T; if b is moreover cofinal in the height of the tree, we say that b is a *cofinal branch*.

By a result of König, all $\aleph_0$-trees have cofinal branches, and by results of Aronszajn mentioned in [50] there are trees on $\aleph_1$ which do not have cofinal branches. As we will discuss later on, a related question for larger cardinals is independent of ZFC (modulo large cardinals).

Definition 3.4. Let κ be a regular cardinal. We say that a κ-tree T is a *κ-Aronszajn tree* if it has no cofinal branches.

We define a useful strengthening of the notion of an Aronszajn tree for a successor cardinal.

Definition 3.5. Let κ be an infinite cardinal. We say that a κ^+-Aronszajn tree T is *special* if it is an union of κ many antichains, where $A \subseteq T$ is an antichain if all distinct elements of A are pairwise incomparable in $<_T$.

The notion of a special tree has several equivalent definitions. It is easy to see that our definition is equivalent to requiring that there exists a function f from T to κ which is injective on chains in T. An argument due to Jensen [46] shows that the existence of a special κ^+-tree is equivalent to the existence of the so called *weak square* $\square^*_\kappa$ (see Definition 3.8, and Cummings [8] for more details).[3]

If there are no κ-Aronszajn trees, we may view κ as having a certain form of compactness for κ-trees: all κ trees are compact in the sense that having branches of all heights below κ implies that there is a cofinal branch.

Definition 3.6. Let κ be an uncountable regular cardinal. We say that κ has *the tree property* if there are no κ-Aronszajn trees, or equivalently every κ-tree has a cofinal branch. We write $\mathsf{TP}(\kappa)$ if κ has the tree property.

Let us now introduce a related notion for the special trees at κ^+. As we mentioned above, the fact that there are no special κ^+-trees is equivalent to the failure of $\square^*_\kappa$, so strictly speaking no new terminology is needed. However, we find it useful to formulate the notion with the direct reference to trees. Before we give the definition of the weak tree property let us for completeness give the definitions of the square principle and the weak square principle.

Definition 3.7. Assume κ is an uncountable cardinal. The *square principle* $\square_\kappa$ holds if there is a sequence $\langle C_\alpha \,|\, \alpha < \kappa^+, \alpha \text{ a limit ordinal}\rangle$ such as every C_α is a club subset of α, the odertype of C_α is at most κ, and for all α, $C_\beta = C_\alpha \cap \beta$ whenever β is a limit point of C_α.

The *weak square principle* $\square^*_\kappa$ allows us up to κ many guesses at α:

Definition 3.8. Assume κ is an uncountable cardinal. $\square^*_\kappa$ holds if there is a sequence $\langle \vec{C}_\alpha \,|\, \alpha < \kappa^+, \alpha \text{ a limit ordinal}\rangle$ such that $\vec{C}_\alpha$ is a family of at most κ many club subsets of α, the ordertype of each $C \in \vec{C}_\alpha$ is at most κ, and for every α and every $C \in \vec{C}_\alpha$, $C \cap \beta \in \vec{C}_\beta$ whenever β is a limit point of C.

The following concept is a weaker case of the tree property:

Definition 3.9. Let κ be an infinite cardinal. We say that κ^+ has *the weak tree property* if there are no special κ^+-Aronszajn trees. We write $\mathsf{wTP}(\kappa)$ if κ has the weak tree property.

Notice that the difference between the tree property and the weak tree property is quite substantial: being a κ-Aronszajn tree is Π^1_1 over $H(\kappa)$, while being a special κ-Aronszajn tree is just Σ^1_1 over $H(\kappa)$. This distinction contributes to the difference between the consistency strengths of these principles, especially at successive cardinals (see Section 4.1 for more details).

As we already mentioned (see [8]):

Fact 3.10. *For all uncountable κ,*

$$\square^*_\kappa \leftrightarrow \neg\mathsf{wTP}(\kappa^+).$$

[3] There is a generalisation of the square principle which can be formulated for regular and limit cardinals κ, and this generalisation leads naturally to the notion of a special κ-tree for a limit and regular κ. See [71] for more details.

For completness we give here the definition of a Suslin tree even though it is not the main interest of this thesis.

Definition 3.11. Let κ be a regular cardinal. We say that a κ-Aronszajn tree T is a *κ-Suslin tree* if it has no antichain of size κ.

Notice that the notions of a κ^+-special Aronszajn tree and κ^+-Suslin tree are mutually exclusive: no κ^+-tree T can be simultaneously Suslin and special. Also note that if above every node t in T there are two distinct nodes s, s' in T, then having no antichain of size κ already implies that T has no cofinal branch.

3.2 Large cardinals

In this section we list definitions and elementary facts about large cardinals related to this thesis. For definitions of other large cardinals and more details about them, see [48] or [44].

Definition 3.12. We say that an uncountable cardinal κ is *weakly inaccessible* if it is regular and limit. We say that it is *(strongly) inaccessible* if moreover, for all $\lambda < \kappa$, $2^\lambda < \kappa$.

Note that under GCH, the notions of a weakly inaccessible and an inaccessible cardinal coincide.

Definition 3.13. We say that an uncountable cardinal κ is *Mahlo* if it is inaccessible and the set of all regular cardinals below κ is stationary in κ.

Mahloness is a natural strengthening of inaccessibility. It is easy to show that the set of inaccessible cardinals below a Mahlo cardinal κ is stationary below κ.

A strengthening of Mahloness is weak compactness. Originally, the definition of a weakly compact cardinal was arrived at by generalising the compactness property of the classical first-order logic to a certain infinitary logic. In this sense, κ is weakly compact if and only if the relevant infinitary logic is compact. For more details about the original definition consult [48]. We present here two definitions of weak compactness which are equivalent to the original one and are used extensively throughout the text.

Definition 3.14. We say that an uncountable cardinal κ is *weakly compact* if it is inaccessible and all κ-tree has a cofinal branch, i.e. the tree property holds at κ.

Before we state the following fact, recall that if $j : M \to N$ is an elementary embedding from a transitive class M to a transitive class N, then $\kappa \in M$ is called the *critical point of j* if κ is the first ordinal moved by j, i.e. j is the identity below κ and $j(\kappa) > \kappa$.

Fact 3.15. *The following are equivalent for an uncountable cardinal κ:*

(i) κ is weakly compact.
(ii) For every transitive set M with size κ, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \to N$ with critical point κ, where N is transitive with size κ and ${}^{<\kappa}N \subseteq N$.

Weakly compact cardinals are relatively small in the sense that they are consistent with $V = L$. With the next large cardinals, we move to cardinals which imply $V \neq L$.

Definition 3.16. We say that an uncountable cardinal κ is *measurable* if there is a non-principal κ-complete ultrafilter on κ.

Notice that an ultrafilter U on κ naturally defines a two-valued measure on subsets of κ: a set $X \subseteq \kappa$ gets measure 1 if and only if $X \in U$. For this reason, an ultrafilter is often called a *measure*.

Fact 3.17. *The following are equivalent for an uncountable cardinal κ:*

(i) κ is measurable.
(ii) There is an elementary embedding j with critical point κ from V into a transitive class M.

It is easy to see that if $j : V \to M$ is as above, then $H(\kappa^+)$ is always included in M. By a strengthening of this property of j we get the definition of a $H(\mu)$-strong cardinal.

Definition 3.18. We say that an uncountable cardinal κ is $H(\mu)$*-strong*, where μ is a cardinal greater than κ, if there is an elementary embedding $j : V \to M$ with critical point κ from V into a transitive class M such that $j(\kappa) > \mu$ and $H(\mu) \subseteq M$.

Remark 3.19. This concept is called by many different names in literature. Suppose λ is κ^{++} for simplicity and that $2^\kappa = \kappa^+$ holds:[4] then a $H(\kappa^{++})$-strong cardinal in our definition is called $V_{\kappa+2}$-strong in [48], and $V_{\kappa+2}$-hypermeasurable in [7]. Furthermore, in our paper [28] we use the term $H(\mu)$*-hypermeasurable* in place of $H(\mu)$*-strong*.

It is often useful to have witnessing embeddings $j : V \to M$ which provide more information about M.

Definition 3.20. Let $\kappa < \mu$ be infinite cardinals. We say that an embedding $j : V \to M$ with critical point κ is a (κ, μ)*-extender embedding* if and only if

$$M = \{j(f)(\alpha) \,|\, f : \kappa \to V \;\&\; \alpha < \mu\}. \tag{3.2}$$

Strong cardinals are witnessed by extender embeddings with some additional properties:

Fact 3.21. *Assume* GCH *and let $\kappa < \mu$ be infinite cardinals, with μ having cofinality greater than κ. If κ is $H(\mu)$-strong, then there is a (κ, μ)-extender embedding $j : V \to M$ which satisfies the following:*

(i) $\mu < j(\kappa) < \mu^+$,
(ii) ${}^\kappa M \subseteq M$,
(iii) $H(\mu) \subseteq M$.

In the context of the tree property, it is useful to define the following slightly technical notion (it is used for instance in [22]).

[4]With $2^\kappa = \kappa^+$, $V_{\kappa+2}$ is logically equivalent to $H(\kappa^{++})$ in the sense that $H(\kappa^{++})$ is included in M if and only if $V_{\kappa+2}$ is included M.

Definition 3.22. We say that an uncountable cardinal κ is *weakly compact strong* if there is a weakly compact cardinal λ above κ and an elementary embedding j with critical point κ from V into a transitive class M such that $j(\kappa) > \lambda$, $H(\lambda) \subseteq M$ and λ is a weakly compact cardinal in M.

Let us add that it is relatively consistent that there exists a weakly compact strong cardinal κ witnessed by a (κ, λ)-extender embedding j with $\lambda < j(\kappa) < \lambda^+$, where $\lambda > \kappa$ is the least weakly compact cardinal above κ: For instance start with an $H(\lambda^+)$-strong cardinal κ witnessed by $j^* : V \to M^*$ (in M^*, λ is clearly the least weakly compact cardinal above κ) and factor j^* so that $j^* = k \circ j$, where $j : V \to M$ is the desired embedding (note that by elementarity and the fact that $k(\lambda) = \lambda$, λ is still the least weakly compact cardinal in M above κ).

For more details regarding the notion of an extender, and extender embeddings, please see [9].

We shall briefly review a weakening of the notion of a strong cardinal which is important as it provides an exact consistency strength for certain questions concerning the failure of GCH at a measurable cardinal, or the failure of GCH at a singular strong limit cardinal.

If U and W are two normal measures at κ, let us write $U <_M W$ if U is an element of the ultrapower generated by W (we call $<_M$ the *Mitchell order*). It can be shown that $<_M$ is well-founded and may be used to assign to κ its Mitchell order $o(\kappa)$ which is at most $(2^\kappa)^+$. For our purposes we just mention that if GCH holds and κ is $H(\kappa^{++})$-strong, then $o(\kappa) = \kappa^{++}$. The Mitchell order can be generalised to extenders – combinatorial objects which are used to construct extender embeddings in Definition 3.20 (see [48] for more details). We will not give many details, just say that in this generalised order $o(\kappa)$ is not limited to $(2^\kappa)^+$, and in particular if κ is $H(\kappa^{+n})$-strong for some $2 \leq n < \omega$ and GCH holds, then $o(\kappa) = \kappa^{+n}$. It can be shown that the exact consistency strength of a measurable cardinal κ with $2^\kappa = \kappa^{+n}$, $2 \leq n < \omega$, is exactly $o(\kappa) = \kappa^{+n}$ (see [33] for more details).

The difference between $o(\kappa) = \kappa^{+n}$ and $H(\kappa^{+n})$-strongness is not very big and is related to the notion of a *tall cardinal*. If $\kappa < \lambda$ are cardinals, we say that κ is λ-*tall* if there is an embedding $j : V \to M$ with critical point κ such that $\lambda < j(\kappa)$ and M is closed under κ-sequences in V. It can be showed that for $2 \leq n < \omega$, $o(\kappa) = \kappa^{+n}$ is equiconsistent with the existence of a κ^{+n}-tall cardinal κ (see [33] for more details).

Finally we move to the largest cardinal in our thesis.

Definition 3.23. We say that an uncountable cardinal κ is λ-*supercompact*, where $\lambda > \kappa$ is a cardinal, if there is an elementary embedding j with critical point κ from V into a transitive class M such that $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$.

Notice that the extra strength comes from the important property that M is closed under λ-sequences in the universe.

Supercompact cardinals are useful because they can be made "indestructible" in the following sense: As was shown by Laver [51], whenever κ is supercompact, there si a forcing $\mathbb{L}$ of size κ such that in the resulting generic extension, κ remains supercompact in any further forcing extension by a forcing which is κ-directed closed (where $\mathbb{P}$ is κ-directed closed if for every $D \subseteq \mathbb{P}$ of size less than κ, if for all p_1, p_2 in D there is

$e \in D$ such that $e \leq p_1$ and $e \leq p_2$, then there is $p \in \mathbb{P}$, with $p \leq d$ for all $d \in D$). The forcing $\mathbb{L}$ is called the *Laver preparation*, and the supercompact κ in $V[\mathbb{L}]$ is called *Laver-indestructible*.

3.3 Forcing

In this section we review some basic facts about forcing and fix notational conventions. The general reference is Jech's book [44]; the treatment of the iteration of forcing notions follows Baumgartner's paper [4].

3.3.1 Basic conventions

A forcing notion is a partially ordered set $(P, \leq)$ with the greatest element which we denote 1_P. To simplify notation, we will often write P instead of $(P, \leq)$ if the ordering is clear from the context.

A condition p is stronger then q, in symbols $p \leq q$, if it carries more informations. We say that two condition p and q are compatible, in symbols $p \,\|\, q$, if there is an element of the ordering such that it is below both p and q. We say that they are incompatible, if they are not compatible and we denote this by $p \perp q$. We say that $A \subseteq P$ is an *antichain* if all distinct p, q in A are incompatible; an antichain is *maximal* if every p in P is compatible with some element in A.

If $(P, \leq)$ is a forcing notion, we write $V[P]$ to denote a generic extension by P if the concrete generic filter is not important. Sometimes we write $P \Vdash \varphi$ in place of $1_P \Vdash \varphi$.

We say that $(P, \leq)$ is *separative* if $p \not\leq q$ implies that there is some $r \leq p$ which is incompatible with q. Note that if $(P, \leq)$ is separative, then $p \leq q$ is equivalent to p forcing q into the generic filter.

A forcing notion is said to be *non-trivial* if below every condition there are two incompatible extensions. Otherwise a forcing notion is called *trivial*. Note that if $(P, \leq)$ is non-trivial, then a P-generic filter cannot be an element of the universe.

Recall that if $(Q, \leq_Q)$ is a partial order, then we can find a complete Boolean algebra, $(\mathrm{RO}(Q), \leq_{\mathrm{RO}(Q)})$, and a dense embedding (see Definition 3.34) i from Q to $\mathrm{RO}^+(Q) = \{b \in \mathrm{RO}(Q) \mid b > 0_{\mathrm{RO}(Q)}\}$. The algebra $\mathrm{RO}(Q)$ is unique up to isomorphism. If $(Q, \leq_Q)$ is in addition separative, then the mapping i is 1-1 and therefore it is an isomorphism between Q and some dense subset of $\mathrm{RO}^+(Q)$. The uniqueness of the Boolean completion can be used to define a natural notion of the *forcing equivalence* of forcing notions:

Definition 3.24. We say that two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$ are *forcing equivalent* if their Boolean completions are isomorphic.

See Section 3.3.3 for more information about forcing equivalence and related notions.

To obtain all generic extensions it suffices to consider only separative orders. If $(P, \leq)$ is not separative, then it has a separative quotient which is forcing equivalent to P. For more details about separative quotients see [44].

Now we define the notion of *a lottery sum* of forcing notions to provide some counterexamples in Section 3.3.3. The concept of a "sum" of forcing notions has been around for a long time; for more details see [38].

Definition 3.25. Let $\{P_i \mid i \in I\}$ be an indexed set of forcing notions $(P_i, \leq_{P_i})$. We define *the lottery*

$$\bigoplus\{P_i \mid i \in I\} \tag{3.3}$$

as a forcing notion as follows: The underlying set is equal to $\{(i,p) \mid p \in P_i \;\&\; i \in I\} \cup \{1\}$, where 1 is not an element of $\bigcup\{P_i \mid i \in I\}$, and the ordering is such that 1 is the greatest element, and $(i,p) \leq (j,q) \leftrightarrow i = j$ and $p \leq_{P_i} q$.

The intuition is that a $\bigoplus\{P_i \mid i \in I\}$-generic first chooses a forcing notion to force with, and then forces with it.

Finally, we define Cohen forcing at a regular cardinal κ because we will use it to illustrate certain concepts in the following sections. More forcing notions will be defined in Section 3.4.

Definition 3.26. Let κ be a regular cardinal and $\alpha > 0$ an ordinal. The *Cohen forcing* at κ of length α, denoted by $\mathrm{Add}(\kappa, \alpha)$, is the set of all partial functions from $\kappa \times \alpha$ to 2 of size less than κ. The ordering is by reverse inclusion, i.e. $p \leq q \leftrightarrow q \subseteq p$.

Cohen forcing at κ is κ-closed, and if $\kappa^{<\kappa} = \kappa$, then it is also κ^+-Knaster (see Definition 3.27).

3.3.2 Basic properties of forcing notions

In this section we review some basic properties which we will use in the thesis.

Definition 3.27. Let P be a forcing notion and let $\kappa > \aleph_0$ be a regular cardinal. We say that P is:

- κ*-cc* if every antichain of P has size less than κ (we say that P is *ccc* if it is $\aleph_1$-cc).
- κ*-Knaster* if for every $X \subseteq P$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of Y are pairwise compatible.
- κ*-closed* if every decreasing sequence of conditions in P of size less than κ has a lower bound.
- κ*-distributive* if P does not add new sequences of ordinals of length less than κ.

It is easy to check that all these properties – except for the κ-closure – are invariant under forcing equivalence. Regarding the closure, note that for every non-trivial forcing notion P which is κ-closed there exists a forcing-equivalent forcing notion which is not even $\aleph_1$-closed (the completion $\mathrm{RO}^+(P)$ is never $\aleph_1$-closed).

Lemma 3.28. *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P is a forcing notion and $\dot{Q}$ is a P-name for a forcing notion. Then the following hold:*

(i) P is κ-closed and P forces $\dot{Q}$ is κ-closed if and only if $P * \dot{Q}$ is κ-closed.

(ii) P is κ-distributive and P forces $\dot{Q}$ is κ-distributive if and only if $P * \dot{Q}$ is κ-distributive.
(iii) P is κ-cc and P forces $\dot{Q}$ is κ-cc if and only if $P * \dot{Q}$ is κ-cc.

Proof. The proofs are routine; for more details see [44] or [49]. □

An analogous statement (iii) for the Knaster property is not in general true: it may happen that $P * \dot{Q}$ is κ-Knaster, yet P does not force that $\dot{Q}$ is κ-Knaster. Consider the following example: Assume MA (Martin's Axiom) and let $\dot{Q}$ be an $\mathrm{Add}(\aleph_0, 1)$-name for the $\aleph_1$-Suslin tree added by $\mathrm{Add}(\aleph_0, 1)$ (see Jech [44] for details). Then $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is ccc by previous lemma (iii) and as we assume MA, all ccc forcing notions are $\aleph_1$-Knaster. Therefore $\mathrm{Add}(\aleph_0, 1) * \dot{Q}$ is $\aleph_1$-Knaster, but $\mathrm{Add}(\aleph_0, 1)$ forces that $\dot{Q}$ is not $\aleph_1$-Knaster.

If Q is in the ground model, $P * \check{Q}$ is equivalent to $P \times Q$. We state some properties which the product forcing has with respect to the chain condition.

Lemma 3.29. *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions. Then the following hold:*

(i) If P and Q are κ-Knaster, then $P \times Q$ is κ-Knaster.
(ii) If P is κ-Knaster and Q is κ-cc, then $P \times Q$ is κ-cc.

Proof. The proofs are routine using only combinatorial arguments (a forcing argument is not required). □

Note that in general Lemma 3.29 cannot be strengthen to say that the product of two κ-cc forcing notions is κ-cc (consider for instance a Suslin tree T at $\aleph_1$ as a forcing notion; then T is $\aleph_1$-cc, but $T \times T$ has an antichain of size $\aleph_1$).

The following lemma summarises some of the more important forcing properties of a product $P \times Q$ regarding the chain condition.

Lemma 3.30. *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions such that P is κ-Knaster and Q is κ-cc. Then the following holds:*

(i) P forces that Q is κ-cc.
(ii) Q forces that P is κ-Knaster.

Proof. (i). This is easy consequence of Lemmas 3.28(iii) and 3.29(ii).

(ii). We follow the argument from [6], attributed to Magidor. Let $q \in Q$ be a condition which forces that $\{\dot{p}_\alpha \,|\, \alpha < \kappa\}$ is a subset of P of size κ. For each α choose $q_\alpha \le q$ which decides the value of $\dot{p}_\alpha$ and denote this value by p_α. Now, by the κ-Knasterness of P, there is $A \subseteq \kappa$ of size κ such that all conditions in $\{p_\alpha \,|\, \alpha \in A\}$ are pairwise compatible.

To conclude the proof it suffices to show that there is q_α which forces that $B = \{\beta \in A \,|\, q_\beta \in \dot{G}\}$ is unbounded in A. Then if G is a generic filter which contain q_α, the set $\{p_\alpha \,|\, \alpha \in B\}$ is a subset of $\{\dot{p}^G_\alpha \,|\, \alpha < \kappa\}$ of size κ and consists of pairwise compatible conditions.

For contradiction assume that there is no such α. It means that for each $\alpha \in A$ we can find $q^*_\alpha \le q_\alpha$ and $\gamma_\alpha > \alpha$ such that for all $\beta \ge \gamma_\alpha$

$$q^*_\alpha \Vdash q_\beta \notin \dot{G}. \tag{3.4}$$

In particular q^*_α is incompatible with all q_β, $\beta \geq \gamma_\alpha$ and therefore also with all q^*_β, $\beta \geq \gamma_\alpha$. Now, it is easy to construct an unbounded subset A^* of A such that all conditions in $\{q^*_\alpha \mid \alpha \in A^*\}$ are pairwise incompatible. This contradicts the assumption that Q is κ-cc. □

Now we mention some properties of the product of two forcing notions with respect to preservation of κ-distributivity and κ-closure. If P and Q are two κ-distributive forcing notions, then the product $P \times Q$ does not have to be κ-distributive. Again consider a Suslin tree T at $\aleph_1$ as a forcing notion; T is $\aleph_1$-distributive (see Jech [44] for the details), but $T \times T$ may collapse $\aleph_1$ if T is homogeneous (see [17] for the details) and therefore it is not $\aleph_1$-distributive. However, if at least one of P and Q is κ-closed, then the product is κ-distributive. Moreover, if both P and Q are κ-closed then their product is κ-closed.

The following lemma summarises some of the more important properties of the product $P \times Q$ regarding the distributivity and closure.

Lemma 3.31. *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ-closed and Q is κ-distributive. Then the following hold:*

(i) P forces that Q is κ-distributive.
(ii) Q forces that P is κ-closed.

Proof. The proof is routine. □

We can also formulate some results for the product of two forcing notions with respect to preservation of the chain condition and distributivity at the same time. The following lemma appeared in [18].

Lemma 3.32. (Easton) *Let $\kappa > \aleph_0$ be a regular cardinal and assume that P and Q are forcing notions, where P is κ-cc and Q is κ-closed. Then the following hold:*

(i) P forces that Q is κ-distributive.
(ii) Q forces that P is κ-cc.

Proof. For the proof of (i), see [44, Lemma 15.19], (ii) is easy. □

Note that in the previous lemma, we cannot strengthen the condition (i) to get κ-closure (for instance if $P = \mathrm{Add}(\aleph_0, 1)$ and $Q = \mathrm{Add}(\aleph_1, 1)$, then it is easy to check that Q is not $\aleph_1$-closed in $V[P]$).

Easton's lemma 3.32 can be generalised in many ways. Let us state one such generalisation which combines the chain condition and distributivity in a more complicated way (it is probably folklore but we have not found a proof so we give one for the benefit of the reader).[5]

Lemma 3.33. *Let $\kappa > \aleph_0$ be a regular cardinal, let P, R, S be forcing notions and let $\dot{Q}$ be a P-name for a forcing notion. Assume that $P \times R$ is κ-cc and P forces that $\dot{Q}$ is κ-closed. If S is κ-closed, then $(P * \dot{Q}) \times R$ forces that S is κ-distributive.*

Proof. Let use denote $(P * \dot{Q}) \times R$ by Z. Assume for simplicity that the greatest condition in $Z \times S$ forces that $\dot{f} : \kappa' \to \mathrm{ORD}$ is a function in $V[Z][S]$ for some fixed

[5] We intend to use this lemma in a paper under preparation (it is not used for the results in this thesis).

$\kappa' < \kappa$ and some name $\dot{f}$. We will find a stronger condition which will force that this function is already in $V[Z]$. As $\dot{f}$ is arbitrary, this will prove the lemma.

By induction in V, we construct for each $\alpha < \kappa'$ sequences $w^\alpha = \langle ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta) \,|\, \beta < \gamma_\alpha < \kappa \rangle$, of conditions in $Z \times S$ with the following properties:

(i) For each $\beta < \gamma_\alpha$, $w^\alpha_\beta = ((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s^\alpha_\beta)$ decides the value of $\dot{f}(\alpha)$.
(ii) 1_P forces that $\langle \dot{q}^\alpha_\beta \,|\, \beta < \gamma_\alpha \rangle$ is a decreasing sequence of conditions in $\dot{Q}$.
(iii) The set $\{(p^\alpha_\beta, r^\alpha_\beta) \,|\, \beta < \gamma_\alpha\}$ is a maximal antichain in $P \times R$.
(iv) $\langle s^\alpha_\beta \,|\, \beta < \gamma_\alpha \rangle$ forms a decreasing sequence in S.

and for α', $\alpha < \alpha' < \kappa'$:

(v) 1_P forces that $\dot{q}^{\alpha'}_0$ is below each $\dot{q}^\alpha_\beta$, $\beta < \gamma_\alpha$.
(vi) $s^{\alpha'}_0$ is below each s^α_β, $\beta < \gamma_\alpha$.

We first construct the sequence w^0 by induction, ensuring as we go the conditions (i)–(iv) above. Choose $w^0_0 = ((p^0_0, \dot{q}^0_0), r^0_0, s^0_0)$ so that it decides the value of $\dot{f}(0)$. Suppose w^0_β has been constructed for every $\beta < \gamma$; we describe the construction of w^0_γ. If γ is a limit ordinal, first take the lower bound of $\dot{q}^0_\beta$ (denote it $\dot{q}'$) and the lower bound of s^0_β (denote it s'), $\beta < \gamma$. This is possible by condition (ii) and (iv), respectively and from the assumption that $\dot{Q}$ is forced to be κ-closed and S is κ-closed. If γ is a successor cardinal $\delta + 1$, work with $\dot{q}^0_\delta$ as $\dot{q}'$ and s^0_δ as s'.

If possible, choose a condition $((p, \dot{q}), r, s)$ such that p forces that $\dot{q}$ is below $\dot{q}'$, s is below s', (p, r) is incompatible with all the previous elements (p^0_β, r^0_β), $\beta < \gamma$, and crucially $((p, \dot{q}), r, s)$ decides the value of $\dot{f}(0)$. In more detail, if possible first pick any (p', r') incompatible with all the previous pairs (p^0_β, r^0_β), $\beta < \gamma$. Then using the forcing theorem there must be and extension of $((p', \dot{q}'), r', s')$ which decides the value of $\dot{f}(0)$. We denote this extension $((p, \dot{q}), r, s)$ (note that $p \Vdash \dot{q} \leq \dot{q}'$). Set $w^0_\gamma = ((p, \dot{q}''), r, s)$, where $\dot{q}''$ is a name which interprets as $\dot{q}$ below the condition p, and interprets as $\dot{q}'$ below conditions incompatible with p.

If this is not possible, set $\gamma_0 = \gamma$. Note that $\gamma_0 < \kappa$ since $P \times R$ is κ-cc.

The construction of w^α for $\alpha < \kappa'$ proceeds analogously, while ensuring the conditions (v)–(vi).

By the κ-closure of $\dot{Q}$ and S, we can take a single limit of all the conditions appearing in the sequences w^α at the coordinates of $\dot{Q}$ and S – denote these limits $\dot{q}$ and s, respectively. Let $G \times F$ be any $Z \times S$-generic containing $((1_P, \dot{q}), 1_R, s)$. We want to argue that we can define $\dot{f}^{G \times F}$ already in $V[G]$. Let $\alpha < \kappa$ be fixed. By the construction there is a unique pair $(p^\alpha_\beta, r^\alpha_\beta)$ such that $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta)$ is in G. This follows from the construction of the sequences w^α as we ensured that $\{(p^\alpha_\beta, r^\alpha_\beta) \,|\, \beta < \gamma_\alpha\}$ is a maximal antichain in $P \times R$ by condition (iii). Working in $V[G]$, we can define the right value of $\dot{f}(\alpha)$ as the value which is forced by $((p^\alpha_\beta, \dot{q}^\alpha_\beta), r^\alpha_\beta, s)$. □

3.3.3 Comparing forcing notions

In this section we state some facts concerning the comparison of forcing notions. To our knowledge, many of these facts have not been written up in detail in literature, so we include their proofs for the benefit of the reader. The books [49] and [2] are a general reference for this section.

For the purposes of this section, we assume (unless we say otherwise) that our forcing notions are non-trivial and separative.

Recall that we say that $(P, \leq_P)$ and $(Q, \leq_Q)$ are forcing equivalent if their Boolean completions are isomorphic (Definition 3.24).

It is easy to see that forcing-equivalence implies the following weaker property:[6]

> $(*)$ for every P-generic G over V there exists a Q-generic H over V in $V[G]$ such that $V[G] = V[H]$, and conversely.

In this section we will discuss several concepts related to the relationship between two forcing notions $(P, \leq_P)$ and $(Q, \leq_Q)$; these concepts will be formulated in terms of existence of certain functions from P to Q (and conversely) and also in terms of model-theoretic conditions which are weakenings of the condition $(*)$.

Definition 3.34. We say that a function $i : P \to Q$ between partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *dense embedding* if it is order-preserving, $i(p) \perp i(p')$ whenever $p \perp p'$, and the range of i is dense in Q.

It is easy to check that the existence of a dense embedding implies forcing equivalence.

Definition 3.35. We say that a function $\pi : P \to Q$ between $(P, \leq_P)$ and $(Q, \leq_Q)$ is a *projection* if it is order-preserving, $\pi(1_P) = 1_Q$, and

$$\text{for all } p \in P \text{ and all } q \leq_Q \pi(p) \text{ there is } p' \leq_P p \text{ such that } \pi(p') \leq_Q q. \tag{3.5}$$[7]

Let π be as above and fix a P-generic filter G. If $D \subseteq Q$ is open dense in Q then $\pi^{-1}{}''D$ is open dense in P and it is easy to see that $\pi''G$ generates a Q-generic filter. Let us denote this generic filter by H. The forcing P can be decomposed into a two-step iteration of Q followed by a quotient forcing P/H defined as follows:

$$P/H = \{p \in P \mid \pi(p) \in H\}. \tag{3.6}$$

Now, it holds that G is a P/H-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model G is taken as a P-generic filter over V and in second as a P/H-generic filter over $V[H]$.

The converse holds as well. If we first take a Q-generic filter H over V and then a P/H-generic filter G over $V[H]$, then G is a P-generic filter over V and moreover the generic filter H is generated by $\pi''G$.

Definition 3.36. We say that a function $i : Q \to P$ between partial orders $(Q, \leq_Q)$ and $(P, \leq_P)$ is a *complete embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$ and

$$\text{for all } p \in P \text{ there is } q \in Q \text{ such that for all } q' \leq q,\ i(q') \parallel p. \tag{3.7}$$

[6] If P is any forcing notion, then the lottery sum of κ-many copies of P, where $\kappa \geq (2^{|P|})^+$, yields a non-equivalent forcing notion which however satisfies the model-theoretic condition $(*)$.

[7] Note that the condition $\pi(1_P) = 1_Q$ together with (3.5) ensures that the range of π is dense in Q.

Analogues of facts mentioned for projections following Definition 3.35 hold also for complete embeddings. Let i be as in the definition above and fix a P-generic filter G. If $D \subseteq Q$ is predense in Q then $i``D$ is predense in $P$ and $i^{-1}``G$ is a Q-generic filter. Let us denote this generic filter by H and in $V[H]$ define a quotient forcing as follows:

$$P/H = \{p \in P \mid \forall q \in H(p \parallel i(q))\}. \tag{3.8}$$

Then G is a P/H-generic filter over $V[H]$ and $V[G] = V[H][G]$, where in the first model G is taken as a P-generic over V and in second as a P/H-generic over $V[H]$.

The converse direction holds as well. If we first take a Q-generic filter H over V and define the quotient forcing P/H and then take a P/H-generic filter G over $V[H]$, then G is P-generic over V and moreover the generic filter H is equal to $i^{-1}``G$.

Remark 3.37. In general, the quotient forcings (3.6) and (3.8) of two separative forcings do not have to be separative. Consider the following easy example using Cohen forcing $\mathrm{Add}(\kappa, \alpha)$ (see Definition 3.26). Let κ be a regular cardinal and $0 < \beta < \alpha$ be ordinals. Then it is easy to see that π from $\mathrm{Add}(\kappa, \alpha)$ to $\mathrm{Add}(\kappa, \beta)$ defined by $\pi(p) = p \restriction (\kappa \times \beta)$ is a projection. Let G be an $\mathrm{Add}(\kappa, \beta)$-generic filter over V. Then

$$\mathrm{Add}(\kappa, \alpha)/G = \{p \in \mathrm{Add}(\kappa, \alpha) \mid p \restriction (\kappa \times \beta) \in G\}. \tag{3.9}$$

It follows that all conditions in $\mathrm{Add}(\kappa, \beta)$ which are in G are in $\mathrm{Add}(\kappa, \alpha)/G$ and each condition p in the quotient $\mathrm{Add}(\kappa, \beta)$ is compatible with all conditions in G. Thus two arbitrary conditions $q_0 \neq q_1$ in G witness that $\mathrm{Add}(\kappa, \alpha)/G$ is not separative. The argument can be modified for complete embeddings as well.

Complete embeddings have the following equivalent – and often more useful – characterisation.

Definition 3.38. We say that a function $i : Q \to P$ between partial orders $(Q, \leq_Q)$ and $(P, \leq_P)$ is a *regular embedding* if it is order-preserving, $i(q) \perp i(q')$ whenever $q \perp q'$, and $i``A$ is a maximal antichain in P, whenever A is maximal in Q.

Lemma 3.39. *Let $(Q, \leq_Q)$ and $(P, \leq_P)$ be two partial orders. Then a function i from Q to P is a complete embedding if and only if it is a regular embedding.*

Proof. Assume that i is a complete embedding between Q and P. Let $A \subseteq Q$ be a maximal antichain and p in P be given. We will show that there is $a \in A$ such that $i(a) \parallel p$, hence $i``A$ is maximal. As p is in P there is $q \in Q$ such that for all $q' \leq q$, $i(q') \parallel p$ by (3.7). Since A is maximal in Q, there is $a \in A$ such that $a \parallel q$, hence there is $q' \leq q$ such that $q' \leq a$. Therefore $i(q') \leq i(a)$ and $i(q') \parallel p$. Hence $i(a) \parallel p$.

For the converse direction assume that i is a regular embedding between Q and P. Let p in P be given and assume for contradiction that for all $q \in Q$ there is $q' \leq q$ such that $i(q') \perp p$. Then the set

$$D = \{q \in Q \mid i(q) \perp p\} \tag{3.10}$$

is dense in Q. Let $A \subseteq D$ be a maximal antichain, then by the definition of a regular embedding $i``A$ is maximal in P, hence there exists $a \in A$ such that $i(a) \parallel p$. This is a contradiction as a is also in D and therefore $i(a) \perp p$. ∎

It would be tempting to claim that a projection from $(P, \leq_P)$ to $(Q, \leq_Q)$ ensures the existence of a complete embedding from $(Q, \leq_Q)$ to $(P, \leq_P)$ and conversely. But in general we need to use the Boolean completions of P and Q.

Lemma 3.40. *Let $(Q, \leq_Q)$ and $(P, \leq_P)$ be two partial orders. Then the following hold:*

(i) If there is a complete embedding from Q to P, then there is a projection from P to $\mathrm{RO}^+(Q)$.
(ii) If there is a projection from P to Q, then there is a complete embedding from Q to $\mathrm{RO}^+(P)$.

Proof. (i). Let i be a complete embedding from Q to P. Let us define a function π from P to $\mathrm{RO}^+(Q)$ by

$$\pi(p) = \bigvee\{q \in Q \mid \forall q' \leq q(i(q') \parallel p)\}. \tag{3.11}$$

First note that π is defined correctly for all $p \in P$ by (3.7). Moreover, for all q in Q it holds:

$$\pi(i(q)) = q. \tag{3.12}$$

To verify (3.12) denote $Q_p = \{q \in Q \mid \forall q' \leq q\ (i(q') \parallel p)\}$ for $p \in P$. To show that $\pi(i(q)) \leq q$, note that for $q^* \in Q_{i(q)}$ it holds that q^* is below q, otherwise there exists $q' \leq q^*$, which is incompatible with q, by separativity of Q. However, as i is a complete embedding, it holds $i(q') \perp i(q)$, which is a contradiction with the assumption that q^* is in $Q_{i(q)}$. On the other hand, for every $q' \leq q$ it holds that $i(q') \leq i(q)$. Therefore q is in $Q_{i(q)}$.

The function π is a projection. The order preservation follows since $Q_{p'} \subseteq Q_p$ whenever $p' \leq p$. Since all conditions are compatible with the weakest condition $1_{\mathrm{RO}^+(Q)}$, $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$.

Assume that $b < \pi(p)$ (if $b = \pi(p)$ the condition (3.5) is satisfied trivially). Since $\pi(p) = \bigvee\{q \in Q \mid \forall q' \leq q\ (i(q') \parallel p)\}$, there is $q \in Q$ such that $q \leq b$ and $i(q)$ is compatible with p. Hence there is $p^* \in P$ below both $i(q)$ and p. The rest now follows as $\pi(p^*) \leq \pi(i(q))$ and $\pi(i(q)) = q$ by (3.12).

(ii). Let π be a projection from P to Q. Let us define a function i from Q to $\mathrm{RO}^+(P)$ by

$$i(q) = \bigvee\{p \in P \mid \pi(p) \leq q\}. \tag{3.13}$$

First note that i is defined correctly for all $q \in Q$ as π is dense. We will show that the function i is a complete embedding. Clearly, i is order preserving as $\{p \in P \mid \pi(p) \leq q'\} \subseteq \{p \in P \mid \pi(p) \leq q\}$ whenever $q' \leq q$. Assume that $i(q) \parallel i(q')$ for q, $q' \in Q$ we will show that $q \parallel q'$. As we work with a complete Boolean algebra, $i(q) \parallel i(q')$ is equivalent to:

$$i(q) \wedge i(q') = \bigvee\{p \wedge p' \mid \pi(p) \leq q \ \&\ \pi(p') \leq q'\} \neq 0_{\mathrm{RO}^+(P)}. \tag{3.14}$$

Therefore there are p and p' in P such that $p \wedge p' \neq 0_{\mathrm{RO}^+(P)}$, $\pi(p) \leq q$ and $\pi(p') \leq q'$. By density of P in $\mathrm{RO}^+(P)$, there is $p^* \in P$ below $p \wedge p'$ and as π is order preserving $\pi(p^*)$ is below both q and q'.

To conclude that i is a complete embedding, it suffices by Lemma 3.39 to verify that the image of a maximal antichain is maximal. Let A be a maximal antichain in Q, and

$p \in P$ be given (it is enough to consider elements of P as P is dense in $\mathrm{RO}^+(P)$). As A is maximal, there is $a \in A$ such that a and $\pi(p)$ are compatible. Hence there is $q \in Q$ which is below a and $\pi(p)$. By (3.5), there is $p' \leq p$, such that $\pi(p') \leq q$. Since $i(a) = \bigvee\{p \in P \,|\, \pi(p) \leq q\}$ and $\pi(p') \leq q$, we conclude $p' \leq i(a)$. Therefore the antichain i"A is maximal. □

A natural method for obtaining projections between $(P, \leq_P)$ and $(Q, \leq_Q)$ is model-theoretic: it suffices to show that every generic extension $V[G]$, where G is P-generic, contains a Q-generic filter H.

Lemma 3.41. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every P-generic filter G over V, there is in $V[G]$ a Q-generic filter over V. Let $\dot{H}$ be a P-name such that $1_P \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[8] Then the following hold:*

(i) Define $\pi : P \to \mathrm{RO}^+(Q)$ by

$$\pi(p) = \bigwedge\{b \in \mathrm{RO}^+(Q) \,|\, p \Vdash b \in \dot{H}\}. \tag{3.15}$$

Set $b_Q = \pi(1_P) = \bigwedge\{b \in \mathrm{RO}^+(Q) \,|\, 1_P \Vdash b \in \dot{H}\}$. Let $\mathrm{RO}^+(Q)\restriction b_Q$ denote the partial order $\{b \in \mathrm{RO}^+(Q) \,|\, b \leq b_Q\}$. Then

$$\pi : P \to \mathrm{RO}^+(Q)\restriction b_Q \text{ is a projection.} \tag{3.16}$$

(ii) Moreover, π can be defined just using $-Q = \{-q \,|\, q \in Q\}$:

$$\pi(p) = \bigwedge\{-q \,|\, q \in Q \;\&\; p \Vdash -q \in \dot{H}\} = \bigwedge\{-q \,|\, q \in Q \;\&\; p \Vdash q \notin \dot{H}\}. \tag{3.17}$$

Proof. (i). First, we argue that π is well defined, i.e. $\pi(p) > 0_{\mathrm{RO}(Q)}$ for all $p \in P$. To see this, denote:

$$H_p = \{b \in \mathrm{RO}^+(Q) \,|\, p \Vdash b \in \dot{H}\}. \tag{3.18}$$

If $\pi(p) = \bigwedge H_p = 0_{\mathrm{RO}^+(Q)}$, then $D = \{b \in \mathrm{RO}^+(Q) \,|\, \exists h \in H_p(h \perp b)\}$ is dense. Therefore if G contain p, then $H_p \subseteq H = \dot{H}^G$ and also $H \cap D \neq \emptyset$, hence H contains two incompatible elements. This is a contradiction with the assumption that $\dot{H}$ is forced to be an $\mathrm{RO}^+(Q)$-generic filter by P.

Notice also that $\pi(p) = \bigwedge H_p = a$ is forced by p into $\dot{H}$: Consider the following dense set:

$$D = \{b \in \mathrm{RO}^+(b) \,|\, b \leq a \vee \exists h \in H_p(h \perp b)\}. \tag{3.19}$$

If G contains p, but H does not contain a, then H must meet D in some element incompatible with some element in H_p. This is a contradiction. Therefore p forces $\pi(p)$ into $\dot{H}$.

Now, we show that π is a projection. The preservation of the ordering is easy. We check the condition (3.5), i.e. for every $p \in P$ and every $c \leq \pi(p)$, there is $p' \leq p$ such that $\pi(p') \leq c$. Let p and c be given. If $c = \pi(p)$, we are trivially done. So suppose $c < \pi(p)$. If for every $p' \leq p$, $p' \nVdash c \in \dot{H}$, then $p \Vdash \pi(p) - c \in \dot{H}$, which contradicts the fact that

[8] Notice that π defined below depends on the specific name $\dot{H}$ we choose.

$\pi(p)$ is the infimum of $H_p = \{b \in \mathrm{RO}^+(\mathbb{Q}) \mid p \Vdash b \in \dot{H}\}$. It follows that there is some $p' \leq p$, $p' \Vdash c \in \dot{H}$. Then $\pi(p') \leq c$ as required.

(ii). Let p be fixed and let a_p denote $\bigwedge\{-q \mid q \in Q \mathbin{\&} p \Vdash -q \in \dot{H}\}$. We wish to show that $\pi(p)$ as in (3.15) is equal to a_p. Clearly $\pi(p) \leq a_p$. For the converse first notice that

$$\pi(p) = \bigwedge\{-q \mid q \in Q \mathbin{\&} \pi(p) \leq -q\}. \tag{3.20}$$

This follows from the fact that each element b of $\mathrm{RO}^+(Q)$ can be expressed as a supremum of elements of Q which are below b. Let as denote $\{-q \mid q \in Q \mathbin{\&} \pi(p) \leq -q\}$ by $-Q_p$. To conclude the prove it is enough to show that $-Q_p$ is a subset of $\{-q \mid q \in Q \mathbin{\&} p \Vdash -q \in \dot{H}\}$, i.e. to prove that if $\pi(p) \leq -q$ then $p \Vdash -q \in \dot{H}$. However, we already prove that p forces $\pi(p)$ into $\dot{H}$, therefore if $-q \geq \pi(p)$ then $p \Vdash -q \in \dot{H}$. □

Lemma 3.42. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partial orders. Assume that for every P-generic filter G over V, there is in $V[G]$ a Q-generic filter over V. Let $\dot{H}$ be a $\mathrm{RO}^+(P)$-name such that $1_{\mathrm{RO}^+(P)} \Vdash$ "$\dot{H}$ is a $\mathrm{RO}^+(Q)$-generic filter".[9] Then the following hold:*

(i) Define $i : \mathrm{RO}^+(Q) \to \mathrm{RO}^+(P)$ by

$$i(b) = \bigvee\{a \in \mathrm{RO}^+(P) \mid a \Vdash b \in \dot{H}\}. \tag{3.21}$$

Set $b_Q = \bigwedge\{b \in \mathrm{RO}^+(Q) \mid 1_{\mathrm{RO}^+(P)} \Vdash b \in \dot{H}\}$. Let $\mathrm{RO}^+(Q) \upharpoonright b_Q$ denote the partial order $\{b \in \mathrm{RO}^+(Q) \mid b \leq b_Q\}$. Then

$$i : \mathrm{RO}^+(Q) \upharpoonright b_Q \to \mathrm{RO}^+(P) \text{ is a complete embedding,} \tag{3.22}$$

where (3.21) implies $i(b_Q) = 1_{\mathrm{RO}^+(P)}$.

(ii) Let $Q \upharpoonright b_Q$ denote the partial order $(Q \cap \mathrm{RO}^+(Q) \upharpoonright b_Q) \cup \{b_Q\}$. Then $i' = i \upharpoonright (Q \upharpoonright b_Q)$ from $Q \upharpoonright b_Q$ to $\mathrm{RO}^+(P)$ is a complete embedding.

(iii) Moreover, i' can be defined using only the conditions in P:

$$i'(q) = \bigvee\{p \in P \mid p \Vdash q \in \dot{H}\}. \tag{3.23}$$

Proof. (i). First notice that i is correctly defined below b_Q, i.e. for $b \leq b_Q$ the set $\{a \in \mathrm{RO}^+(P) \mid a \Vdash b \in \dot{H}\}$ is nonempty. Let us denote this set by $\mathrm{RO}^+(P)_b$. If $b = b_Q$ then $i(b) = 1_{\mathrm{RO}^+(P)}$ by density argument as in (3.19). Assume that $b < b_Q$. If $\mathrm{RO}^+(P)_b$ is empty then there is no $a \in \mathrm{RO}^+(P)$, $a \Vdash b \in \dot{H}$, i.e. $1_{\mathrm{RO}^+(P)} \Vdash b \notin \dot{H}$. Then $1_{\mathrm{RO}^+(P)}$ forces $-b \wedge b_Q$ to be in $\dot{H}$ and this is a contradiction as we defined b_Q to be the infimum of the conditions in $\mathrm{RO}^+(Q)$ which are forced into $\dot{H}$ by $1_{\mathrm{RO}^+(P)}$.

Further notice that $i(b)$ forces b into $\dot{H}$. If not, then there is a below $i(b)$ which forces that b is not in $\dot{H}$ but as a is below $i(b) = \bigvee\{a \in \mathrm{RO}^+(P) \mid a \Vdash b \in \dot{H}\}$, there is $a_0 \leq a$ which forces b into $\dot{H}$. This is a contradiction.

If $b \leq b'$, then every $a \in \mathrm{RO}^+(P)$ which forces $b \in \dot{H}$, forces b' in $\dot{H}$ as well, since $\dot{H}$ is forced to be a generic filter, therefore i is order preserving. The preservation of incompatibility is easy, as compatible conditions cannot force two incompatible conditions into a filter.

[9] Notice that i defined below depends on the specific name $\dot{H}$ we choose.

To finish the proof, it suffices by Lemma 3.39 to show that the image of a maximal antichan is maximal. Let A be a maximal antichain in $\mathrm{RO}^+(Q)$ and let b in $\mathrm{RO}^+(P)$ be given. As A is a maximal antichain and $\dot{H}$ is forced to be a generic filter, there has to be $a \in A$ and $b' \leq b$ such that $b' \Vdash a \in \dot{H}$. But since $i(a) = \bigvee\{b \in \mathrm{RO}^+(P) \,|\, b \Vdash a \in \dot{H}\}$, $b' \leq i(a)$ and hence $b \| i(a)$; therefore i"A is maximal.

(ii) This follows from Lemma 3.45(i).

(iii). Let q be fixed and let a_q denote $\bigvee\{p \in P \,|\, p \Vdash q \in \dot{H}\}$. We show that $i'(q)$ as in (3.21) is equal to a_q. Clearly $a_q \leq i'(q)$. For the converse, as $i'(q)$ is an element of $\mathrm{RO}^+(P)$ and P is dense in $\mathrm{RO}^+(P)$, $i'(q) = \bigvee\{p \in P \,|\, p \leq i'(q)\}$; but all conditions below $i'(q)$ have to force q in $\dot{H}$, and therefore $i'(q) \leq a_q$. □

Remark 3.43. Note that in the previous two lemmas, Lemma 3.41 and Lemma 3.42, we cannot in general require $\pi(1_P) = 1_{\mathrm{RO}^+(Q)}$ or $i(1_Q) = 1_{\mathrm{RO}^+(P)}$, respectively. Consider the lottery sum of $\mathrm{Add}(\aleph_0, 1)$ and $\mathrm{Add}(\aleph_1, 1)$. It is easy to see that every $\mathrm{Add}(\aleph_0, 1)$-generic filter adds a generic filter for the lottery but only below a condition which chooses $\mathrm{Add}(\aleph_0, 1)$.

We conclude this section by further facts about projections and complete embeddings.

Lemma 3.44. *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $\pi : P \to Q$ is a projection.*

(i) If P' is dense in P, then $\pi \restriction P' : P' \to Q$ is a projection.
(ii) (a) If P is dense in P', then there is $\pi' \supseteq \pi$ such that $\pi' : P' \to \mathrm{RO}^+(Q)$ is a projection.
(b) If P' is forcing equivalent with P. Then there is a projection $\pi' : P' \to \mathrm{RO}^+(Q)$.
(iii) Let $\dot{R}$ be a P-name for a forcing notion. Then π naturally extends to a projection $\pi' : P * \dot{R} \to Q$.

Proof. (i). Obvious.

(ii)(a). For $p' \in P'$ define

$$\pi'(p') = \bigvee\{\pi(p) \,|\, p \in P \,\&\, p \leq p'\}. \tag{3.24}$$

By density of P in P', $\{\pi(p) \,|\, p \leq p'\}$ is non-empty for every p' and therefore $\pi'(p')$ is in $\mathrm{RO}^+(Q)$. If $p' \leq q'$ are in P', then clearly $\pi'(p') \leq \pi'(q')$. Suppose $p' \in P'$ is arbitrary and $b \leq \pi'(p')$. By the definition of $\pi'(p)$, there is $b' \leq b$ such that for some $p \leq p'$, $p \in P$, $b' \leq \pi(p)$. It follows there is some $q \leq p \leq p'$, $q \in P$, such that $\pi(q) = \pi'(q) \leq b' \leq b$ as desired.

(ii)(b). As P is dense in $\mathrm{RO}^+(P)$, by the previous item there is a projection π^* from $\mathrm{RO}^+(P)$ to $\mathrm{RO}^+(Q)$. Since P' is forcing equivalent to P, P' is dense in $\mathrm{RO}^+(P)$, and $\pi' = \pi^* \restriction P'$ is a projection from P' to $\mathrm{RO}^+(Q)$ by (i).

(iii). Define

$$\pi'(p, \dot{r}) = \pi(p), \tag{3.25}$$

for every $(p, \dot{r})$ in $P * \dot{R}$. If $(p_1, \dot{r}_1) \leq (p_2, \dot{r}_2)$, then in particular $p_1 \leq p_2$, and so $\pi'(p_1, \dot{r}_1) \leq \pi'(p_2, \dot{r}_2)$ because π is order-preserving. If $(p, \dot{r})$ is arbitrary and $b \leq \pi'(p, \dot{r}) = \pi(p)$, then since π is a projection, there is $p' \leq p$ such that $\pi(p') \leq b$. Since $(p', \dot{r}) \leq (p, \dot{r})$, $\pi'(p', \dot{r}) \leq b$ is as required. □

Lemma 3.45. *Assume $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders and $i : Q \to P$ is a complete embedding.*

(i) If Q' is dense in Q, then $i \restriction Q' : Q' \to P$ is a complete embedding.

(ii) (a) If Q is dense in Q', then there is $i' \supseteq i$ such that $i' : Q' \to \mathrm{RO}^+(P)$ is a complete embedding.

(b) If Q' is forcing equivalent with Q, then there is a complete embedding $i' : Q' \to \mathrm{RO}^+(P)$.

(iii) Let $\dot{R}$ be a P-name for a forcing notion. Then i naturally extends to a complete embedding $i' : Q \to P * \dot{R}$.

Proof. (i). Obvious.

(ii)(a). For $q' \in Q'$ define

$$i'(q') = \bigvee \{i(q) \,|\, q \in Q \;\&\; q \leq q'\}. \tag{3.26}$$

By density of Q in Q', $\{i(q) \,|\, q \leq q'\}$ is non-empty for every q' and therefore $i'(q')$ is in $\mathrm{RO}^+(P)$. If $q'_0 \leq q'_1$ in Q', then clearly $i'(q'_0) \leq i'(q'_1)$.

Assume that $i'(q'_0)$ is compatible with $i'(q'_1)$, then

$$i'(q'_0) \wedge i'(q'_1) = \bigvee \{i(q_0) \wedge i(q_1) \,|\, q_0, q_1 \in Q \;\&\; q_0 \leq q'_0 \;\&\; q_1 \leq q'_1\} \neq 0_{\mathrm{RO}(P)}. \tag{3.27}$$

Therefore there are $q_0 \leq q'_0$ and $q_1 \leq q'_1$ such that $i(q_0)$ and $i(q_1)$ are compatible. By the definition of complete embedding, q_0 is compatible with q_1. Hence $q'_0 \,\|\, q'_1$, as $q_0 \leq q'_1$ and $q_1 \leq q'_1$.

Suppose $b \in \mathrm{RO}^+(P)$ is arbitrary. Then there is $p \in P$, $p \leq b$, by density of P in $\mathrm{RO}^+(P)$. Therefore there is $q \in Q$ so that for all $q^* \in Q$ such that $q^* \leq q$, $i(q^*)$ is compatible with p, hence with b. Now, we need to show that for all $q' \in Q'$ such that $q' \leq q$, $i'(q')$ is compatible with b. Let $q' \leq q$, $q' \in Q'$, be given and denote $Q_{q'} = \{i(q) \,|\, q \in Q \;\&\; q \leq q'\}$ so that $i'(q') = \bigvee Q_{q'}$. As all conditions in $Q_{q'}$ are compatible with b, and so is $i'(q')$.

(ii)(b). By (a) and the fact that Q is dense in $\mathrm{RO}^+(Q)$ we conclude that there is a complete embedding i^* from $\mathrm{RO}^+(Q)$ to $\mathrm{RO}^+(P)$. Since Q' is forcing equivalent to Q, Q' is dense in $\mathrm{RO}^+(Q)$, hence $i' = i^* \restriction Q'$ is a complete embedding from Q' to $\mathrm{RO}^+(P)$ by (i).

(iii). Define

$$i'(q) = (i(q), 1_{\dot{R}}). \tag{3.28}$$

If $q_0 \leq q_1$, then $i'(q_0) = (i(q_0), 1_{\dot{R}}) \leq (i(q_1), 1_{\dot{R}}) = i'(q_1)$ because i is order-preserving. The same argument holds for the preservation of incompatibility. Let $(p, \dot{r})$ is arbitrary. Then there is $q \in Q$ such that for all $q' \leq q$, $i(q') \,\|\, p$ and therefore for all $q' \leq q$, $i'(q')$ is compatible with $(p, \dot{r})$. □

3.4 Forcing notions used in the thesis

Apart from the usual Cohen forcing (see Definition 3.26), we also use Mitchell forcing (in several variants) and Prikry forcing (with and without interleaved collapses). In this section we briefly review their definitions and basic properties.

3.4.1 Mitchell forcing

Mitchell forcing was first defined by Mitchell in [55]. The presentation of Mitchell forcing, which we use in this thesis, was introduced by Abraham in [1]. In this section we review the basic properties and state several facts which will be used in the thesis. All proofs can be found in [1].

Before we define Mitchell forcing, let us fix the following convention: if p is a condition in $\mathrm{Add}(\kappa,\alpha)$ and $\beta<\alpha$, let $p\restriction\beta$ denote the restriction of p to $\mathrm{Add}(\kappa,\beta)$.

Definition 3.46. Let κ be a regular cardinal and $\lambda>\kappa$ an inaccessible cardinal. *Mitchell forcing* at κ of length λ, denoted by $\mathbb{M}(\kappa,\lambda)$, is the set of all pairs (p,q) such that p is in Cohen forcing $\mathrm{Add}(\kappa,\lambda)$ and q is a function with $\mathrm{dom}(q)\subseteq\lambda$ of size at most κ and for every $\beta\in\mathrm{dom}(q)$, it holds:

$$1_{\mathrm{Add}(\kappa,\beta)}\Vdash q(\beta)\in\dot{\mathrm{Add}}(\kappa^+,1), \tag{3.29}$$

where $\dot{\mathrm{Add}}(\kappa^+,1)$ is the canonical $\mathrm{Add}(\kappa,\beta)$-name for Cohen forcing at κ^+.

A condition (p,q) is stronger than (p',q') if

(i) $p\le p'$,
(ii) $\mathrm{dom}(q)\supseteq\mathrm{dom}(q')$ and for every $\beta\in\mathrm{dom}(q')$, $p\restriction\beta\Vdash q(\beta)\le q'(\beta)$.

Assuming that $\kappa^{<\kappa}=\kappa$ and $\lambda>\kappa$ is an inaccessible cardinal, Mitchell forcing $\mathbb{M}(\kappa,\lambda)$ is λ-Knaster and κ-closed. Moreover, it collapses the cardinals in the open interval (κ^+,λ) to κ^+ and forces $2^\kappa=\lambda=\kappa^{++}$. The preservation of the cardinals (in particular of κ^+) is shown by means of the product analysis due to Abraham [1].

Let $\mathbb{T}$ be defined as follows:

$$\mathbb{T}=\{(\emptyset,q)\mid(\emptyset,q)\in\mathbb{M}(\kappa,\lambda)\}. \tag{3.30}$$

The ordering on $\mathbb{T}$ is the one induced from $\mathbb{M}(\kappa,\lambda)$. It is clear that $\mathbb{T}$ is κ^+-closed in V. We will call $\mathbb{T}$ the *term forcing* (of Mitchell forcing).

It is is easy to see that the function

$$\pi:\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}\to\mathbb{M}(\kappa,\lambda) \tag{3.31}$$

which maps $(p,(\emptyset,q))$ to (p,q) is a projection. Since the product $\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}$ preserves κ^+, so does the forcing $\mathbb{M}(\kappa,\lambda)$. Another consequence of this product analysis (i.e. of the existence of the projection π) is that $\mathbb{M}(\kappa,\lambda)$ is forcing-equivalent to $\mathrm{Add}(\kappa,\lambda)*\dot{\mathbb{Q}}_{\mathbb{M}}$, where $\dot{\mathbb{Q}}_{\mathbb{M}}$ is a forcing notion which is forced to be κ^+-distributive.

There are natural projections from Mitchell forcing of length λ to Mitchell forcings of shorter lengths and a projection to Cohen forcing $\mathrm{Add}(\kappa,\lambda)$. For the first claim, define a function $\sigma^{\lambda,\alpha}$ from $\mathbb{M}(\kappa,\lambda)$ to $\mathbb{M}(\kappa,\alpha)$, where α is an ordinal between κ and λ, as follows: $\sigma^{\lambda,\alpha}((p,q))=(p\restriction\alpha,q\restriction\alpha)$. For the second claim, define a function ρ from $\mathbb{M}(\kappa,\lambda)$ to $\mathrm{Add}(\kappa,\lambda)$ by $\rho((p,q))=p$. It is easy to see that $\sigma^{\lambda,\alpha}$ and ρ are projections.

Remark 3.47. Notice that the term forcing $\mathbb{T}$ collapses the cardinals between κ^+ and λ: Suppose $\kappa^{<\kappa}=\kappa$ and λ is inaccessible. As $\mathbb{T}$ is κ^+-closed, Cohen forcing $\mathrm{Add}(\kappa,\lambda)$ is still κ^+-Knaster and κ-closed in $V[\mathbb{T}]$. In particular, it does not collapse cardinals over $V[\mathbb{T}]$ (so it must be $\mathbb{T}$ which collapses the cardinals).

The term forcing analysis carries over to quotients given by the projection $\sigma^{\lambda,\alpha}$ whenever α is an inaccessible cardinal between κ and λ. Let G_α be an $\mathbb{M}(\kappa,\alpha)$-generic filter and define in $V[G_\alpha]$ the quotient $\mathbb{M}(\kappa,\lambda)/G_\alpha$ as follows:

$$\mathbb{M}(\kappa,\lambda)/G_\alpha = \{(p,q) \in \mathbb{M}(\kappa,\lambda) \mid (p \restriction \alpha, q \restriction \alpha) \in G_\alpha\}. \tag{3.32}$$

Regarding this quotient, we can now analogously define the term forcing $\mathbb{T}_\alpha$ in $V[G_\alpha]$:

$$\mathbb{T}_\alpha = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}(\kappa,\lambda)/G_\alpha\}. \tag{3.33}$$

And also a function π_α from $\mathrm{Add}(\kappa, \lambda - \alpha) \times \mathbb{T}_\alpha$ to $\mathbb{M}(\kappa,\lambda)/G_\alpha$ by $\pi_\alpha((p,(\emptyset,q))) = (p,q)$.

Fact 3.48. *Let α be inaccessible and G_α an $M(\kappa,\alpha)$-generic filter. Then in $V[G_\alpha]$ the following hold:*

(i) *π_α is a projection from $\mathrm{Add}(\kappa,\lambda-\alpha) \times \mathbb{T}_\alpha$ to $\mathbb{M}(\kappa,\lambda)/G_\alpha$.*
(ii) *$\mathbb{T}_\alpha$ is κ^+-closed in $V[G_\alpha]$.*

Using these facts, one can show the following:

Theorem 3.49. (Mitchell) *Assume $\kappa^{<\kappa} = \kappa$. If λ is a weakly compact cardinal, then $\mathbb{M}(\kappa,\lambda)$ forces the tree property at $\lambda = \kappa^{++}$; if λ is just a Mahlo cardinal, $\mathbb{M}(\kappa,\lambda)$ forces the weak tree property at $\lambda = \kappa^{++}$.*

For completeness, consider the quotient of $\mathrm{Add}(\kappa,\lambda) \times \mathbb{T}$ after $\mathbb{M}(\kappa,\lambda)$. Let G be $\mathbb{M}(\kappa,\lambda)$-generic. We define

$$\mathbb{Q}_\mathbb{T} = (\mathrm{Add}(\kappa,\lambda) \times \mathbb{T})/G = \{(p,(\emptyset,q)) \in \mathrm{Add}(\kappa,\lambda) \times \mathbb{T} \mid (p,q) \in G\}. \tag{3.34}$$

Assuming $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, $\mathbb{Q}_\mathbb{T}$ adds a special λ-Aronszajn tree: As $\mathbb{M}(\kappa,\lambda) * \dot{\mathbb{Q}}_\mathbb{T}$ is forcing equivalent to $\mathrm{Add}(\kappa,\lambda) \times \mathbb{T}$, it suffices to show that the latter forcing adds a special λ-Aronszajn tree. Since $\mathbb{T}$ is κ^+-closed, it still holds in the generic extension by $\mathbb{T}$ that $2^\kappa = \kappa^+$; it also holds that $\lambda = \kappa^{++}$ by Remark 3.47. Therefore in $V[\mathbb{T}]$ there exists a special λ-Aronszajn tree T, and the forcing $\mathrm{Add}(\kappa,\lambda)$ is λ-Knaster. By Fact 4.3, it follows that $\mathrm{Add}(\kappa,\lambda)$ does not add cofinal branches to the tree T; therefore T is a special λ-Aronszajn tree in $V[\mathbb{T}][\mathrm{Add}(\kappa,\lambda)]$.

Remark 3.50. Note that the generic for the Cohen part of $\mathbb{Q}_\mathbb{T}$ is already added by the $\mathbb{M}(\kappa,\lambda)$-generic G; therefore the forcing $\mathbb{Q}_\mathbb{T}$ is forcing equivalent over $V[G]$ to the forcing poset $\{(\emptyset,q) \in \mathbb{T} \mid (\emptyset,q) \in G\}$.

The following lemma summarises the preservation of the chain conditions and distributivity by Mitchell forcing.

Lemma 3.51. *Assume $\kappa \geq \aleph_0$ is regular and $\lambda > \kappa$ is inaccessible. Assume P is κ^+-cc, Q is κ^+-closed and R is κ^+-Knaster. Then the following holds:*

(i) *$P \times \mathbb{M}(\kappa,\lambda)$ forces that Q is κ^+-distributive.*
(ii) *$Q \times \mathbb{M}(\kappa,\lambda)$ forces that P is κ^+-cc.*
(iii) *$R \times \mathbb{M}(\kappa,\lambda)$ forces that P is κ^+-cc.*

Proof. (i). Let $\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}$ be the product forcing which projects onto $\mathbb{M}(\kappa,\lambda)$, where $\mathbb{T}$ is the κ^+-closed term forcing. The product $\mathbb{T}\times Q$ is κ^+-distributive over $V[P\times\mathrm{Add}(\kappa,\lambda)]$ by Easton's lemma 3.32, and thus

$$\text{(3.35)} \quad \text{all } \kappa\text{-sequences of ordinals in } V[P\times\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}\times Q] \text{ are already in } V[P\times\mathrm{Add}(\kappa,\lambda)].$$

There is a natural projection

$$\pi : P\times\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}\times Q \to P\times\mathbb{M}(\kappa,\lambda)\times Q. \tag{3.36}$$

If there were a condition r in $P\times\mathbb{M}(\kappa,\lambda)\times Q$ forcing a counterexample to the κ^+-distributivity of Q over $V[P\times\mathbb{M}(\kappa,\lambda)]$, one could pick a generic filter F for $P\times\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}\times Q$ such that for some $r'\in F$, $\pi(r')\le r$. In $V[F]$, the κ-sequence of ordinals forced by r to violate the κ^+-distributivity would contradict (3.35).

(ii) and (iii). It suffices to argue that P is κ^+-cc in the generic extension by $Q\times\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}$ for (ii), and $R\times\mathrm{Add}(\kappa,\lambda)\times\mathbb{T}$ for (iii). This is easy to show using the Easton's lemma 3.32. Note that the assumption of Knasterness for R ensures that $R\times\mathrm{Add}(\kappa,\lambda)$ is still κ^+-Knaster. □

3.4.2 Prikry forcing

Prikry forcing was devised by Prikry [58] and is used to add a cofinal ω-sequence of ordinals to a measurable cardinal κ, without adding any bounded subset of κ or collapsing any cardinals. It is a robust forcing which does not require any cardinal-arithmetic assumptions (besides the measurability of κ) to behave properly.

Definition 3.52. Let κ be a measurable cardinal and U a normal ultrafilter on κ. *Prikry forcing* at κ, denoted by $\mathrm{Prk}_U(\kappa)$, is the set of all pairs (s,A) such that

(i) s is a finite subset of κ;
(ii) $A\in U$;
(iii) $\min(A)>\max(s)$.

The ordering is defined by $(s,A)\le(t,B)$ if

(i) s is an end-extension of t;
(ii) $A\subseteq B$;
(iii) $s\setminus t\subseteq B$.

We say that (s,A) *directly extends* (t,B), and write it as

$$(s,A)\le^*(t,B) \tag{3.37}$$

if $(s,A)\le(t,B)$ and $s=t$.

Prikry forcing $\mathrm{Prk}_U(\kappa)$ satisfies the following important property (*Prikry property*): If φ is a sentence in the forcing language and (s,A) is a condition, then there exists a direct extension $(s,B)\le^*(s,A)$ which decides φ.

Prikry forcing $\mathrm{Prk}_U(\kappa)$ is κ^+-Knaster since all conditions with the same s are compatible. The forcing is not even $\aleph_1$-closed in $\le$, but it is κ-closed in the ordering $\le^*$. The closure of $\le^*$ together with the Prikry property ensures that no bounded subsets of κ are added, and therefore all cardinals (and cofinalities) below κ are preserved.

3.4.3 Prikry forcing with collapses

Prikry forcing $\mathrm{Prk}_U(\kappa)$ defined in the previous section changes the cofinality of a large cardinal κ to ω. Prikry forcing with collapses, which we define now, collapses κ to become $\aleph_\omega$.

Let us recall the following collapsing forcing notions:

Definition 3.53. Let κ be a regular cardinal and $\lambda > \kappa$ a cardinal. The collapse forcing, denoted by $\mathrm{Coll}(\kappa, \lambda)$, is the set of all partial functions from κ to λ of size less than κ. The ordering is by reverse inclusion, i.e. $p \leq q \leftrightarrow q \subseteq p$.

Notice that $\mathrm{Coll}(\kappa, \kappa)$ is equivalent to $\mathrm{Add}(\kappa, 1)$. $\mathrm{Coll}(\kappa, \lambda)$ is κ-closed, and if $\lambda^{<\kappa} = \lambda$, the size of $\mathrm{Coll}(\kappa, \lambda)$ is λ. In particular, under GCH, $\mathrm{Coll}(\kappa, \lambda)$ collapses λ to κ (and preserves all other cardinals).

Definition 3.54. Let κ be a regular cardinal and $\lambda > \kappa$ a cardinal. *Lévy collapse*, denoted by $\mathrm{Coll}(\kappa, <\lambda)$, is the set of all partial functions p from $\lambda \times \kappa$ to λ of size less than κ such that $p(\alpha, \beta) < \alpha$ for every $(\alpha, \beta) \in \mathrm{dom}(p)$. The ordering is by reverse inclusion, i.e. $p \leq q \leftrightarrow q \subseteq p$.

If λ is inaccessible, $\mathrm{Coll}(\kappa, <\lambda)$ is λ-cc and preserves λ so that λ becomes the successor of κ in the generic extension. $\mathrm{Coll}(\kappa, <\lambda)$ is equivalent to the product of $\mathrm{Coll}(\kappa, |\alpha|)$ for $\alpha < \lambda$ with support of size less than κ.

Prikry forcing with collapses $\mathrm{PrkCol}(U, G^g)$ which we define next can be described as follows: it simultaneously adds an ω-sequence cofinal in κ, and collapses all but finitely many cardinals between the points in the ω-sequence, ensuring $\kappa = \aleph_\omega$ in the generic extension. It is important that the two tasks (adding an ω-sequence and collapsing) are performed at the same time; it is known that performing them one by one may to lead to collapsing (see [41] for more details).

Prikry forcing with collapses was introduced in [31]. Our presentation follows [31].

Definition 3.55. Let κ be a measurable cardinal, U a normal measure at κ, and $j_U : V \to M$ the ultrapower embedding generated by U. *Prikry forcing with collapses*, denoted by $\mathrm{PrkCol}(U, G^g)$, is determined by U and a "guiding generic" G^g, which is a $\mathrm{Coll}(\kappa^{+k}, < j(\kappa))^M$-generic filter over M, for some natural number $1 < k < \omega$.

A condition $r = (s, F, H)$ in $\mathrm{PrkCol}(U, G^g)$ is defined as follows:

(i) s is a finite sequence of the form $(f_0, \alpha_1, \ldots, \alpha_{m-1}, f_{m-1})$, where the α_i's are inaccessible cardinals below κ (we call s the *lower part*);
(ii) $\alpha_i < \alpha_j$ for $i < j < m$;
(iii) $f_0 \in \mathrm{Coll}(\aleph_0, < \alpha_1)$ and $f_i \in \mathrm{Coll}(\alpha_i^{+k}, < \alpha_{i+1})$ for $0 < i < m-1$ and $f_{m-1} \in \mathrm{Coll}(\alpha_{m-1}^{+k}, <\kappa)$;
(iv) $A \in U$ and $\min(A) > \alpha_{m-1}$;
(v) F is a function defined on A such that $F(\alpha) \in \mathrm{Coll}(\alpha^{+k}, <\kappa)$ (α inaccessible);
(vi) $[F]_U$, the equivalence class of F in M, is in G^g.

The ordering is defined by $((f_0, ..., \alpha_{m-1}, f_{m-1}), A, F) \leq ((h_0, ..., \beta_{n-1}, h_{n-1}), B, H)$ if

(i) $n \leq m$;

(ii) $f_0 \leq h_0$, and $\alpha_i = \beta_i$ and $f_i \leq h_i$ for $0 < i < n$ (note this implies $\beta_{n-1} < \alpha_n$ if $m - 1 \geq n$);
(iii) $A \subseteq B$;
(iv) $\alpha_i \in B$ for $n \leq i < m$;
(v) $F(\alpha) \leq H(\alpha)$ for $\alpha \in A$;
(vi) $f_i \leq H(\alpha_i)$ for $n \leq i < m$.

If $n = m$ in the definition above, we say that the first condition *directly extends* the second condition. Notice that unlike in $\mathrm{Prk}_U(\kappa)$, the direction extension may change the elements in the lower part s by strengthening the collapsing conditions.

The forcing satisfies the Prikry condition: any sentence in the forcing language is decided by a direct extension (this crucially uses the fact that G^g is a generic filter over M).

$\mathrm{PrkCol}(U, G^g)$ is κ^+-Knaster because all elements in G^g are compatible, and so the compatibility is determined by the lower part. A product-like analysis and the Prikry property are used to show that below κ only the cardinals explicitly collapsed by $\mathrm{PrkCol}(U, G^g)$ are collapsed.

3.5 The continuum function

One of the motivations of our research is to study the continuum function – the function which maps κ to 2^κ – with respect to large-cardinal properties. In this section we briefly review some basic facts regarding the continuum function.

The behaviour of the continuum function differs at regular and singular cardinals. It is known that at regular cardinals the continuum function is very easily changed by forcing, as was shown by Easton [18].

Definition 3.56. We say that a proper class function F from regular cardinals to cardinals is an *Easton function* if for all κ and λ it holds:

(i) $\kappa < \lambda \to F(\kappa) \leq F(\lambda)$;
(ii) $\mathrm{cf}(F(\kappa)) > \kappa$.

Note that the continuum function satisfies the requirements of the Easton function with $F(\kappa) = 2^\kappa$. By the following theorem, the conditions (i) and (ii) of Definition 3.56 are the only restrictions which ZFC puts to the continuum function on regular cardinals.

Theorem 3.57. (Easton [18]) *Let M be a countable transitive model of* ZFC + GCH *and let F be a definable Easton function*[10] *in M. Then there is cofinality-preserving generic extension $M[G]$ of M, where the Easton function F is the continuum function on regular cardinals, i.e. for all regular cardinals $\kappa \in M[G]$:*

$$M[G] \models F(\kappa) = 2^\kappa. \tag{3.38}$$

The situation for singular cardinals is quite different. It is known that ZFC puts non-trivial restrictions on the values of 2^κ for singular κ. The following result of Silver appeared in [64].

[10] The definability is not essential; but a more careful formulation must be used to avoid this assumption.

Theorem 3.58. (Silver) *Let κ be a singular cardinal with an uncountable cofinality. If $2^\alpha = \alpha^+$ for stationarily many cardinals α below κ, then $2^\kappa = \kappa^+$.*

The assumption of the uncountable cofinality is necessary in Silver's theorem. Already $\aleph_\omega$ can be the first cardinal where the GCH fails under some large cardinal assumptions (see Gitik [31]). However, there are some non-trivial bounds on $2^{\aleph_\omega}$ in ZFC: by a remarkable result of Shelah (see [62]), if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega}$ is strictly less than the minimum of $\aleph_{(2^{\aleph_0})^+}$ and $\aleph_{\omega_4}$.

The restrictions on the continuum function on a singular cardinal κ with uncountable cofinality are similar to restrictions which large cardinals put on the continuum function: an easy argument shows that if κ is a measurable cardinal and GCH holds on a set in some normal measure on κ, then GCH holds at κ (notice the subtle difference: in Silver's theorem we have a stationary set while for the measurable cardinal we have a measure-one set). More details regarding the continuum function and large cardinals can be found in [24].

For the results in our thesis (Sections 6 and 7) it is relevant to consider the following statement $(*)_n$ whose consistency strength provides a lower bound for the results in our thesis:

$(*)_n$ There is a singular strong limit cardinal μ of countable cofinality such that $2^\mu = \mu^{+n}$, where $2 \leq n < \omega$.

Notice that $(*)_n$ implies the failure of GCH at κ; since κ is a singular strong limit cardinal, we may also say that $(*)_n$ implies the failure of SCH – the Singular Cardinal Hypothesis – a weakening of GCH formulated only for singular cardinals and which says that if κ is a strong limit singular cardinal, then $2^\kappa = \kappa^+$.[11]

$(*)_n$ is equiconsistent with the existence of a large cardinal κ with the Mitchell order $o(\kappa) = \kappa^{+n}$ (see [31, 32, 33] and Section 3.2 for more details). We will work with the stronger assumptions of the existence of a supercompact cardinal (Section 6), and of a strong cardinal κ of an appropriate degree (Section 7). More precisely, in Section 7 we will start with an $H(\lambda^{+n})$-strong cardinal κ, where λ is the least weakly compact cardinals above κ and $1 \leq n < \omega$. While the assumption of $H(\lambda^{+n})$-strongness is not optimal, it is quite close to being optimal, and is certainly much weaker than supercompactness (see [30] for more details regarding the exact consistency strength of the tree property at the double successor of a singular strong limit cardinal).

Remark 3.59. Let us add that the assumption of supercompactness in Section 6 may be weakened to a strong cardinal of an appropriate degree by methods in Section 7 at the price of a more complicated argument.[12] In fact, the assumption of strongness may be further weakened and formulated in terms of tall cardinals of an appropriate degree (but this would still be not optimal according to [30], so we will not give details). See Section 4.1 for more details regarding the background for the tree property.

[11] SCH may also be formulated for singular cardinals which are not strong limit; see [44] for more details.

[12] While a supercompact cardinal κ can be made indestructible with GCH still holding above and at κ by [51], the analogous construction in Section 7 (see [40] for more details) non-trivially enlarges the size of 2^{κ^+} destroying the strong-limitness of the weakly compact cardinal λ (see Section 7.3.1 for more details).

4 Background information on the tree property

In Section 4.1 we briefly review the existing results on the tree property. In Section 4.2 we discuss "branch lemmas" which mention properties which are useful when showing that certain forcings do not add branches to trees. The typical use of these lemmas is illustrated in Section 4.3 with a brief review of the basic construction (due to Mitchell) which ensures the tree property at $\aleph_2$.

4.1 The tree property at successor cardinals

The first construction which showed that it is consistent to have the tree property at a successor cardinal is due to Mitchell [55].[13] Starting with regular cardinals $\omega \leq \kappa < \lambda$, Mitchell found a forcing notion which does the following: with GCH, it collapses cardinals in the interval (κ^+, λ) (and no other cardinals), and whenever λ is Mahlo, then the weak tree property holds at $2^\kappa = \lambda = \kappa^{++}$, and whenever λ is weakly compact, then the tree property holds at $2^\kappa = \lambda = \kappa^{++}$. It is important for further development to notice that κ itself is regular, and so Mitchell's construction achieves the tree property at a double successor of a regular cardinal – thus leaving aside successors of singulars, and double successor of singulars. The large cardinal assumptions are optimal in the sense that if the tree property holds at some κ, then κ is weakly compact in L, and if the weak tree property holds at κ, then κ is Mahlo in L. Mitchell [55] gives an argument that the weak tree property can be forced at two successive cardinals, such as $\aleph_2$ and $\aleph_3$, starting with just two Mahlo cardinals. He left it open whether it is consistent to have the tree property at two successive cardinals.

Abraham [1] solved the question by finding a forcing notion which ensures the tree property at $\aleph_2$ and $\aleph_3$, with $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$. Abraham started with a supercompact cardinal and a weakly compact cardinal above it. While the assumption might seem too strong at the first glance,[14] the paper gives an argument (due to Magidor) that two weakly compact cardinals certainly do not suffice since having the tree property at successive cardinals implies the existence of $0^\sharp$. This lower bound was later improved to the level of Woodin cardinals (see [21] for more details).

Another development was the result of Cummings and Foreman [10] who generalised Abraham's construction and obtained a model where the tree property holds at every $\aleph_n$ for $1 < n < \omega$, and $2^{\aleph_m} = \aleph_{m+2}$ for $0 \leq m < \omega$ (and GCH elsewhere). They left open whether one can extend the interval of cardinals with the tree property further, in particular to include $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.

We will leave $\aleph_{\omega+1}$ aside for a moment and focus on $\aleph_{\omega+2}$. Since $\aleph_\omega$ is strong limit in the model in [10] and GCH holds at $\aleph_\omega$, the tree property necessarily fails at $\aleph_{\omega+2}$. In the second part of [10] (attributed to Foreman), they give an argument which shows how to get the tree property at κ^{++} for a strong limit singular κ with countable cofinality (starting with a supercompact κ and a weakly compact above it). They also claim that their construction generalises to collapse κ to $\aleph_\omega$, and ensure the tree property at $\aleph_{\omega+2}$. However, they provided no argument, and in hindsight it does not seem that an easy

[13] The modern presentation of Mitchell's forcing is due to Abraham [1], and it is the one we use in this thesis (see Section 3.4.1 for more details about Mitchell forcing).

[14] It is used only once to lift an embedding using a master condition argument.

modification of their argument for κ^{++} with the tree property generalises to $\aleph_{\omega+2}$: The problem is that Prikry forcing with collapses prevents the use of the type of forcing they used in [10].[15] Today, there are four different arguments available for the tree property at $\aleph_{\omega+2}$ (to our knowledge): the first one is the construction of Friedman and Halilović [22], followed by Gitik's construction in [30], the construction in [15] due to Cummings and others, and our present construction (see Theorem 2.4). The construction in [22] is completely different from the argument of Cummings and Foreman in [10]: first, it uses just a weakly compact strong cardinal, and second it uses Sacks iteration at κ of length λ, followed by Prikry forcing with collapses, to achieve the desired goal.[16] The constructions in [15] and our construction in Theorem 2.4 are similar, but differ in important aspects. They both use Mitchell forcing followed by Prikry forcing with collapses. However Theorem 2.4 uses only a strong cardinal of a suitable degree, while [15] uses a supercompact cardinal.[17] Furthermore, the construction in Theorem 2.4 achieves any desired finite gap at $\aleph_\omega$. Regarding the Gitik's construction, it proceeds from a sequence of short extenders and it is optimal with respect to the large cardinal assumptions (however, it is not known how to generalise it to achieve a larger gap than 2 at $\aleph_\omega$).

Remark 4.1. The optimal lage cardinal assumption for the tree property at $\aleph_{\omega+2}$ is close to a weakly compact strong cardinal.[18] We will not give too many details, but let us say that it is not so important that we start with a weakly compact strong embedding – a tall embedding $j : V \to M$ which sends κ above a weakly compact cardinal λ in M would also suffice – the issue is whether we need to assume the existence of one big extender, or a sequence of short extenders would suffice. As it turns out, the optimal large cardinal strength is indeed formulated with a sequence of short extenders as we mentioned above (see [30]).

Let us return to the case of the tree property at $\aleph_{\omega+1}$. Let us first note that in all the models we discussed, with the tree property at $\aleph_{\omega+2}$, the tree property at $\aleph_{\omega+1}$ fails. However, by itself the tree property at $\aleph_{\omega+1}$ is achievable as shown by Magidor and Shelah in [54] (the key ingredient of the construction is a theorem in ZFC, proved in [54], which says that if λ is a singular limit of strongly compact cardinals, then the tree property holds at λ^+). However, the methods in [54] force SCH at $\aleph_\omega$, so the tree property at $\aleph_{\omega+2}$ fails in the model in [54].

A lot of recent research has been focused on combining the above-mentioned results and obtaining the tree property at all regular cardinals in the interval $[\aleph_2, \aleph_{\omega+2}]$. There has been an important progress, but the main question is still unanswered (see for instance [56, 66, 75, 68, 76, 67] for more details). With the natural goal being to force the tree property at all regular cardinals, it is also important to consider singular cardinals with uncountable cofinality; there has been some important progress here as well (see Sinapova

[15] See Footnote 2 for more details.

[16] The use of Sacks forcing enforces a direct method of proof: there is no "product-style" analysis along the lines of (3.31). A common restriction related to an iteration with support κ applies: it is possible to achieve only gap 2 at $\aleph_\omega$, i.e. $2^{\aleph_\omega} = \aleph_{\omega+2}$. In retrospect, the use of Sacks forcing probably makes the argument more complicated than the methods for the Mitchell-like forcings (unless we want to achieve some sort definability result together with the tree property – in this setting an iteration is the primary option; see Section 8.4 for more details).

[17] The paper [68] by Sinapova and Unger contains an argument for the tree property at κ^{++} for a large strong limit κ of countable cofinality, with gap 3.

[18] Note that $\aleph_\omega$ violates SCH, so lower bounds described in Section 3.5 apply.

[65] for $\aleph_{\omega_1+1}$ and Golshani and Mohammadpour [37] for κ^{++}, κ singular with uncountable cofinality for more details).

Remark 4.2. The results reviewed so far work with $\aleph_\omega$ being strong limit. If we relax this requirement, then it is consistent that both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ have the tree property by a result of Fontanella and Friedman [19]. It is an intriguing question whether $\aleph_\omega$ can at all be strong limit with the tree property holding at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$, especially because the tree property can consistently hold at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ with $\aleph_{\omega^2}$ strong limit (see Sinapova and Unger [68]). It may very well be that $\aleph_\omega$ is a special case, whose properties are governed by theorems provable in ZFC (as is the bound on $2^{\aleph_\omega}$ identified by Shelah).

Stepping back to the weak tree property (or equivalently to the failure of the weak square principle), it turns out that killing all special Aronszajn trees is much easier than killing all Aronszajn trees. As we already said, Mitchell [55] gave a proof of the tree weak property holding, for instance, at $\aleph_2$ and $\aleph_3$. This construction was generalised by Unger [74] to all cardinals at the interval $[\aleph_2, \aleph_\omega)$, starting with infinitely many Mahlo cardinals.

Moving on to our thesis and research, it is important to state that all the results reviewed so far did not specifically control the continuum function, and therefore achieve the least possible gap at the relevant cardinal: if the tree property holds at κ^{++}, then $2^\kappa = \kappa^{++}$. It is therefore natural to ask whether one can control the continuum function on regular cardinals in the presence of the tree property as freely as in the case of the usual Easton theorem.

4.2 Trees and forcing

An essential step in showing that some forcing notions force the tree property is to argue that they do not add cofinal branches to certain trees. Fact 4.3 is due to Baumgartner (see [4]) and Fact 4.4 is due to Silver (see [1] for more details; a proof with $\lambda = \omega$ is in [49, Chapter VIII, §3]) .

Fact 4.3. *Let κ be a regular cardinal and assume that P is a κ-Knaster forcing notion. If T is a tree of height κ, then forcing with P does not add cofinal branches to T.*

Fact 4.4. *Let κ, λ be regular cardinals and $2^\kappa \geq \lambda$. Assume that P is a κ^+-closed forcing notion. If T is a λ-tree, then forcing with P does not add cofinal branches to T.*

These facts can be generalised as follows (see Unger [73] and [72]).

Fact 4.5. *Let κ be a regular cardinal and assume that P is a forcing notion such that square of P, $P \times P$, is κ-cc. If T is a tree of height κ, then forcing with P does not add cofinal branches to T.*

Fact 4.6. *Let $\kappa < \lambda$ be regular cardinals and $2^\kappa \geq \lambda$. Assume that P and Q are forcing notions such that P is κ^+-cc and Q is κ^+-closed. If T is a λ-tree in $V[P]$, then forcing with Q over $V[P]$ does not add cofinal branches to T.*

Moreover, the previous fact can be generalised as follows.

Fact 4.7. *Let $\kappa < \xi \leq \lambda$ be regular cardinals and $2^\kappa \geq \lambda$. Assume that P and Q are forcing notions such that P is ξ-cc and Q is ξ-closed. If T is a λ-tree in $V[P]$, then Q does not add cofinal branches to T.*

Note that the previous fact says something new only if ξ is inaccessible: if ξ is equal to μ^+ for some $\mu \geq \kappa$, we can apply Fact 4.6.

4.3 Mitchell's argument for the tree property

In this section we briefly review the original argument of Mitchell [55] in the presentation of Abraham [1] to illustrate the basic idea of obtaining the tree property at the double successor of a regular cardinal.

Assuming GCH, let κ be a regular cardinal and $\lambda > \kappa$ a weakly compact cardinal. We will argue that $\mathbb{M}(\kappa, \lambda)$ forces the tree property at $\lambda = \kappa^{++}$ and makes $2^\kappa = \lambda$.

Suppose for contradiction there is a name $\dot{T}$ and a condition p which forces that $\dot{T}$ is a λ-Aronszajn tree. Let G be an $\mathbb{M}(\kappa, \lambda)$-generic filter which contains p, and let $k : M \to N$ be a weakly compact embedding with critical point λ such that M contains all relevant parameters, in particular the name $\dot{T}$ (we can also assume that $\dot{T}$ is in N). Suppose $G * H$ is $k(\mathbb{M}(\kappa, \lambda))$-generic filter and let k^* be the lifting of k to

$$k^* : M[G] \to N[G][H].$$

The tree $T = (\dot{T})^G$ is in both $M[G]$ and $N[G]$, and T has a cofinal branch b in $N[G][H]$ since $k^*(T)$ (which is an element of $N[G][H]$) contains T as an initial segment, and has nodes of height λ. The argument is finished by arguing that b is already in $N[G]$, and hence in $M[G]$ – but this is a contradiction since we assumed that T is a λ-Aronszajn tree in $V[G]$, and so also in $M[G]$. The fact that b cannot be added by H uses the facts reviewed in Section 4.2 and a the product and quotient analysis of Mitchell forcing: the forcing adding H regularly embeds into a product of a κ^+-Knaster forcing and a κ^+-closed forcing over $N[G]$.

If λ is just a Mahlo cardinal, a similar argument is used that the weak tree property holds at λ in $V[\mathbb{M}(\kappa, \lambda)]$.

Let us mention that obtaining the tree property at successive cardinals is much more difficult than obtaining it on a single cardinal: the problem is that if we naively forced with $\mathbb{M}(\kappa, \lambda)$ and $\mathbb{M}(\kappa^+, \lambda^*)$, where $\lambda < \lambda^*$ are weakly compact, then the forcings "overlap" and do not ensure the tree property. This problem cannot be resolved by a clever trick as the consistency strength of two successive cardinals with the tree property is at least on the level of Woodin cardinals ([21]).

The overlapping occurs even if we wish to get the tree property at cardinals μ and μ^{++} for some regular μ. However, in this situation one can argue that $\mathbb{M}(\kappa, \lambda) \times \mathbb{M}(\lambda, \lambda^*)$ does ensure the tree property, where $\lambda < \lambda^*$ are weakly compact (λ will become μ and λ^* will become μ^{++} in the notion in the previous sentence); see Section 5 for more details.

If we focus on the weak tree property, the situation is different: obtaining two successive cardinals with the weak tree property can be done by an easy modification of Mitchell forcing starting with two Mahlo cardinals (see Mitchell [55] and Unger [74]).

Remark 4.8. In our results we generalise Mitchell's construction in several aspects (see Section 2 for a review of the results and the modifications of the original argument).

5 The tree property below $\aleph_\omega$

In this section we study the continuum function below $\aleph_\omega$ in connection with the tree property and the weak tree property at cardinals $\aleph_n$ for $1 < n < \omega$. The results are joint with Radek Honzik and were submitted as [42].

The structure of the section is as follows. First, in Theorem 5.5, we deal for simplicity with a single cardinal and show that the tree property at $\aleph_2$ is compatible with $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ (we use "gap three" for concreteness, there is nothing particular about it).[19] Theorem 5.5 is generalised in Theorem 5.7 where we show (starting with infinitely many weakly compact cardinals) that the tree property at every even cardinal larger than $\aleph_1$ below $\aleph_\omega$ is compatible with any continuum function which satisfies $2^{\aleph_{2n}} \geq \aleph_{2n+2}$, $n < \omega$. In Theorem 5.14, we formulate an analogous result for the weak tree property: starting with infinitely many Mahlo cardinals, we show that the weak tree property at every $\aleph_n$, $1 < n < \omega$, is compatible with any continuum function which satisfies $2^{\aleph_n} \geq \aleph_{n+2}$ for $n < \omega$. We focus on the case when $\aleph_\omega$ is a strong limit cardinal in the resulting model, but the method of the proof is not limited to that configuration.

Note that we use only modest large cardinal assumptions, i.e. weakly compact cardinals and Mahlo cardinals, and therefore we cannot get two successive cardinals with the tree property as this requires large cardinals on the level of one Woodin cardinal (see [21]), and all known arguments require supercompactness.

5.1 Large $2^{\aleph_0}$ and $2^{\aleph_1}$ with $\mathsf{TP}(\aleph_2)$

In this section we provide a proof of a special case of Theorem 5.7. It illustrates the main idea behind the construction with more clarity than the proof of Theorem 5.7, which needs to deal with infinitely many cardinals.

We assume that the reader is familiar with the usual argument which shows that $\mathbb{M}(\kappa, \lambda)$ forces the tree property at λ, whenever κ is regular, $\kappa < \lambda$ and λ is weakly compact. For the proof see a quick review in Section 4.3, or papers [55] or [1].

For concreteness of the construction in this section we will force "gap three" on $\aleph_0$ and $\aleph_1$, i.e. get $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ with the tree property at $\aleph_2$. Other values of the continuum functions are easily obtainable; see Theorem 5.7.

Let κ be a weakly compact cardinal. Denote

$$\mathbb{P} = \mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa^+) \times \mathrm{Add}(\aleph_1, \kappa^{++}). \tag{5.1}$$

Remark 5.1. Note that $\mathbb{M}(\aleph_0, \kappa)$ forces $2^{\aleph_0} = \aleph_2$, and therefore to increase the value of $2^{\aleph_1}$, we need to use some kind of product because the forcing $\mathrm{Add}(\aleph_1, 1)$ defined in $V[\mathbb{M}(\aleph_0, \kappa)]$ collapses $2^{\aleph_0}$ to $\aleph_1$ (by a density argument, every subset of ω occurs as a segment in a generic filter g for $\mathrm{Add}(\aleph_1, 1)$, and therefore g yields a surjection from $\aleph_1$ onto $2^{\aleph_0}$).

Lemma 5.2. *Assume* GCH*. In* $V[\mathbb{P}]$*,* $\kappa = \aleph_2$*,* $2^{\aleph_0} = \aleph_3$*,* $2^{\aleph_1} = \aleph_4$*.*

Proof. Let I denote the open interval of cardinals between $\aleph_1$ and κ. It suffices to show that in $V[\mathbb{P}]$ the cardinals in I are collapsed, and no other cardinals are collapsed. $\mathbb{M}(\aleph_0, \kappa)$

[19] This result for $2^{\aleph_0}$ already follows from the "indestructibility" results presented in [72].

collapses cardinals in I, but no other cardinals, see paragraph following Definition 3.46 of Mitchell forcing. The forcing $\mathrm{Add}(\aleph_0,\kappa^+)$ is ccc, and therefore no more cardinals are collapsed in $V[\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,\kappa^+)]$. By Lemma 3.51, the forcing $\mathrm{Add}(\aleph_1,\kappa^{++})$ is $\aleph_1$-distributive in $V[\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,\kappa^+)]$. Since the whole product $\mathrm{Add}(\aleph_0,\kappa^+)\times\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_1,\kappa^{++})$ is κ-Knaster, $\mathrm{Add}(\aleph_0,\kappa^+)\times\mathbb{M}(\aleph_0,\kappa)$ forces that $\mathrm{Add}(\aleph_1,\kappa^{++})$ is κ-cc (see Section 3.3.2). It follows that the forcing $\mathrm{Add}(\aleph_1,\kappa^{++})$ applied over $V[\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,\kappa^+)]$ preserves all cardinals not in I. This finishes the proof. □

Lemma 5.3. *Assume $\mathbb{P}$ forces that $\dot{T}$ is a κ-Aronszajn tree, where $\dot{T}$ is a nice name for a subset of κ.*[20] *Then there are $\kappa\subseteq A\subseteq\kappa^+$ and $\kappa\subseteq B\subseteq\kappa^{++}$ both of size κ such that $\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,A)\times\mathrm{Add}(\aleph_1,B)$ forces that $\dot{T}$ is a κ-Aronszajn tree.*

Proof. Since all forcings composing $\mathbb{P}$ are κ-Knaster, it follows that the product $\mathbb{P}$ is κ-cc (in fact κ-Knaster; see Section 3.3.2). It follows that $\dot{T}$ is of the form $\{\{\check{\alpha}\}\times A_\alpha \mid \alpha<\kappa\}$, where each A_α is an antichain of size less than κ. Let us write a condition in $\mathbb{P}$ as $\bar{p}=((p,q),p_0,p_1)$, where (p,q) is in $\mathbb{M}(\aleph_0,\kappa)$, p_0 is in $\mathrm{Add}(\aleph_0,\kappa^+)$ and p_1 is in $\mathrm{Add}(\aleph_1,\kappa^{++})$. The set A is defined by $\kappa\cup A'$ where A' contains all β such there is some $\alpha<\kappa$ and some $\bar{p}$ in A_α such that (n,β), for some $n<\omega$, is in the domain of the condition p_0 which is in $\bar{p}$. Similarly for B and p_1's. □

Corollary 5.4. *If $\mathbb{P}$ adds a κ-Aronszajn tree, then so does*

$$\mathbb{P}|\kappa =_{df} \mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_1,\kappa). \tag{5.2}$$

Proof. Any bijection between A and κ determines an isomorphism between $\mathrm{Add}(\aleph_0,A)$ and $\mathrm{Add}(\aleph_0,\kappa)$, and similarly for B. □

Theorem 5.5. (GCH) *Assume κ is weakly compact and $\mathbb{P}$ is as in (5.1). Then in $V[\mathbb{P}]$, $2^{\aleph_0}=\aleph_3$, $2^{\aleph_1}=\aleph_4$, and $\mathsf{TP}(\aleph_2)$.*

Proof. The fact that $2^{\aleph_0}=\aleph_3$ and $2^{\aleph_1}=\aleph_4$ is proved in Lemma 5.2. It remains to verify the tree property at $\aleph_2$. By Corollary 5.4, it suffices to show that $\mathbb{P}|\kappa$ cannot add a κ-Aronszajn tree. Suppose for contradiction there is a condition (r_1,r_2) in $\mathbb{P}|\kappa=\mathbb{M}(\aleph_0,\kappa)\times[\mathrm{Add}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_1,\kappa)]$ which forces there is a κ-Aronszajn tree $\dot{T}$.

Let $j:\mathscr{M}\to\mathscr{N}$ be a weakly compact embedding with critical point κ where $\mathscr{M}$ and $\mathscr{N}$ are transitive models of ZFC^- closed under $<\kappa$-sequences, and $\mathscr{M}$ contains all parameters required for the argument (in particular, the forcing $\mathbb{P}|\kappa$ and name $\dot{T}$).

Let $G*(H_1\times H_2)$ denote a generic filter over V for

$$\mathbb{M}(\aleph_0,\kappa)*[\mathrm{Add}(\aleph_0,j(\kappa)-\kappa)\times\dot{\mathbb{T}}_\kappa], \tag{5.3}$$

where the product $\mathrm{Add}(\aleph_0,j(\kappa)-\kappa)\times\mathbb{T}_\kappa$ projects to $j(\mathbb{M}(\aleph_0,\kappa))/G$. Denote by $G*H$ the $j(\mathbb{M}(\aleph_0,\kappa))$-generic filter obtained from $G*(H_1\times H_2)$, using the projection π_κ appearing in Fact 3.48(i); note that we have automatically $j''G\subseteq G*H$ because j is the identity on the conditions in $G\subseteq\mathbb{M}(\aleph_0,\kappa)$. Assume further that $r_1\in G$.

Now we can lift j in $V[G*(H_1\times H_2)]$ to

$$j:\mathscr{M}[G]\to\mathscr{N}[G*H]. \tag{5.4}$$

[20] We will identify κ-trees with subsets of κ (every κ-tree is isomorphic to (κ,R) for some some binary relation R).

Let $x^* \times y^*$, with $x^* = x_0 \times x_1$ and $y^* = y_0 \times y_1$, be $V[G * (H_1 \times H_2)]$-generic for $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa))$, with $x_0 \times y_0$ being $\mathrm{Add}(\aleph_1, \kappa) \times \mathrm{Add}(\aleph_0, \kappa)$-generic over $V[G * (H_1 \times H_2)]$ so that

$$j\text{"}(x_0 \times y_0) \subseteq x^* \times y^*. \tag{5.5}$$

The inclusion (5.5) is possible because j is the identity on the conditions in $x_0 \times y_0$. Assume further that $r_2 \in x_0 \times y_0$.

Remark 5.6. It is worth noting that $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa))$ lives in $V[G]$ (actually already in V), so $x^* \times y^* \times H_1 \times H_2$ is a generic filter over $V[G]$ for the product forcing $\mathrm{Add}(\aleph_1, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa)) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times \mathbb{T}_\kappa$, and therefore x^*, y^*, H_1, and H_2 are mutually generic over $V[G]$.

Now we can lift j in $V[G * (H_1 \times H_2)][x^* \times y^*]$ to

$$j : \mathscr{M}[G][x_0 \times y_0] \to \mathscr{N}[G][H][x^* \times y^*]. \tag{5.6}$$

Recall that we have put the name $\dot{T}$ into $\mathscr{M}$; we can assume that $\dot{T}$ is a nice name for a subset of κ, and is therefore present also in $\mathscr{N}$. Since (r_1, r_2) is in $G * (x_0 \times y_0)$, $T = \dot{T}^{G*(x_0 \times y_0)}$ is a κ-Aronszajn tree in $\mathscr{M}[G][x_0 \times y_0]$, and also in $\mathscr{N}[G][x_0 \times y_0]$. As $j(T)$ is a $j(\kappa)$-tree, it has nodes of height κ. Since $T = j(T) \restriction \kappa$, the last sentence implies that T has a cofinal branch in $\mathscr{N}[G][H][x^* \times y^*]$.

By Remark 5.6, the relevant filters are mutually generic over $V[G]$, and hence also over $\mathscr{N}[G]$, and therefore we can write

$$\mathscr{N}[G][H][x^* \times y^*] \subseteq \mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2]. \tag{5.7}$$

We finish the proof by showing that the generic filter $x_1 \times y_1 \times H_1 \times H_2$ cannot add a cofinal branch to T, and therefore any such branch existing in the models in (5.7) must already exist in $\mathscr{N}[G][x_0 \times y_0]$, which contradicts our initial assumption that T is a κ-Aronszajn tree in $\mathscr{N}[G][x_0 \times y_0]$.

Let P_1 denote the forcing $\mathrm{Add}(\aleph_1, j(\kappa) - \kappa) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa) \times \mathrm{Add}(\aleph_0, j(\kappa) - \kappa)$ which adds the generic filter $x_1 \times y_1 \times H_1$. As the square of P_1 is isomorphic to P_1, it suffices to show by Fact 4.5 that $\mathbb{P}|\kappa$ forces P_1 to be κ-cc. This follows from the fact that both $\mathbb{P}|\kappa$ and P_1 are κ-Knaster, and therefore $\mathbb{P}|\kappa \times P_1$ is κ-cc (in fact κ-Knaster), and so $\mathbb{P}|\kappa$ forces that P_1 is κ-cc (see Section 3.3.2). Hence there are no new cofinal branches in T in

$$\mathscr{N}[G][y_0][x_0][x_1][y_1][H_1]. \tag{5.8}$$

Now we show that H_2 cannot add a cofinal branch either, which will finish the proof. The term forcing $\mathbb{T}_\kappa$ is $\aleph_1$-closed in $\mathscr{N}[G]$, and by Lemma 3.51 (with P being trivial), the forcing $\mathrm{Add}(\aleph_1, j(\kappa))$ (which adds $x_0 \times x_1$) is $\aleph_1$-distributive in $\mathscr{N}[G]$, and therefore does not add new countable sequences; this implies that that T_κ is still $\aleph_1$-closed in $\mathscr{N}[G][x_0][x_1]$. We can therefore apply Fact 4.6 over the the model $\mathscr{N}[G][x_0][x_1]$ with $P = \mathrm{Add}(\aleph_0, j(\kappa))$ (note that P is isomorphic to the $\aleph_0$-Cohen forcing which adds $y_0 \times y_1 \times H_1$) and $Q = \mathbb{T}_\kappa$. Thus there are no new cofinal branches in T in the model

$$\mathscr{N}[G][x_0][x_1][y_0][y_1][H_1][H_2] = \mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2]. \tag{5.9}$$

This finishes the proof. □

5.2 Main theorems

In this section, we prove a more general version of Theorem 5.5, both for the tree property (Theorem 5.7), and the weak tree property (Theorem 5.14).

5.2.1 The tree property

Let $\kappa_1 < \kappa_2 < \cdots$ be an ω-sequence of weakly compact cardinals with limit λ. Let κ_0 denote $\aleph_0$. In Theorem 5.7, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the tree property at all even aleph's.

Let A denote the set $\{\kappa_i \,|\, i < \omega\} \cup \{\kappa_i^+ \,|\, i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+1}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)); see also Definition 3.56.

Theorem 5.7. *Assume* GCH *and let $\langle \kappa_i \,|\, i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is an $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_{2n}$, and $\kappa_n^+ = \aleph_{2n+1}$.
(ii) For all $0 < n < \omega$, the tree property holds at $\aleph_{2n}$.
(iii) The continuum function on $A = \{\aleph_n \,|\, n < \omega\}$ is controlled by f.

Proof. Let $\mathbb{P}$ be a reverse Easton iteration of Cohen forcings $\mathrm{Add}(\alpha, 1)$ for every inaccessible $\alpha < \lambda$. We will see that $\mathbb{P}$ ensures that the weak compactness of the κ_i's is preserved at a certain stage of the argument (see the paragraph after (5.21)).

Let $\dot{\mathbb{M}}(\kappa_n, \kappa_{n+1})$ denote a $\mathbb{P}$-name for Mitchell forcing which makes $2^{\kappa_n} = \kappa_{n+1}$ and forces the tree property at κ_{n+1}. Let $\dot{\mathbb{Q}}$ be a name for the full-support product of Mitchell forcings in $V[\mathbb{P}]$:

$$\dot{\mathbb{Q}} \text{ is a name for } \prod_{n<\omega} \dot{\mathbb{M}}(\kappa_n, \kappa_{n+1}). \tag{5.10}$$

Finally, let $\dot{\mathbb{R}}$ be a $\mathbb{P}$-name for the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to the cardinals in A):

$$\dot{\mathbb{R}} \text{ is a name for } \prod_{n<\omega} (\dot{\mathrm{Add}}(\kappa_n, f(\kappa_n)) \times \dot{\mathrm{Add}}(\kappa_n^+, f(\kappa_n^+)). \tag{5.11}$$

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{P} * (\dot{\mathbb{Q}} \times \dot{\mathbb{R}}). \tag{5.12}$$

We leave it as an exercise for the reader to verify that $\mathbb{S}$ preserves all cardinals in A and forces the prescribed continuum function (it is a routine generalisation of Lemma 5.2

using the product analysis of $\mathbb{M}(\kappa, \lambda)$ in Section 3.4.1 and Lemma 3.51). We will check that the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$.

Let us work in $V[\mathbb{P}]$ for simplicity of notation (so that we can remove all "dots" from the forcing notions).

Let us write $\mathbb{R}^0(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$, $\mathbb{R}^1(n) = \mathrm{Add}(\kappa_n^+, f(\kappa_n^+))$, and $\mathbb{R}(n) = \mathbb{R}^0(n) \times \mathbb{R}^1(n)$. Thus $\mathbb{R} = \prod_{n<\omega} \mathbb{R}(n)$.

Let us denote for $0 < n < \omega$:

$$\mathbb{S}(n) = \mathbb{R}^0(n+1) \times \prod_{m \le n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m), \tag{5.13}$$

and

$$\mathbb{S}(n)_{\mathrm{tail}} = \mathbb{R}^1(n+1) \times \prod_{m > n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m > n+1} \mathbb{R}(m), \tag{5.14}$$

so that $\mathbb{Q} \times \mathbb{R} = \mathbb{S}(n) \times \mathbb{S}(n)_{\mathrm{tail}}$.

Suppose for contradiction that $\mathbb{S}$ adds a κ_{n+1}-Aronszajn tree T (for simplicity, we assume that the weakest condition forces the existence of such a tree; otherwise we would work below a condition which forces it). Then T is added by

$$\mathbb{P} * \dot{\mathbb{S}}(n) \tag{5.15}$$

because $\mathbb{S}(n)_{\mathrm{tail}}$ is κ_{n+1}^+-closed in $V[\mathbb{P}]$, and using Lemma 3.51, viewing $\mathbb{S}(n)$ as a product of a κ_{n+1}^+-cc forcing and $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2})$, it follows that $\mathbb{S}(n)_{\mathrm{tail}}$ is still κ_{n+1}^+-distributive in $V[\mathbb{P}]$ over $\mathbb{S}(n)$, and hence does not add any κ_{n+1}-trees.

The forcing $\mathbb{S}(n)$ is κ_{n+2}-Knaster in $V[\mathbb{P}]$ (because all the forcings making up the product are κ_{n+2}-Knaster, and this property is preserved by products), and therefore T has in $V[\mathbb{P}]$ a $\mathbb{S}(n)$-name $\dot{T}$ which can be taken to be a $<\kappa_{n+2}$-sequence of elements in $V[\mathbb{P}]$ (without loss of generality, a nice name for a subset of κ_{n+1}). This name – having size less than κ_{n+2} in $V[\mathbb{P}]$ – is already present in $\mathbb{P}(< \kappa_{n+2})$ (the iteration $\mathbb{P}$ below κ_{n+2}) because $\mathbb{P}(<\kappa_{n+2})$ forces its tail in $\mathbb{P}$ to be κ_{n+2}-closed. It follows that

$$\mathbb{P}(<\kappa_{n+2}) * \dot{\mathbb{S}}(n) \tag{5.16}$$

already adds T (note that $\dot{\mathbb{S}}(n)$ can be taken to be a $\mathbb{P}(<\kappa_{n+2})$-name as all the conditions in this forcing have size less than κ_{n+2}, so the expresion (5.16) is meaningful).

Let us define in $V[\mathbb{P}(<\kappa_{n+2})]$:

$$\mathbb{S}(n)^- = \prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m). \tag{5.17}$$

Thus we can write the forcing in (5.16) as

$$\mathbb{P}(<\kappa_{n+2}) * (\dot{\mathbb{M}}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1) \times \dot{\mathbb{S}}(n)^-). \tag{5.18}$$

This forcing is equivalent to

$$\mathbb{P}(<\kappa_{n+2}) * (\dot{\mathbb{M}}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1)) * \dot{\mathbb{S}}(n)^-. \tag{5.19}$$

because $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1)$ does not change $H(\kappa_{n+1})$ where the conditions in the rest of the forcing live.

We claim that T is in fact added by

$$\mathbb{P}(<\kappa_{n+2}) * \dot{\mathrm{Add}}(\kappa_{n+1}, 1) * \dot{\mathbb{S}}(n)^-. \tag{5.20}$$

This is true because T has a name in the forcing

$$\mathbb{P}(<\kappa_{n+2}) * (\dot{\mathrm{Add}}(\kappa_{n+1}, \kappa_{n+2}) \times \dot{\mathbb{R}}^0(n+1)) * \dot{\mathbb{S}}(n)^- \tag{5.21}$$

of size at most κ_{n+1} (using the product analysis of $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2})$ which we discussed in Section 3.4.1) and therefore a name in the forcing (5.20).

$\mathbb{P}(<\kappa_{n+2}) * \dot{\mathrm{Add}}(\kappa_{n+1}, 1)$ preserves the weak compactness of κ_{n+1} (since in $\mathbb{P}(<\kappa_{n+2})$ we prepared by Cohen forcing at inaccessibles below κ_{n+1}; this may be shown for instance by lifting a weakly compact embedding, see [9] for more details), so it remains to show that $\mathbb{S}(n)^-$ forces the tree property at κ_{n+1} for a weakly compact κ_{n+1}.

Work in $V[\mathbb{P}(<\kappa_{n+2})]$ and let us write $\mathbb{S}(n)^-$ as:

$$\mathbb{S}(n)^- = \mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{S}_1 \times \mathbb{S}_2 \times \mathbb{S}_3, \tag{5.22}$$

where

- $\mathbb{S}_1 = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1})$,
- $\mathbb{S}_2 = \mathbb{R}^0(n) \times \prod_{m<n} \mathbb{R}(m)$, and
- $\mathbb{S}_3 = \mathbb{R}^1(n)$.

These forcings have the following basic properties which are relevant for the proof:

(a) $\mathbb{M}(\kappa_n, \kappa_{n+1})$ is κ_{n+1}-Knaster, and there is a projection onto it from the product forcing $\mathrm{Add}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}^n$, where $\mathbb{T}^n$ is a κ_n^+-closed term forcing.
(b) $\mathbb{S}_1$ is κ_n-Knaster, and bounded in $H(\kappa_{n+1})$.
(c) $\mathbb{S}_2$ is κ_n^+-Knaster.
(d) $\mathbb{S}_3$ is κ_n^+-closed.

Denote $\kappa_{n+1} = \kappa$. Exactly as in the proof of Theorem 5.5, using the fact that the whole product $\mathbb{S}(n)^-$ is κ-cc (using the productivity of the Knaster property), if $\mathbb{S}(n)^-$ adds a κ-Aronszajn tree, then so does the forcing

$$\mathbb{M}(\kappa_n, \kappa) \times \mathbb{S}_1 \times \mathbb{S}_2|\kappa \times \mathbb{S}_3|\kappa, \tag{5.23}$$

where $\mathbb{S}_2|\kappa$ and $\mathbb{S}_3|\kappa$ denote the restrictions of all of the Cohen products in $\mathbb{S}_2$ and $\mathbb{S}_3$ to length κ. In more detail, for $\mathbb{R}^i(m)$, $i < 2, m \le n$, let us write $\mathbb{R}^0(m)|\kappa = \mathrm{Add}(\kappa_m, \kappa)$ and $\mathbb{R}^1(m)|\kappa = \mathrm{Add}(\kappa_m^+, \kappa)$, and $\mathbb{R}(m)|\kappa = \mathbb{R}^0(m)|\kappa \times \mathbb{R}^1(m)|\kappa$. Then $\mathbb{S}_2|\kappa$ denotes the forcing $\mathbb{R}^0(n)|\kappa \times \prod_{m<n} \mathbb{R}(m)|\kappa$, and $\mathbb{S}_3|\kappa$ denotes $\mathbb{R}^1(n)|\kappa$. The fact that already the forcing in (5.23) adds the tree follows exactly as in Lemma 5.3 and Corollary 5.4 with appropriate reformulations.

Recall that we work in $V[\mathbb{P}(<\kappa_{n+2})]$ where by our assumption we have a name $\dot{T}$ for a κ-Aronszajn tree in the forcing (5.23). Let

$$j : \mathscr{M} \to \mathscr{N} \tag{5.24}$$

be a weakly compact embedding with critical point κ, where $\mathscr{M}$ contains all the relevant parameters (in particular, the forcing from (5.23) and the name $\dot{T}$ (which we view as a nice name for a subset of κ – and is therefore also in $\mathscr{N}$ as it is a subset of $H(\kappa)$)). Note that $\mathscr{M}$ and $\mathscr{N}$ are closed under sequences of length $<\kappa$ in $V[\mathbb{P}(<\kappa_{n+2})]$, so in particular the forcings $j(\mathbb{S}_2|\kappa)$ and $j(\mathbb{S}_3|\kappa)$ mean in $\mathscr{N}$ and in $V[\mathbb{P}(<\kappa_{n+2})]$ the same thing as the conditions in these forcings are sequences of length $<\kappa$ (for the same reason, $\mathbb{S}_2|\kappa$ and $\mathbb{S}_3|\kappa$ denote the same forcing in $\mathscr{M}$, $\mathscr{N}$ and $V[\mathbb{P}(<\kappa_{n+2})]$).

Pursuing the analogy with Theorem 5.5, and the notation in that proof, consider the model $\mathscr{N}[G][x_0][y_0][x_1][H_2][H_1][y_1]$, where in our case we have:

(a) $G = G_0 \times G_1$ is $\mathbb{M}(\kappa_n, \kappa) \times \mathbb{S}_1$-generic.
(b) x_0 is $\mathbb{S}_3|\kappa$-generic.
(c) x_1 is such that $x_0 \times x_1$ is $j(\mathbb{S}_3|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{S}}_3$: $j(\mathbb{S}_3|\kappa) = \mathbb{S}_3|\kappa \times \hat{\mathbb{S}}_3$.
(d) y_0 is $\mathbb{S}_2|\kappa$-generic.
(e) y_1 is such that $y_0 \times y_1$ is $j(\mathbb{S}_2|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{S}}_2$: $j(\mathbb{S}_2|\kappa) = \mathbb{S}_2|\kappa \times \hat{\mathbb{S}}_2$.
(f) H_1 is $\mathrm{Add}(\kappa_n, j(\kappa) - \kappa)$-generic.
(g) H_2 is $\mathbb{T}^n_\kappa$-generic, where $\mathbb{T}^n_\kappa$ is the term forcing which is κ_n^+-closed in $\mathscr{N}[G_0]$.

Recall that we assume for simplicity that the weakest condition forces that $\dot{T}$ is a κ-Aronszajn tree; otherwise we would need to choose $G_0 \times G_1 \times x_0 \times y_0$ below a condition wich forces it.

Remark 5.8. Note that the product $\mathbb{S}_1 \times j(\mathbb{S}_3|\kappa) \times j(\mathbb{S}_2|\kappa)$ lives in $V[\mathbb{P}(<\kappa_{n+2})[G_0]$ (actually already in $V[\mathbb{P}(<\kappa_{n+2})]$), so that $G_1 \times x_0 \times x_1 \times y_0 \times y_2 \times H_1 \times H_2$ is a generic filter for a product forcing over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$, and therefore all these generic filters are mutually generic over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$.

Let us write

$$V^* = V[\mathbb{P}(<\kappa_{n+2})][G][x_0 \times x_1 \times y_0 \times y_1 \times H_1 \times H_2]. \tag{5.25}$$

The conditions (a)–(g) above guarantee we can lift j in V^*:

$$j : \mathscr{M}[G][x_0][y_0] \to \mathscr{N}[G][x_0][y_0][x_1][y_1][H], \tag{5.26}$$

where H is generic for the quotient $j(\mathbb{M}(\kappa_n, \kappa))/G_0$ over $V[\mathbb{P}(<\kappa_{n+2})][G_0]$ and $j\text{"}G_0 \subseteq G_0 * H$ (this is ensured exactly as in the proof of Theorem 5.5).

Since $\dot{T}$ is also present in $\mathscr{N}$, the κ-Aronszajn tree $T = \dot{T}^{G \times x_0 \times y_0}$ is present in $\mathscr{N}[G][x_0][y_0]$; since $j(T)$ restricted to κ is T, the embedding (5.26) ensures that $j(T)$ has a node of height κ, and therefore T has a cofinal branch, in $\mathscr{N}[G][x_0][y_0][x_1][y_1][H]$. We will argue this is not possible as the larger model $\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1][H_2]$ cannot obtain a new cofinal branch over $\mathscr{N}[G][x_0][y_0]$.

First note that $P_1 = \hat{\mathbb{S}}_3 \times \hat{\mathbb{S}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa) - \kappa)$ (which adds the generic $x_1 \times y_1 \times H_1$) is isomorphic to its square. By Fact 4.5 it therefore suffices to show that $P_0 = \mathbb{M}(\kappa_n, \kappa) \times \mathbb{S}_1 \times \mathbb{S}_3|\kappa \times \mathbb{S}_2|\kappa$ (which adds the generic $G \times x_0 \times y_1$) forces that P_1 is κ-cc to conclude that there are no new cofinal branches in T in

$$\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1].$$

This follows by the productivity of Knaster forcings as both P_0 and P_1 are κ-Knaster.

It remains to show that H_2 cannot add a cofinal branch to T either. Denote $P = \mathbb{S}_1 \times \mathbb{S}_2 \times \hat{\mathbb{S}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa) - \kappa)$.

Claim 5.9. *The following hold.*

(i) $\mathbb{T}^n_\kappa$ *is* κ_n^+*-closed in* $\mathscr{N}[G_0][x_0][x_1]$.
(ii) P *is* κ_n^+*-cc in* $\mathscr{N}[G_0][x_0][x_1]$.

Proof. (i). $\mathbb{T}^n_\kappa$ is κ_n^+-closed in $\mathscr{N}[G_0]$, and by Lemma 3.51 (with P being trivial), $\mathscr{N}[G_0]$ and $\mathscr{N}[G_0][x_0][x_1]$ have the same $<\kappa_n^+$-sequences of ordinals. Now the claim follows.

(ii). We will show that P is forced to be κ_n^+-cc by

$$\mathrm{Add}(\kappa_n, \kappa) \times \mathbb{T}^n \times \mathbb{S}_3|\kappa \times \hat{\mathbb{S}}_3. \tag{5.27}$$

This suffices as there is a projection from the forcing (5.27) to the forcing $\mathbb{M}(\kappa_n, \kappa) \times \mathbb{S}_3|\kappa \times \hat{\mathbb{S}}_3$ (which adds $G_0 \times x_0 \times x_1$).

We use Easton's lemma 3.32: $P_2 = \mathbb{T}^n \times \mathbb{S}_3|\kappa \times \hat{\mathbb{S}}_3$ is κ_n^+-closed and $\mathrm{Add}(\kappa_n, \kappa) \times P$ is κ_n^+-cc, and therefore P_2 forces $\mathrm{Add}(\kappa_n, \kappa) \times P$ to be κ_n^+-cc, and so $P_2 \times \mathrm{Add}(\kappa_n, \kappa)$ forces P to be κ_n^+-cc. In some detail, P_2 forces that $\mathrm{Add}(\kappa_n, \kappa) \times P$ is κ_n^+-cc if and only if P_2 forces that $\mathrm{Add}(\kappa_n, \kappa)$ forces that P is κ_n^+-cc, which is equivalent to $P_2 \times \mathrm{Add}(\kappa_n, \kappa)$ forcing that P is κ_n^+-cc. □

Recall that our tree T is in $\mathscr{N}[G][x_0][y_0]$. Denote

$$P = \mathbb{S}_1 \times \mathbb{S}_2 \times \hat{\mathbb{S}}_2 \times \mathrm{Add}(\kappa_n, j(\kappa) - \kappa) \text{ and } Q = \mathbb{T}^n_\kappa. \tag{5.28}$$

Consider now the P-generic filter $G_1 \times y_0 \times y_1 \times H_1$ over the model $\mathscr{N}[G_0][x_0][x_1]$. It gives rise to the model $\mathscr{N}[G_0][x_0][x_1][G_1][y_0][y_1][H_1]$ which is equal to $\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1]$, and extends $\mathscr{N}[G][x_0][y_0]$, and therefore contains the tree T. Now let us apply Fact 4.6 over the model $\mathscr{N}[G_0][x_0][x_1]$ with P and Q fixed in (5.28). By Claim 5.9, P is κ_n^+-cc and Q is κ_n^+-closed in $\mathscr{N}[G_0][x_0][x_1]$. It follows that that H_2 (the generic filter for Q) does not add new cofinal branches to T over

$$\mathscr{N}[G][x_0][y_0][x_1][y_1][H_1]. \tag{5.29}$$

This finishes the proof. □

A more succinct formulation of Theorem 5.7 is as follows:

Corollary 5.10. (GCH) *Assume there are infinitely many weakly compact cardinals. Let* f *be a function from* ω *to* ω *which satisfies*

(i) For all $m, n < \omega$, $m < n \to f(m) \leq f(n)$.
(ii) $f(2n) \geq 2n + 2$ *for all* $n < \omega$.

Then there is a model where the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$, *and the continuum function below* $\aleph_\omega$ *obeys* f*: i.e.* $2^{\aleph_n} = \aleph_{f(n)}$ *for all* $n < \omega$.

Remark 5.11. Let us remark that the technique in the proof of Theorem 5.7 is not limited to having $\aleph_\omega$ strong limit: the values of $2^{\aleph_n}$ for $n < \omega$ can be bigger than $\aleph_\omega$ in the resulting model (subject to the usual restrictions on the continuum function). This follows from the fact that the argument for the tree property at $\kappa_{n+1} = \kappa$ reduces to the forcing in (5.23) which uses only a portion of $\mathbb{R}$ of size κ; thus $\mathbb{R}$ can be chosen to force $2^{\aleph_n}$ arbitrarily high without a material change in the argument.

5.2.2 The weak tree property

For the sake of completeness, we also address the question of the weak tree property and the continuum function below $\aleph_\omega$.

Let $\kappa_2 < \kappa_3 < \cdots$ be an ω-sequence of Mahlo cardinals with limit λ. Let κ_0 denote $\aleph_0$, and κ_1 denote $\aleph_1$. In Theorem 5.14, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the weak tree property at all $\aleph_n$, $n \geq 2$.

Let A denote the set $\{\kappa_i \,|\, i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+2}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)).

The following natural modification of Mitchell forcing first appeared in [74].

Definition 5.12. Let $0 \leq n < \omega$ be given. We define $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ as the collection of pairs (p, q) which satisfy the same conditions as in $\mathbb{M}(\kappa_n, \kappa_{n+2})$ with the difference that instead of $\mathrm{Add}(\kappa_n^+, 1)$ for collapsing, we use $\mathrm{Add}(\kappa_{n+1}, 1)$, and the size of the domain of q is now $<\kappa_{n+1}$.

In particular, $\mathbb{M}(\kappa_n, \kappa_{n+2})$ is equal to $\mathbb{M}(\kappa_n, \kappa_n^+, \kappa_{n+2})$. By an analysis similar to Abraham [1] (for more details se Section 3.4.1), one can show that $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ is a projection of the product

$$\mathrm{Add}(\kappa_n, \kappa_{n+2}) \times \mathbb{T}^n, \tag{5.30}$$

where $\mathbb{T}^n$ is a term forcing which is κ_{n+1}-closed and κ_{n+2}-cc (see [74], Lemma 4.7; see also [44], Theorem 16.30; it is easy to see that the forcing is actually κ_{n+2}-Knaster by the same argument).

We get the following analogue of Lemma 3.51.

Lemma 5.13. *Assume $\aleph_0 \leq \kappa < \mu < \lambda$ are regular cardinals, $\nu^{<\kappa} < \mu$ for all $\nu < \mu$, and λ is inaccessible. Assume P is μ-cc and Q is μ-closed. Then $P \times \mathbb{M}(\kappa, \mu, \lambda)$ forces that Q is μ-distributive.*

Proof. The proof is the same as in Lemma 3.51 with the modification that $P \times \mathrm{Add}(\kappa, \lambda)$ is now μ-cc (since by our cardinal-arithmetic assumption, $\mathrm{Add}(\kappa, \lambda)$ is μ-Knaster by the usual Δ-system argument) and the term part of Mitchell forcing is μ-closed. □

Note that we use Lemma 5.13 with GCH in the present context, so the cardinal-arithmetic assumptions are automatically satisfied.

The following theorem is a generalisation of Theorem 4.11 in [74].

Theorem 5.14. *Assume GCH and let $\langle \kappa_i \,|\, i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is an $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_n$,

(ii) The continuum function on $A = \{\aleph_n \,|\, n < \omega\}$ is controlled by f.
(iii) The weak tree property holds at every $\aleph_n$, $2 \leq n < \omega$.

Proof. Set $\mathbb{Q}$ to be the full support product

$$\mathbb{Q} = \prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2}). \tag{5.31}$$

Let $\mathbb{R}$ be the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to cardinals in A):

$$\mathbb{R} = \prod_{n<\omega} \mathrm{Add}(\kappa_n, f(\kappa_n)). \tag{5.32}$$

For simplicity of notation, let us write $\mathbb{R}(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$.

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{Q} \times \mathbb{R}. \tag{5.33}$$

Again, we leave it as an exercise for the reader to verify that the cardinals in A are preserved, $\kappa_n = \aleph_n$, and the continuum function below $\aleph_\omega$ is controlled by f. The proof is basically the same as in [74] using the usual Easton-style analysis, the product analysis of the forcing $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ in (5.30) and Lemma 5.13.

Let $n < \omega$ be fixed. We show that there are no special κ_{n+2}-Aronszajn trees in $V[\mathbb{S}]$.

Let us denote:

$$\mathbb{S}(n) = \prod_{m\leq n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m\leq n+1} \mathbb{R}(m), \tag{5.34}$$

and

$$\mathbb{S}(n)_{\mathrm{tail}} = \prod_{m>n+2} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m>n+2} \mathbb{R}(m), \tag{5.35}$$

so that

$$\mathbb{S} = \mathbb{S}(n) \times \mathbb{R}(n+2) \times \mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{S}(n)_{\mathrm{tail}}. \tag{5.36}$$

Suppose for contradiction $\mathbb{S}$ adds a special κ_{n+2}-Aronszajn tree (we assume for simplicity that the weakest condition forces it; otherwise we would work below an appropriate condition). Then also the forcing

$$\mathbb{S}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \times \mathbb{T}^{n+1} \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \times \mathbb{T}^{n+2} \times \mathbb{S}(n)_{\mathrm{tail}} \tag{5.37}$$

adds a special κ_{n+2}-Aronszajn tree because it projects onto $\mathbb{S}$. Denote the tree T.

Then T is added by

$$\mathbb{S}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \times \mathbb{T}^{n+1} \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \tag{5.38}$$

because $\mathbb{T}^{n+2} \times \mathbb{S}(n)_{\mathrm{tail}}$ is κ_{n+3}-closed in V, and therefore by Easton's lemma 3.32 κ_{n+3}-distributive over the forcing (5.38) which is κ_{n+3}-cc by the productivity of the Knaster property.

We finish the proof by arguing that the forcing in (5.38) cannot add T (and its specialising function g which maps T to the cardinal predecessor of κ_{n+2} (i.e. κ_{n+1}) in the generic extension by (5.38)). In the interest of further simplification of notation, we will not introduce variables for generic filters, but we will use the convention that $V[P]$ denotes a generic extension by a forcing P whenever the exact generic filter is irrelevant (recall that we assume that the weakest condition forces that $\dot{T}$ is a κ_{n+2}-Aronszajn tree).

Let us work in

$$V^* = V[\mathbb{R}(n+2) \times \mathbb{T}^{n+1} \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})]. \tag{5.39}$$

V^* still satisfies that κ_{n+2} is a Mahlo cardinal and $\mathbb{S}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$ is κ_{n+2}-cc (this follows by Easton's lemma 3.32 since $\mathbb{R}(n+2) \times \mathbb{T}^{n+1} \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})$ is κ_{n+2}-closed). Note that since κ_{n+2} is a Mahlo cardinal in V^* (and in particular inaccessible), T together with its specialising function g cannot be present already in V^*.

Since T is in the generic extension over V by the forcing in (5.38), there are in V^* some $\mathbb{S}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$-names $\dot{T}$ and $\dot{g}$ for the tree T and the function g witnessing its specialisation.[21] We can identify both $\dot{T}$ and $\dot{g}$ with a name for a subset of κ_{n+2}. Since the forcing $\mathbb{S}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$ is κ_{n+2}-cc, we may assume that already $\mathbb{S}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})$ adds the tree and the specialising function, where $\mathbb{S}(n)|\kappa_{n+2}$ is the forcing

$$\prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n+1} \mathbb{R}(m)|\kappa_{n+2}, \tag{5.40}$$

where $\prod_{m \le n+1} \mathbb{R}(m)|\kappa_{n+2}$ is the restriction of Cohen forcings to length κ_{n+2} (see (5.23) and what follows for more details on the notation). This follows from the fact that the names $\dot{T}$ and $\dot{g}$ refer to up to κ_{n+2}-many Cohen coordinates in the forcing $\mathbb{S}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$, so we can proceed as in Lemma 5.3 with suitable modifications.

Since κ_{n+2} is a Mahlo cardinal, there is a V-inaccessible δ, $\kappa_{n+1} < \delta < \kappa_{n+2}$, such that $T \restriction \delta, g \restriction \delta$ (the restrictions of T and g to the subtree of T of height δ) are added over V^* by the following forcing

$$\prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \times \prod_{m \le n+1} \mathbb{R}(m)|\delta \times \mathrm{Add}(\kappa_{n+1}, \delta). \tag{5.41}$$

Notice that $\mathbb{R}(n+1)|\delta$ denotes the forcing $\mathrm{Add}(\kappa_{n+1}, \delta)$, and since $\mathrm{Add}(\kappa_{n+1}, \delta)$ is isomorphic to its square, we may replace $\prod_{m \le n+1} \mathbb{R}(m)|\delta$ by $\prod_{m \le n} \mathbb{R}(m)$ in (5.41):

$$\mathbb{S}_0 = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \times \prod_{m \le n} \mathbb{R}(m)|\delta \times \mathrm{Add}(\kappa_{n+1}, \delta). \tag{5.42}$$

The forcing $\mathbb{S}_0$ in (5.42) is δ-cc – and therefore in particular preserves the regularity of δ – using the productivity of the Knaster property. The inaccessible δ is found as follows: as T and g restricted to $\beta < \kappa_{n+2}$ have size $<\kappa_{n+2}$, there is a closed unbounded subset in κ_{n+2} of ordinals $\sigma(\beta)$ such that $T \restriction \beta$ and $g \restriction \beta$ are added by the forcing restricted to $\sigma(\beta)$, where β ranges over ordinals between κ_{n+1} and κ_{n+2}. By the Mahloness of κ_{n+2}, this closed unbounded set has an inaccessible fixed point (which we denote δ).

[21] We assume for simplicity that the weakest condition forces that there is a special κ_{n+2}-Aronszajn tree; if not, choose the generic filter to contain this condition.

In (5.46) below, we will work with the product

$$\mathbb{S}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{5.43}$$

With the current definition of $\mathbb{S}(n)|\kappa_{n+2}$ in (5.40), the product (5.43) is equal to

$$\prod_{m\leq n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m\leq n} \mathbb{R}(m)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{5.44}$$

Since Cohen forcing is isomorphic to its square, the forcing in (5.44) is isomorphic to (5.45):

$$\prod_{m\leq n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m\leq n} \mathbb{R}(m)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}). \tag{5.45}$$

In order to simplify notation, we will from now on identify (5.43) with (5.45).

We finish the proof by arguing that the forcing from the model $V^*[\mathbb{S}_0]$ to

$$V^*[\mathbb{S}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})] \tag{5.46}$$

cannot add a cofinal branch to $T\restriction\delta$. This will be a contradiction for the following reason. $T\restriction\delta$ has a cofinal branch in the final model since it is a tree of height κ_{n+2} and therefore has nodes of height $\delta < \kappa_{n+2}$. If the forcing from $V^*[\mathbb{S}_0]$ to $V^*[\mathbb{S}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})]$ cannot add such a branch, it must be present already in $V^*[\mathbb{S}_0]$, but this is impossible: by the choice of δ and the properties of the specialisation function g, g restricted to such a branch yields an injective function from δ into the cardinal predecessor of δ (i.e. κ_{n+1}) in the forcing extension $V^*[\mathbb{S}_0]$ and therefore the cardinal δ would be collapsed in $V^*[\mathbb{S}_0]$ (which we argued below (5.42) is not the case). See also [55], Section 5, for more details.

Let us denote by $\mathbb{T}^n_\delta$ the term forcing which is κ_{n+1}-closed in the extension $V[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$, and hence also in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$ using the κ_{n+2}-closure of the forcing from V to V^*, such that in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$,

$$\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})/\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \tag{5.47}$$

is a projection of $\mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \mathbb{T}^n_\delta$.

Thus it suffices to show that over $V^*[\mathbb{S}_0]$, the forcing

$$\mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \mathbb{T}^n_\delta \times \prod_{m\leq n} \mathbb{R}(m)|(\kappa_{n+2} - \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta) \tag{5.48}$$

does not add a cofinal branch to $T\restriction\delta$, where $\prod_{m\leq n} \mathbb{R}(m)|(\kappa_{n+2} - \delta)$ is the restriction of Cohen forcings to the interval $[\delta, \kappa_{n+2})$ (see (5.23) and what follows for details on the notation).

We can now finish the proof analogously to Theorem 5.7.[22] Denote

$$P_1 = \mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \prod_{m\leq n} \mathbb{R}(m)|(\kappa_{n+2} - \delta) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta), \tag{5.49}$$

[22] It is immaterial to the argument whether we work in a generic extension of M as in Theorem 5.7, and discuss the ordinals $\kappa < j(\kappa)$, or work in a generic extension of V, and discuss the ordinals $\delta < \kappa_{n+2}$. Note that the proof of Theorem 5.7 could also have been formulated with some $\delta < \kappa$ without mentioning an elementary embedding.

and

$$Q = \mathbb{T}^n_\delta. \tag{5.50}$$

Note that $P_1 \times Q$ is the forcing from (5.48).

P_1 is isomorphic to its square and is δ-cc in $V^*[\mathbb{S}_0]$ by the productivity of the Knaster property. It follows that P_1 cannot add a cofinal branch to $T \restriction \delta$ by Fact 4.5. It remains to show that Q cannot add a cofinal branch to $T \restriction \delta$ over the model $V^*[\mathbb{S}_0][P_1]$.

Let

$$\begin{aligned} P = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m\le n} \mathbb{R}(m)|\delta \\ \times \operatorname{Add}(\kappa_n, \kappa_{n+2} - \delta) \times \prod_{m\le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta). \end{aligned} \tag{5.51}$$

Now we state the analogue of Claim 5.9 (we explicitly spell out all the relevant forcings for better orientation even though the expression could be simplified: in particular, $\operatorname{Add}(\kappa_{n+1}, \delta) \times \operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)$ is isomorphic to $\operatorname{Add}(\kappa_{n+1}, \kappa_{n+2})$).

Claim 5.15. *The following hold.*

(i) Q in (5.50) is κ_{n+1}-closed in

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]. \tag{5.52}$$

(ii) P in (5.51) is κ_{n+1}-cc in

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]. \tag{5.53}$$

Proof. (i). By Lemma 5.13, $\operatorname{Add}(\kappa_{n+1}, \delta) \times \operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)$ is κ_{n+1}-distributive over the model $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$, where Q is κ_{n+1}-closed, and therefore Q stays κ_{n+1}-closed in the model (5.52).

(ii). As in the proof of Claim 5.9(ii), use the product analysis of $\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)$ and show the κ_{n+1}-cc of P using the productivity of the Knaster property and Easton's lemma 3.32. □

Now the proof can be finished by applying Fact 4.7 to P and Q from Claim 5.15 over the model $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)]$ (notice that in this model it is true that $\delta = \kappa_{n+1}^+$ and $2^{\kappa_n} = \delta$, so the cardinal assumptions in Fact 4.7 are satisfied). In more detail: the tree $T \restriction \delta$ is in the generic extension $V^*[\mathbb{S}_0]$, and hence also in

$$V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \delta)][\operatorname{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)][P] = V^*[\mathbb{S}_0][P_1] \tag{5.54}$$

which extends $V^*[\mathbb{S}_0]$ (note that (5.54) holds by the fact that the forcings over the model V^* on the left-hand side and the right-hand side of the equation are composed of the product of the same forcing notions, just suitably regrouped).

Now the proof is finished by applying Q over the model (5.54) using Fact 4.7. □

As with Theorem 5.7, we may formulate a more succinct version of Theorem 5.14:

Corollary 5.16. (GCH) *Assume there are infinitely many Mahlo cardinals. Let f be a function from ω to ω which satisfies* *(i)* *For all* $m, n < \omega$, $m < n \to f(m) \leq f(n)$,

(ii) $f(n) > n + 1$ *for all* $n < \omega$.

Then there is a model where the weak tree property holds at every $\aleph_n$, $1 < n < \omega$, *and the continuum function below* $\aleph_\omega$ *obeys* f*: i.e.* $2^{\aleph_n} = \aleph_{f(n)}$ *for all* $n < \omega$.

Note that Remark 5.11 also applies in this context.

6 The double successor of a singular cardinal

In this section we study the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality. The results in this section are joint with Sy-David Friedman and Radek Honzik, and were submitted as [27].

In [10], Cummings and Foreman showed that starting from a Laver-indestructible supercompact cardinal κ and a weakly compact $\lambda > \kappa$, one can construct a generic extension where $2^\kappa = \lambda = \kappa^{++}$, κ is a singular strong limit cardinal with cofinality ω, and the tree property holds at κ^{++}. It is natural to try to generalize this result in at least two directions.

First, one can ask whether – in addition to the properties identified in the previous paragraph – κ can equal $\aleph_\omega$. Cummings and Foreman suggested in [10] that this is possible, but did not provide any details. A model with the tree property at $\aleph_{\omega+2}$, with $\aleph_\omega$ strong limit, was first constructed by Friedman and Halilović in [22], moreover from a significantly lower large cardinal assumption of strength.[23] Shortly afterwards, Gitik, answering a question posed in [22], showed in [30] that the same result can be proved form a weaker and optimal assumption.

Second, one can ask whether it is possible to have 2^κ greater than κ^{++} with the tree property at κ^{++}. Using a variant of Mitchell forcing, Friedman and Halilović [23] proved that starting from a sufficiently strong κ, one can keep the measurability of κ together with $2^\kappa > \kappa^{++}$ and the tree property at κ^{++}.

In this section, we generalize [10] in the second direction. In Theorem 6.1, we prove that starting from a Laver-indestructible supercompact κ and a weakly compact λ above, one can find a forcing extension where κ is strong limit singular with cofinality ω, $2^\kappa = \kappa^{+3} = \lambda^+$, and the tree property holds at κ^{++}. In Theorem 6.25 we give an outline of a generalisation in which the gap $(\kappa, 2^\kappa)$ can be arbitrarily large: $2^\kappa = \mu$ for any cardinal $\mu > \lambda$ with cofinality greater than κ. The method of the proof is based on the argument in [10], with reference to [73] which fills a gap in the final stage of that argument.

6.1 Gap three

Recall that a supercompact cardinal κ is *Laver-indestructible* if it remains supercompact in any forcing extension by a forcing which is κ-directed closed (where $\mathbb{P}$ is κ-directed closed if for every $D \subseteq \mathbb{P}$ of size less than κ, if for all p_1, p_2 in D there is $e \in D$ such that $e \le p_1$ and $e \le p_2$, then there is $p \in \mathbb{P}$, with $p \le d$ for all $d \in D$).

Theorem 6.1. *Assume* GCH *and let κ be a Laver-indestructible supercompact cardinal and $\kappa < \lambda$, λ weakly compact. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ *preserves cardinals* $\le \kappa^+$ *and* $\ge \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \;\&\; 2^\kappa = \lambda^+ \;\&\; \mathrm{cf}(\kappa) = \omega \;\&\;$ *κ is strong limit*$)$.
(iii) $V[\mathbb{R}] \models \mathsf{TP}(\lambda)$.

The proof will be given in a sequence of lemmas, and is divided into two stages. Stage

[23] The technique of proof in [22] used Sacks forcing to obtain the tree property, unlike the proof in [10] which is based on a Mitchell-style analysis.

1 defines $\mathbb{R}$, verifies some basic properties for (i) and (ii) of Theorem 6.1 and shows that if $\mathbb{R}$ adds an Aronszajn tree on λ, then already a regular subforcing, which we denote $\mathbb{R}^*$, adds an Aronszajn tree on λ. The forcing $\mathbb{R}^*$ is designed to be very similar to the forcing used in [10]. In stage 2, we show that indeed $\mathbb{R}^*$ allows a very similar analysis to [10] (with correction according to [73]), and therefore cannot add an Aronszajn tree on λ, which finishes the proof.

6.1.1 Stage 1

Definition 6.2. Let $\mathbb{P}$ denote the Cohen forcing $\mathrm{Add}(\kappa, \lambda^+)$ and for $\alpha < \lambda^+$, let $\mathbb{P}|\alpha$ denote $\mathrm{Add}(\kappa, \alpha)$.

The following lemma will be useful.

Lemma 6.3. *Let $\dot{U}$ be a $\mathbb{P}$-name such that*

$$1_{\mathbb{P}} \Vdash \dot{U} \text{ is a normal measure on } \kappa. \tag{6.1}$$

Then there is a set A of unboundedly many $\alpha < \lambda^+$ containing its limit points of cofinality $>\kappa$ such that for every $\alpha \in A$ and every $\mathbb{P}$-generic filter G,

$$\dot{U}^G \cap V[G|\alpha] \in V[G|\alpha]. \tag{6.2}$$

Proof. Let $\alpha_0 < \lambda^+$ be given, we show how to find $\alpha \geq \alpha_0$ in A. Let $\langle \dot{x}_i \,|\, i < \nu < \lambda^+ \rangle$ be some enumeration of all nice $\mathbb{P}|\alpha_0$-names for subsets of κ. Note that there are at most λ-many such names so we can indeed choose $\nu < \lambda^+$. For every $i < \nu$, let X_i be a maximal antichain in $\mathbb{P}$ of conditions deciding the statement $\dot{x}_i \in \dot{U}$; by the κ^+-cc of $\mathbb{P}$, the size of X_i is at most κ. Let $\beta_0 \geq \alpha_0$ be such that the supports of all conditions in $\bigcup_{i<\nu} X_i$ are contained in β_0. Repeat this procedure κ^+-many times, building an increasing chain of ordinals and let $\alpha = \sup\{\beta_k \,|\, k < \kappa^+\}$, $\mathrm{cf}(\alpha) = \kappa^+$. Now, if $\dot{x}$ is a $\mathbb{P}|\alpha$-name for a subset of κ, then there is some $\alpha' < \alpha$ such that all coordinates mentioned by $\dot{x}$ are below α'; it follows that $\dot{x}$ was considered in the construction, together with a maximal antichain X in $\mathbb{P}$ of conditions deciding the statement $\dot{x} \in \dot{U}$. Using these $\dot{x}$'s and X's, one can build a $\mathbb{P}|\alpha$-name $\dot{U}_\alpha$ such that for every nice $\mathbb{P}|\alpha$-name $\dot{x}$ for a subset of κ:

$$\dot{x}^{G|\alpha} = \dot{x}^G \in \dot{U}^G \Leftrightarrow \dot{x}^{G|\alpha} \in \dot{U}_\alpha^{G|\alpha}. \tag{6.3}$$

It is clear that if A is defined to be the set of $\alpha < \lambda^+$ constructed as above and α with cofinality $>\kappa$ is a limit point of A, then $\alpha \in A$.[24] □

Fix temporarily a $\mathbb{P}$-generic filter G. Denote $\dot{U}^G = U$. For any $\alpha \in A$ such that $\lambda < \alpha < \lambda^+$ there is a $\mathbb{P}_\alpha$-name, which we denote by $\dot{U}_\alpha$, such that

$$(\dot{U}_\alpha)^{G|\alpha} = U \cap V[G|\alpha]. \tag{6.4}$$

Let us write U_α for $U \cap V[G|\alpha]$. Let us fix $\beta \in A, \lambda < \beta$. This β is going to be fixed for the remainder of the proof.

[24] We mention the closure of A because the current proof is directly applicable to Lemma 6.5 with a set B, where the closure is relevant to ensure $B^* \subseteq B$ for a certain set B^* defined in Section 6.1.2, 2nd paragraph. The closure will not be used for A, though.

For $\alpha \leq \lambda$, let $\mathrm{Even}(\alpha)$ denote the set of even ordinals below α. For $\alpha \leq \lambda$, let us write $\mathbb{P}|\mathrm{Even}(\alpha)$ to denote the Cohen forcing $\mathrm{Add}(\kappa, \mathrm{Even}(\alpha))$ which only mentions coordinates indexed by even ordinals. Let ι be a bijection between β and $\mathrm{Even}(\lambda)$; ι naturally generates an isomorphism between $\mathbb{P}|\beta$ and $\mathbb{P}|\mathrm{Even}(\lambda)$ which we also denote ι. Let us further extend the domain of ι to all $\mathbb{P}|\beta$-names, and also to $\mathbb{P}|\beta$-generic filters, in the obvious way.

Since $1_{\mathbb{P}|\beta} \Vdash \dot{U}_\beta$ is a measure, we have $1_{\mathbb{P}|\mathrm{Even}(\lambda)} \Vdash \iota(\dot{U}_\beta)$ is a measure.

Remark 6.4. Note that ι generates a $\mathbb{P}|\mathrm{Even}(\lambda)$-generic filter $\iota(G|\beta)$ such that $V[G|\beta] = V[\iota(G|\beta)]$, and

$$U_\beta = (\dot{U}_\beta)^{G|\beta} = \iota(\dot{U}_\beta)^{\iota(G|\beta)}. \tag{6.5}$$

However, it is not true that $\iota(\dot{U}_\beta)^{G|\mathrm{Even}(\lambda)} = U_\beta$, where $G|\mathrm{Even}(\lambda)$ is the $\mathbb{P}|\mathrm{Even}(\lambda)$-generic filter composed of the Cohen generics on the even coordinates of G below λ. The reason is that $V[G|\mathrm{Even}(\lambda)]$ is a proper submodel of $V[\iota(G|\beta)] = V[G|\beta]$.

The proof of the following lemma is the same as for Lemma 6.3.

Lemma 6.5. *There is a set B of unboundedly many $\alpha < \lambda$ containing its limit points of cofinality $>\kappa$ such that for every $\alpha \in B$ and every $\mathbb{P}|\mathrm{Even}(\lambda)$-generic filter H,*

$$\iota(\dot{U}_\beta)^H \cap V[H|\mathrm{Even}(\alpha)] \in V[H|\mathrm{Even}(\alpha)], \tag{6.6}$$

where $H|\mathrm{Even}(\alpha)$ is the restriction of H to $\mathbb{P}|\mathrm{Even}(\alpha)$.

Let us write $\dot{U}^\iota_\alpha$ for the natural (i.e. obtained from the construction in the proof of Lemma 6.5) $\mathbb{P}|\mathrm{Even}(\alpha)$-name for the measure $\iota(\dot{U}_\beta)^H \cap V[H|\mathrm{Even}(\alpha)]$.

We fix the following notation: Denote $\hat{A} = (A \cap [\beta, \lambda^+)) \cup \{\lambda^+\}$. For every $\gamma \in \hat{A}$, let $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$ denote the Cohen forcing $\mathrm{Add}(\kappa, \gamma)$ followed by Prikry forcing $\mathbb{Q}_\gamma$ defined with respect to the measure $\dot{U}_\gamma$ (where we identify $\dot{U}_{\lambda^+}$ with $\dot{U}$ and $\mathbb{P}|\lambda^+ * \mathbb{Q}_{\lambda^+}$ with $\mathbb{P} * \mathbb{Q}$). For $\alpha \in B$, where B is as in Lemma 6.5, let $\mathbb{Q}^\iota_\alpha$ be a $\mathbb{P}|\mathrm{Even}(\alpha)$-name for Prikry forcing defined with the $\mathbb{P}|\mathrm{Even}(\alpha)$-name $\dot{U}^\iota_\alpha$. Let us also define $\mathbb{Q}^\iota_\lambda$ as Prikry forcing with the measure $\iota(\dot{U}_\beta)$ in $\mathbb{P}|\mathrm{Even}(\lambda)$.

The following lemma defines certain projections which will be used later on.

Lemma 6.6. *The following hold.*

(i) For every $\gamma < \delta$ in $\hat{A}$, there is a projection

$$\sigma^\delta_\gamma : \mathbb{P}|\delta * \mathbb{Q}_\delta \to \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma). \tag{6.7}$$

(ii) For every γ in $\hat{A}$ and every $\alpha \in B$, there is a projection

$$\sigma^\gamma_\alpha : \mathbb{P}|\gamma * \mathbb{Q}_\gamma \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha). \tag{6.8}$$

(iii) For $\gamma \in A \cap (\beta, \lambda^+)$ and $\alpha \in B$, let $\hat{\sigma}^\gamma_\alpha$ be the extension of σ^γ_α to the Boolean completion of $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$ obtained according to Lemma 3.44(ii)(b):

$$\hat{\sigma}^\gamma_\alpha : \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma) \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha). \tag{6.9}$$

Then the projections commute:

$$\sigma^{\lambda^+}_\alpha = \hat{\sigma}^\gamma_\alpha \circ \sigma^{\lambda^+}_\gamma. \tag{6.10}$$

Proof. (i). Let $G * x$ be a $\mathbb{P}|\delta * \mathbb{Q}_\delta$-generic filter,[25] where x is an ω-sequence cofinal in κ. By the geometric condition for Prikry genericity,[26] and the fact that $\dot{U}_\gamma$ is the restriction of $\dot{U}_\delta$, it is clear that $G|\gamma * x$ is $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$-generic. The result follows by Lemma 3.41.

(ii). Let $G * x$ be a $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$-generic filter, where x is an ω-sequence cofinal in κ. By (6.5) and the geometric condition for the generic filters for Prikry forcings,

$$\iota(G|\beta) * x \text{ is } \mathbb{P}|\mathrm{Even}(\lambda) * \mathbb{Q}^\iota_\lambda\text{-generic.} \tag{6.11}$$

Substituting $H = \iota(G|\beta)$ in Lemma 6.5, for every $\alpha \in B$, $\mathbb{Q}^\iota_\alpha$ is a forcing in $V[H|\mathrm{Even}(\alpha)]$ defined with respect to the restriction of the measure U; it follows that $H|\mathrm{Even}(\alpha) * x$ is a generic filter for $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha$ existing in $V[G * x]$. The result again follows by Lemma 3.41.

(iii). σ^γ_α is correctly defined by Lemma 3.44(ii)(b). Let us fix $(p,(s,\dot{A}))$ in $\mathbb{P} * \mathbb{Q}$ and let us denote

$$b_\alpha = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha) \mid (p,(s,\dot{A})) \Vdash b \in \dot{G}_\alpha\},$$

$$b_\gamma = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma) \mid (p,(s,\dot{A})) \Vdash b \in \dot{G}_\gamma\},$$

and

$$b^\gamma_\alpha = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha) \mid b_\gamma \Vdash b \in \dot{G}_\alpha\},$$

where $\dot{G}_\gamma$ and $\dot{G}_\alpha$ are the canonical names for the generic filters. The intuition is that the Boolean value b_α (and similarly b_γ and b^γ_α) corresponds to a condition $(\iota(p|\beta)|\alpha,(s,\dot{C}))$ for some $\dot{C}$ which is the intersection of all elements in $\dot{U}_\alpha$ in $V^{\mathbb{P}|\mathrm{Even}(\alpha)*\mathbb{Q}^\iota_\alpha}$ which contain $\dot{A}$; the problem is that this condition in general may not exist in $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha$, and it is necessary to use the more abstract Boolean names.

We show that $b_a = b^\gamma_\alpha$.

To argue for $b^\gamma_\alpha \leq b_\alpha$, notice that we can identify every element of $\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha)$ with an element b of $\mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma)$ by virtue of the projection $\hat{\sigma}^\gamma_\alpha$; now if $(p,(s,\dot{A}))$ forces b into $\dot{G}_\alpha$, then clearly $(p,(s,\dot{A}))$ forces b into $\dot{G}_\gamma$. In particular b_γ forces b into $\dot{G}_\alpha$, and so $b^\gamma_\alpha \leq b_\alpha$.

Conversely, b_γ can be identified with an element of $\mathrm{RO}^+(\mathbb{P} * \mathbb{Q})$, and under this identification $(p,(s,\dot{A})) \leq b_\gamma$. It follows that if b_γ forces $b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha)$ into $\dot{G}_\alpha$, so does $(p,(s,\dot{A}))$, and hence $b_\alpha \leq b^\gamma_\alpha$. □

We are now ready to define the main forcing $\mathbb{R}$.

Definition 6.7. Conditions in $\mathbb{R}$ are triples (p,q,r) which satisfy the following (where B is as in Lemma 6.5):

(i) (p,q) is a condition in $\mathbb{P} * \mathbb{Q}$.
(ii) r is a function with $\mathrm{dom}(r) \subseteq B$ and $|\mathrm{dom}(r)| \leq \kappa$ such that for every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a nice $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha$-name and:

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1). \tag{6.12}$$

[25] We abuse notation here and identify $G * x$ with the generic filter which it determines.
[26] The geometric condition characterises the genericity for Prikry forcing: a cofinal ω-sequence in κ determines a generic filter if and only if it is eventually contained in every element of the measure used to define the forcing.

The ordering is defined as follows: $(p', q', r') \leq (p, q, r)$ if the following hold:

(i) $(p', q') \leq (p, q)$ in $\mathbb{P} * \mathbb{Q}$.
(ii) $\mathrm{dom}(r) \subseteq \mathrm{dom}(r')$ and for every $\alpha \in \mathrm{dom}(r)$,

$$\sigma_\alpha^{\lambda^+}(p', q') \Vdash_{\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\iota)} r'(\alpha) \leq r(\alpha). \tag{6.13}$$

The following lemmas identify the basic properties of $\mathbb{R}$.

Define $\mathbb{U}$ to consist of all elements of $\mathbb{R}$ of the form $(1, 1, r)$, with the induced partial ordering. Let $\nu : (\mathbb{P} * \mathbb{Q}) \times \mathbb{U} \to \mathbb{R}$ be given by $\nu((p, q), (1, 1, r)) = (p, q, r)$.

Lemma 6.8. *The following hold:*

(i) $\mathbb{P} * \mathbb{Q}$ *has a dense subset which has the* κ^+*-Knaster property.*
(ii) $\mathbb{U}$ *is* κ^+*-closed.*
(iii) ν *is a projection which commutes with the natural projections from* $\mathbb{R}$ *and* $(\mathbb{P}*\mathbb{Q}) \times \mathbb{U}$ *to* $\mathbb{P} * \mathbb{Q}$ *(so that in a natural way* $V[\mathbb{P} * \mathbb{Q}] \subseteq V[\mathbb{R}] \subseteq V[(\mathbb{P} * \mathbb{Q}) \times \mathbb{U}]$*).*
(iv) $V[\mathbb{R}]$ *and* $V[\mathbb{P} * \mathbb{Q}]$ *have the same* κ*-sequences.*

Proof. (i). Let Z contain all conditions of the form $(p, (\check{s}, \dot{A}))$; then Z is dense in $\mathbb{P} * \mathbb{Q}$ and has the κ^+-Knaster property. (ii)–(iii) are obvious. Regarding (iv), by (i), (ii) and the Easton's lemma 3.32, $\mathbb{U}$ is κ^+-distributive over $V[\mathbb{P} * \mathbb{Q}]$; then (iv) follows by (iii). □

Lemma 6.9. *The following hold:*

(i) $\mathbb{R}$ *has the* λ*-Knaster property.*
(ii) $\mathbb{R}$ *collapses cardinals in the interval* (κ^+, λ) *(and no other cardinals), making* κ^{++} *in* $V[\mathbb{R}]$ *equal to* λ*. In* $V[\mathbb{R}]$*,* $2^\kappa = \lambda^+ = \kappa^{+3}$*.*

Proof. (i). Let $Y = \{(p_\alpha, q_\alpha, r_\alpha) \mid \alpha < \lambda\}$ be a set of conditions in $\mathbb{R}$ of size λ. We wish to find a subset Y' of size λ which consists of pairwise compatible conditions. By a Δ-system argument there is a cofinal $a \subseteq \lambda$ such that $\{(p_\alpha, q_\alpha) \mid \alpha \in a\}$ is a family of pairwise compatible conditions in $\mathbb{P} * \mathbb{Q}$. By another Δ-system argument, there is a cofinal $a' \subseteq a$, and a root $r \subseteq B$ of size $\leq \kappa$, such that for all $\alpha, \beta \in a'$, $\alpha \neq \beta$, $\mathrm{dom}(r_\alpha) \cap \mathrm{dom}(r_\beta) = r$. By the inaccessibility of λ, the number of nice $\mathbb{P}|\mathrm{Even}(\gamma) * \mathbb{Q}_\gamma^\iota$-names, $\gamma \in r$, for conditions in $\mathrm{Add}(\kappa^+, 1)$ is less than λ. Hence there is a cofinal $a'' \subseteq a'$ such that if α, β are in a'', then for all $\gamma \in r$, $r_\alpha(\gamma) = r_\beta(\gamma)$. It follows $Y' = \{(p_\alpha, q_\alpha, r_\alpha) \mid \alpha \in a''\}$ is as required.

(ii). Obvious. □

We will need to consider truncations of $\mathbb{R}$, which we define next.

Definition 6.10. Let $\gamma \in A$, and $\lambda < \beta < \gamma$. Conditions in $\mathbb{R}|\gamma$ are triples (p, q, r) which satisfy the following:

(i) (p, q) is a condition in $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$, where $\mathbb{Q}_\gamma$ is Prikry forcing defined with respect to the measure $\dot{U}_\gamma$.
(ii) r is a function with $\mathrm{dom}(r) \subseteq B$ and $|\mathrm{dom}(r)| \leq \kappa$ such that for every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a nice $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\iota$-name and:

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\iota \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1). \tag{6.14}$$

The ordering is defined as for $\mathbb{R}$, but using the projections σ_α^γ, $\alpha \in B$.

Lemma 6.11. *Let γ be in A and $\beta < \gamma < \lambda^+$. There is a projection from $\mathbb{R}$ to* $\mathrm{RO}^+(\mathbb{R}|\gamma)$.

Proof. First notice that $\mathbb{R}|\gamma$ is densely embeddable in $\hat{\mathbb{R}}|\gamma$, which is defined as $\mathbb{R}|\gamma$ but with elements of $\mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma)$ instead of $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$, and with the projection $\hat{\sigma}^\gamma_\alpha$. Because of the commutativity $\sigma^{\lambda^+}_\alpha = \hat{\sigma}^\gamma_\alpha \circ \sigma^{\lambda^+}_\gamma$, see Lemma 6.6(iii), it is easy to check that in $V^{\mathbb{R}}$, we can find a generic for $\hat{\mathbb{R}}|\gamma$. □

We now show that if $\mathbb{R}$ adds an Aronszajn tree on λ, then a truncation $\mathbb{R}|\beta^*$ for a certain β^* must add an Aronszajn tree on λ.

Before we give the lemma, let us define some terminology. Let (p,q) be a condition in $\mathbb{P} * \mathbb{Q}$; without loss of generality, q is of the form $(s, \dot{E})$ for some finite subset s of κ and some nice $\mathbb{P}$-name $\dot{E}$ for a subset of κ. We say that a coordinate $\alpha < \lambda^+$ is in the support of (p,q) if α is in the support of p or in the support of some p' which occurs in the nice name $\dot{E}$.

Lemma 6.12. *Suppose $\mathbb{R}$ forces that there is an Aronszajn tree on λ. Then for some β^* in A, $\beta < \beta^*$, $\mathbb{R}|\beta^*$ forces there is an Aronszajn tree on λ.*

Proof. Let $\dot{T}$ be a nice name for a subset of λ which in some natural way corresponds to an Aronszajn tree on λ, which we assume exists in $V^{\mathbb{R}}$. $\dot{T}$ is of the form $\bigcup\{\{\alpha\} \times K_\alpha \mid \alpha < \lambda\}$, where K_α for $\alpha < \lambda$ is an antichain in $\mathbb{R}$. By the λ-Knaster property, $|K_\alpha| < \lambda$ for every $\alpha < \lambda$. It follows there are at most λ many coordinates $\alpha < \lambda^+$ which are in the support of (p,q) such that for some r, $(p,q,r) \in \bigcup_{\alpha<\lambda} K_\alpha$ (we say that α is *in the support of $\dot{T}$*). Hence we can choose β^* in A such that $\beta < \beta^*$, and $\mathbb{R}|\beta^*$ forces that $\dot{T}'$ is an Aronszajn tree on λ, for some name $\dot{T}'$ which is naturally obtained from $\dot{T}$. □

Suppose now that $\mathbb{R}$ does force that there is an Aronszajn tree on λ and let us fix β^* as above (we will later show that the assumption that $\mathbb{R}$ adds an Aronszajn tree on λ leads to a contradiction).

Let ι^* be an isomorphism between $\mathbb{P}|\beta^*$ and $\mathbb{P}|\lambda$; choose ι^* so that it extends ι (the fixed isomorphism between $\mathbb{P}|\beta$ and $\mathbb{P}|\mathrm{Even}(\lambda)$). This implies $\iota(\dot{U}_\beta) = \iota^*(\dot{U}_\beta)$, and therefore the measure $\iota^*(\dot{U}_{\beta^*})$ is forced to extend the measure $\iota(\dot{U}_\beta)$. More precisely, if $G|\beta^*$ is $\mathbb{P}|\beta^*$-generic, then the following hold:

(i) $\bar{G} = \iota^*(G|\beta^*)$ is $\mathbb{P}|\lambda$-generic and its restriction to its even coordinates, to be denoted as $\bar{G}|\mathrm{Even}(\lambda)$, is equal to $\iota(G|\beta)$ (and $\bar{G}|\mathrm{Even}(\lambda)$ is $\mathbb{P}|\mathrm{Even}(\lambda)$-generic).
(ii) The measure $\iota(\dot{U}_\beta)^{\bar{G}|\mathrm{Even}(\lambda)}$ in $V[\bar{G}|\mathrm{Even}(\lambda)]$ is extended by the measure $\iota^*(\dot{U}_{\beta^*})^{\bar{G}}$ in $V[\bar{G}]$.

Define $\mathbb{Q}^{\iota^*}_\lambda$ as Prikry forcing in $\mathbb{P}|\lambda$ with the measure $\iota^*(\dot{U}_{\beta^*})$.

Lemma 6.13. *(i) ι^* extends to an isomorphism from $\mathbb{P}|\beta^* * \mathbb{Q}_{\beta^*}$ onto $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda$.*
(ii) For every $\alpha \in B$, $\sigma^\lambda_\alpha = \sigma^{\beta^*}_\alpha \circ (\iota^*)^{-1}$ is a projection

$$\sigma^\lambda_\alpha : \mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha). \tag{6.15}$$

Proof. (i). Let us view $\mathbb{Q}_{\beta^*}$ as a collection of conditions $(p,(s,\dot{A}))$, where $\dot{A}$ is a nice name. It is clear that we can naturally extend ι^* so that $\iota^*(\dot{A})$ is a nice name in $\mathbb{P}|\lambda$. Moreover, since ι^* is an isomorphism, $\mathbb{P}|\beta^*$ forces that $\dot{A}$ is in $\dot{U}_{\beta^*}$ if and only if $\iota(\dot{A})$ is in $\iota^*(\dot{U}_{\beta^*})$.

(ii). This is clear because $(\iota^*)^{-1}$ is an isomorphism. □

Let us define the following variant of $\mathbb{R}$, and call it $\mathbb{R}^*$:

Definition 6.14. Conditions in $\mathbb{R}^*$ are triples (p, q, r) which satisfy the following:

(i) (p, q) is a condition in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda$.
(ii) r is a function with $\mathrm{dom}(r) \subseteq B$ and $|\mathrm{dom}(r)| \leq \kappa$ such that for every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a nice $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha$-name and:

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1). \tag{6.16}$$

The ordering is defined by means of the projections σ^λ_α, $\alpha \in B$.

Lemma 6.15. *$\mathbb{R}|\beta^*$ and $\mathbb{R}^*$ are isomorphic.*

Proof. Define $f : \mathbb{R}|\beta^* \to \mathbb{R}^*$ by assigning to $(p, (s, \dot{A}), r)$ the condition $(\iota^*(p), (s, \iota^*(\dot{A})), r)$, where $(s, \dot{A})$ is a condition in $\mathbb{Q}_{\beta^*}$. Since σ^λ_α is determined by ι^* and $\sigma^{\beta^*}_\alpha$ (6.15), it is easy to check that f is an isomorphism. □

By Lemma 6.12, it follows that if $\mathbb{R}$ adds an Aronszajn tree on λ, $\mathbb{R}^*$ adds an Aronszajn tree on λ. In Stage 2, we show that this cannot happen.

6.1.2 Stage 2

We verify that the method of [10] can be applied in our case to verify that $\mathbb{R}^*$ does not add an Aronszajn tree at λ. In the argument, we use ideas from [73] to fill some gaps in [10].

In order to carry out the analysis of $\mathbb{R}^*$, we need to be able to define truncations $\mathbb{R}^*|\alpha$ for a large set $B^* \subseteq B$ below λ. First we apply the construction in Lemma 6.5 to the measure $\iota^*(\dot{U}_{\beta^*})$ in $\mathbb{P}|\lambda$, and obtain an unbounded set B^* below λ where the measure $\iota^*(\dot{U}_{\beta^*})$ reflects. Using the closure at points of cofinality $>\kappa$, one can in fact refine to get $B^* \subseteq B$. For $\alpha \in B^*$, define $\mathbb{Q}^{\iota^*}_\alpha$ as Prikry forcing defined with respect to the restriction of the measure $\iota^*(\dot{U}_{\beta^*})$. Denote $\hat{B}^* = B^* \cup \{\lambda\}$. We now proceed as in Lemma 6.6, and in particular using Lemma 3.41, to obtain for every $\alpha < \gamma$ in $\hat{B}^*$ projections:

$$\varrho^\gamma_\alpha : \mathbb{P}|\gamma * \mathbb{Q}^{\iota^*}_\gamma \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha) \tag{6.17}$$

and

$$\hat{\varrho}^\gamma_\alpha : \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}^{\iota^*}_\gamma) \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha), \tag{6.18}$$

which moreover satisfy:

$$\varrho^\lambda_\alpha = \hat{\varrho}^\gamma_\alpha \circ \varrho^\lambda_\gamma. \tag{6.19}$$

Recall we used projections σ^λ_α, $\alpha \in B$, to define the forcing $\mathbb{R}^*$. We show that σ^λ_α is the same projection as ϱ^λ_α for $\alpha \in B^*$, and therefore we can view $\mathbb{R}^*$ as being defined with the projections ϱ^λ_α, $\alpha \in B^*$.

Lemma 6.16. *For $\alpha \in B^*$, $\sigma^\lambda_\alpha = \varrho^\lambda_\alpha$.*

Proof. Let us fix $\alpha \in B^*$ and a condition (p,q) in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda$, and let us temporarily denote $\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha)$ by $B^{\mathrm{Even}(\alpha)}$. Let F be a $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda$-generic filter, F^* a $\mathbb{P}|\beta^* * \mathbb{Q}_{\beta^*}$-generic filter, and let $\dot{F}|\mathrm{Even}(\alpha)$ and $\dot{F}^*|\mathrm{Even}(\alpha)$ be the canonical names for the $B^{\mathrm{Even}(\alpha)}$-generic filters existing in $V[F]$ and $V[F^*]$, respectively. Since ι^* extends ι, it is clear that for every $b \in B^{\mathrm{Even}(\alpha)}$,

$$(p,q) \Vdash b \in \dot{F}|\mathrm{Even}(\alpha) \Leftrightarrow (\iota^*)^{-1}(p,q) \Vdash b \in \dot{F}^*|\mathrm{Even}(\alpha), \tag{6.20}$$

and therefore

$$\varrho^\lambda_\alpha(p,q) = \bigwedge\{b \in B^{\mathrm{Even}(\alpha)} \mid (p,q) \Vdash b \in \dot{F}|\mathrm{Even}(\alpha)\} = \bigwedge\{b \in B^{\mathrm{Even}(\alpha)} \mid (\iota^*)^{-1}(p,q) \Vdash b \in \dot{F}^*|\mathrm{Even}(\alpha)\} = \sigma^\lambda_\alpha(p,q), \tag{6.21}$$

as desired. □

Now we can define truncations $\mathbb{R}|\gamma$ for $\gamma \in B^*$:

Definition 6.17. For $\gamma \in B^*$, define $\mathbb{R}^*|\gamma$ as follows. Conditions in $\mathbb{R}^*|\gamma$ are triples (p,q,r):

(i) (p,q) is a condition in $\mathbb{P}|\gamma * \mathbb{Q}^{\iota^*}_\gamma$.
(ii) r is a function with $\mathrm{dom}(r) \subseteq B^* \cap \gamma$ and $|\mathrm{dom}(r)| \le \kappa$ such that for every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a nice $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha$-name and:

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1). \tag{6.22}$$

The ordering is defined as follows: $(p', q', r') \le (p,q,r)$ if the following hold:

(i) $(p', q') \le (p,q)$ in $\mathbb{P} * \mathbb{Q}$.
(ii) $\mathrm{dom}(r) \subseteq \mathrm{dom}(r')$ and for every $\alpha \in \mathrm{dom}(r)$,

$$\varrho^\gamma_\alpha(p', q') \Vdash_{\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha)} r'(\alpha) \le r(\alpha). \tag{6.23}$$

Lemma 6.18. *For every $\gamma \in B^*$ there exists a projection from $\mathbb{R}^*$ to $\mathrm{RO}^+(\mathbb{R}^*|\gamma)$.*

Proof. It follows as in Lemma 6.11, using the fact that $\sigma^\lambda_\alpha = \varrho^\lambda_\alpha$, $\alpha \in B^*$ (see Lemma 6.16). □

The analysis in Lemma 6.8 can be applied to $\mathbb{R}^*$ straightforwardly. Let $\mathbb{U}^*$ denote the κ^+-closed forcing such that there is a projection from $(\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda) \times \mathbb{U}^*$ to $\mathbb{R}^*$. By arguments similar to Lemma 6.8 and 6.9, for an inaccessible $\alpha \in \hat{B}^*$, $\mathbb{R}^*|\alpha$ preserves all cardinals except in the interval (κ^+, α) and forces $2^\kappa = \alpha$. Moreover, $V[\mathbb{R}^*|\alpha]$ is a submodel of $V[\mathbb{R}^*]$ and every bounded subset of λ in $V[\mathbb{R}^*]$ appears in $V[\mathbb{R}^*|\alpha]$, for some $\alpha \in B^*$.

The existence of $\mathbb{U}^*$ generalizes to the truncations $\mathbb{R}^*|\alpha$, $\alpha \in B^*$.

Lemma 6.19. *Let α be in B^*. Then $\mathbb{R}/(\mathbb{R}|\alpha)$ is in $V[\mathbb{R}|\alpha]$ a projection of $(\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha) \times \mathbb{U}^*_\alpha$ for some κ^+-closed forcing $\mathbb{U}^*_\alpha$ in $\mathbb{R}|\alpha$.*

Proof. Obvious. □

Following [10, Lemma 6.5], and the correction in [73], the proof is finished by showing that for every $\alpha \in B^*$, the product $(\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha) \times (\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha)$ ("the square of $(\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha)$") is κ^+-cc in $V[\mathbb{R}^*|\alpha]$ (this result is stated as Lemma 6.24).

Since the argument in [73] is stated for a different forcing, we provide a self-contained proof of Lemma 6.24. For the proof of Lemma 6.24, we need to prove some preliminary facts (Lemma 6.20 – Lemma 6.23).

Lemma 6.20. *Assume $(p,(s,\dot{A})) \in \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$ and $(q,(t,\dot{B})) \in \mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}$ are arbitrary conditions. Then $(p,(s,\dot{A}))$ forces that $(q,(t,\dot{B}))$ is not a condition in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$ if and only if one of the following conditions holds:*

(i) $q \restriction \alpha$ is incompatible with p,

(ii) $q \restriction \alpha$ is compatible with p, s does not extend t and t does not extend s,

(iii) $q \restriction \alpha$ is compatible with p, s extends t and $q \cup p \Vdash s \setminus t \nsubseteq \dot{B}$,

(iv) $q \restriction \alpha$ is compatible with p, t extends s and $(q \restriction \alpha) \cup p \Vdash t \setminus s \nsubseteq \dot{A}$.

Proof. Notice that $(p,(s,\dot{A})) \Vdash (q,(t,\dot{B})) \notin \mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$ if and only if there is no generic filter $G * x$ such that $(q,(t,\dot{B})) \in G * x$ and $(p,(s,\dot{A})) \in G|\alpha * x$.

From right to left, it is easy to see that each of the conditions (i)–(iv) above rules out the existence of such a generic filter $G * x$.

To prove the implication from left to right, assume that all conditions (i)–(iv) above fail. Then p is compatible with q and it has to hold that either s extends t or t extends s. If s extends t, then $q \cup p \nVdash s \setminus t \nsubseteq \dot{B}$. This means that there is r below $q \cup p$ such that $r \Vdash s \setminus t \subseteq \dot{B}$. Consider the condition $(r,(s,\dot{A} \cap \dot{B}))$ and let $G*x$ be generic filter such that $(r,(s,\dot{A} \cap \dot{B})) \in G * x$. It is easy to verify that $(q,(t,\dot{B})) \in G * x$ and $(p,(s,\dot{A})) \in G|\alpha * x$. The second case, if t extends s, is similar. □

We have just characterised the case when a condition in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}$ is forced out of the quotient. Now, we focus on the case when a condition is forced into the quotient. The following lemma says when a condition is not forced out of the quotient.

Lemma 6.21. *Assume $(p,(s,\dot{A})) \in \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$ and $(q,(t,\dot{B})) \in \mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}$ are arbitrary conditions. If they satisfy the following conditions*

(i) s extends t,

(ii) $p \leq q \restriction \alpha$ and

(iii) $q \cup p \Vdash s \setminus t \subseteq \dot{B}$,

then the following hold:

(i) $(p,(s,\dot{A}))$ does not force that $(q,(t,\dot{B}))$ out of the quotient $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$.

(ii) There is a direct extension of $(p,(s,\dot{A}))$ which forces $(q,(t,\dot{B}))$ into the quotient $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_{\alpha}$.

Proof. (i). It is enough to find $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}$-generic $G * x$ such that $(q,(t,\dot{B}))$ is in $G * x$ and $(p,(s,\dot{A}))$ is in $G|\alpha * x$. Note that $(p,(s,\dot{A}))$ and $(q,(t,\dot{B}))$ are compatible in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_{\lambda}$ as witnessed by $(p \cup q,(s,\dot{A}))$. It suffices to choose $G * x$ such that $(p \cup q,(s,\dot{A}))$ is in $G * x$, and the rest of the argument follows.

(ii). By the Prikry property, there is a direct extension of $(p, (s, \dot{A}))$ which decides the statement "$(q, (t, \dot{B}))$ is in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$". The negative decision contradicts (i) (when applied to the direct extension); it follows that the decision must be positive. □

Lemma 6.22. *Assume $(p, (s, \dot{A}))$ is a condition in $\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$, and $\dot{r}_i$ for $i < 2$, are conditions forced by the weakest condition of $\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$ into the quotient $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$. Then there are $(p', (s', \dot{A}')) \leq (p, (s, \dot{A}))$, $(q_i, (t_i, \dot{B}_i))$ and $\bar{q}_i \leq q_i$, $i < 2$, such that for $i < 2$:*

(i) $(p,' (s', \dot{A}'))$ decides $\dot{r}_i$ to be $(q_i, (t_i, \dot{B}_i))$,

(ii) $(p', (s', \dot{A}'))$ and $(\bar{q}_i, (t_i, \dot{B}_i))$ satisfy the assumptions (i)–(iii) of Lemma 6.21.

Proof. Let $(p', (s', \dot{A}')) \leq (p, (s, \dot{A}))$ be such that it decides the value of $\dot{r}_i$ to be $(q_i, (t_i, \dot{B}_i))$ for $i < 2$. We may assume that s' extends t_i and $p' \leq q_i \restriction \alpha$ for $i < 2$. Since $(p', (s', \dot{A}'))$ forces that $\dot{r}_0$ is in $\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$, the condition (iii) in Lemma 6.20 has to fail, hence there is $\bar{q}_0 \leq p' \cup q_0$ such that $\bar{q}_0$ forces $s' \setminus t_0 \subseteq \dot{B}_0$. Now, if it is necessary we can extend p' to ensure $p' \leq \bar{q}_0 \restriction \alpha$.

Now, we need to deal with $\dot{r}_1 = (q_1, (t_1, \dot{B}_1))$. Since $(p', (s', \dot{A}'))$ forces that $\dot{r}_1$ is in $\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$, the condition (iii) in Lemma 6.20 has to fail. Therefore there is $\bar{q}_1 \leq p' \cup q_1$ such that $\bar{q}_1$ forces $s' \setminus t_1 \subseteq \dot{B}_1$. Again, if it is necessary we can extend p' so that $p' \leq \bar{q}_1 \restriction \alpha$. □

Lemma 6.23. *$(\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda)^2 \times (\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha)$ is κ^+-cc.*

Proof. Obvious. □

Finally we can prove the desired lemma which finishes the proof of Theorem 6.1.

Lemma 6.24. *For every $\alpha \in B^*$, the square of $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$ is κ^+-cc in $V[\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha]$.*

Proof. For contradiction assume that $\{(\dot{r}^0_\beta, \dot{r}^1_\beta) \mid \beta < \kappa^+\}$ is a $\mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$-name for an antichain in $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\iota^*}_\alpha$. By Lemma 6.22, we can find for each $\beta < \kappa^+$ and $i < 2$ conditions $(p_\beta, (s_\beta, \dot{A}_\beta))$, $(q^i_\beta (t^i_\beta, \dot{B}^i_\beta))$ and extensions $\bar{q}^i_\beta \leq q^i_\beta$ which satisfy items (i) and (ii) in Lemma 6.22.

By Lemma 6.23, there are $\beta < \beta' < \kappa^+$ such that p_β is compatible with $p_{\beta'}$ and $\bar{q}^i_\beta$ is compatible with $\bar{q}^i_{\beta'}$ for $i < 2$. This means that $p_\beta \cup p_{\beta'}$ and $\bar{q}^i_\beta \cup \bar{q}^i_{\beta'}$ for $i < 2$ are conditions in $\mathbb{P}|\lambda$. Moreover we may assume that $t^i = t^i_\beta = t^i_{\beta'}$ and $s = s_\beta = s_{\beta'}$ for $i < 2$.

For $i < 2$, the conditions $(p_\beta \cup p_{\beta'}, (s, \dot{A}_\beta \cap \dot{A}_{\beta'}))$ and $(\bar{q}^i_\beta \cup \bar{q}^i_{\beta'}, (t^i, \dot{B}^i_\beta \cap \dot{B}^i_{\beta'}))$ satisfy the assumptions of Lemma 6.21. Therefore, there is an direct extension of $(p_\beta \cup p_{\beta'}, (s, \dot{A}_\beta \cap \dot{A}_{\beta'}))$ (applying Lemma 6.21 twice) which forces the compatibility of $(\dot{r}^0_\beta, \dot{r}^1_\beta)$ and $(\dot{r}^0_{\beta'}, \dot{r}^1_{\beta'})$ into the quotient. This is a contradiction. □

This finishes the proof of Theorem 6.1.

6.2 An arbitrary gap

Theorem 6.25. *Assume* GCH *and let κ be a Laver-indestructible supercompact cardinal, λ a weakly compact cardinal and μ a cardinal of cofinality greater than κ such that $\kappa < \lambda < \mu$. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ *preserves cardinals* $\leq \kappa^+$ *and* $\geq \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \ \& \ 2^\kappa = \mu \ \& \ \mathrm{cf}(\kappa) = \omega \ \& \ \ \kappa$ *is strong limit*$)$.
(iii) $V[\mathbb{R}] \models \mathsf{TP}(\lambda)$.

We will not give a detailed proof, but instead specify what modifications to the proof of Theorem 6.1 are needed to prove Theorem 6.25. Assume the notation is the same as in the proof of Theorem 6.1 unless said otherwise.

Modify the construction in Stage 1 in Section 6.1.1 as follows:

(1) In analogy with Lemma 6.3, find a set $A \subseteq [\mu]^\lambda$ which is unbounded in $[\mu]^\lambda$ and closed under unions of increasing chains of cofinality larger than κ which satisfies:

- For every $x \in A$, $\lambda + 1 \subseteq x$.
- For every $x \in A$, there is a name $\dot{U}_x$ such that in $V[\mathbb{P}|x]$, $\dot{U}_x$ interprets as the restriction of the measure $\dot{U}$ on κ. Let us denote by $\mathbb{P}|x * \mathbb{Q}_x$ the Cohen forcing restricted to x followed by Prikry forcing with the measure $\dot{U}_x$.

(2) Choose an arbitrary $x_0 \in A$ and an isomorphism $\iota : \mathbb{P}|x_0 \to \mathbb{P}|\mathrm{Even}(\lambda)$. Thus $\iota(\dot{U}_{x_0})$ is a measure in $\mathbb{P}|\mathrm{Even}(\lambda)$.

(3) Denote $\hat{A} = \{y \in A \,|\, x_0 \subseteq y\}$. As in Lemma 6.6, and with the notation naturally modified for the current situation, there is an unbounded set $B \subseteq \lambda$ closed under limits of cofinality larger than κ, and commutative projections

$$\sigma^\mu_y : \mathbb{P} * \mathbb{Q} \to \mathrm{RO}^+(\mathbb{P}|y * \mathbb{Q}_y), \text{ for } y \in \hat{A},$$

$$\hat{\sigma}^y_\alpha : \mathrm{RO}^+(\mathbb{P}|y * \mathbb{Q}_y) \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha), \text{ for } y \in \hat{A}, \alpha \in B,$$

and

$$\sigma^\mu_\alpha : \mathbb{P} * \mathbb{Q} \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\iota_\alpha), \text{ for } \alpha \in B$$

with

$$\sigma^\mu_\alpha = \hat{\sigma}^y_\alpha \circ \sigma^\mu_y, \text{ for } y \in \hat{A}, \alpha \in B.$$

Note that we denote by $\mathbb{Q}^\iota_\alpha$ Prikry forcing defined with respect to the restriction of the measure $\iota(\dot{U}_{x_0})$ to $V[\mathbb{P}|\mathrm{Even}(\alpha)]$.

(4) Modify Definition 6.7 of $\mathbb{R}$ to use ι and σ^μ_α, in the sense of the previous paragraph. As in Definition 6.10, define the truncations $\mathbb{R}|y$ for $y \in \hat{A}$.

(5) The key step is to show that if $\dot{T}$ is a λ-Aronszajn tree added by $\mathbb{R}$, then for some $y \in \hat{A}$, $\mathbb{R}|y$ adds an Aronszajn tree on λ and importantly, $\mathbb{R}|y$ is isomorphic to $\mathbb{R}^*$ (which is the same forcing as in Definition 6.14). We argue as follows:

By the λ-Knaster property of (a dense subset of) $\mathbb{R}$, there is $y \in \hat{A}$ such that the support of $\dot{T}$ (see the paragraph after Lemma 6.11) is included in y. Choose a bijection ι^* extending ι, $\iota^* : \mathbb{P}|y \to \mathbb{P}|\lambda$. Denote $\mathbb{Q}^{\iota^*}_\lambda$ Prikry forcing in $V[\mathbb{P}|\lambda]$ defined with respect to the measure $\iota^*(\dot{U}_y)$. As in Lemma 6.13, ι^* extends to an isomorphism between $\mathbb{P}|y * \mathbb{Q}_y$ and $\mathbb{P}|\lambda * \mathbb{Q}^{\iota^*}_\lambda$. Finally, as in Lemma 6.13, $\mathbb{R}|y$ is isomorphic to $\mathbb{R}^*$.

Stage 2 of the argument is exactly the same as in the proof of Theorem 6.1

Remark 6.26. Notice that the cofinality of 2^κ is greater than κ in the final model in Theorem 6.25. It is of interest to ask whether the cofinality of 2^κ can be equal to γ, for some $\omega < \gamma \leq \kappa$, with the tree property at κ^{++} (since κ has countable cofinality it may in principle happen). Our method of proof cannot be easily modified to reach this configuration because κ was regular when we enlarged the size of 2^κ, and therefore the cofinality of 2^κ at that moment had to be greater than κ.

7 The double successor of $\aleph_\omega$

In this section we strengthen the result of the previous section and collapse κ to $\aleph_\omega$, i.e. we show that the tree property at $\aleph_{\omega+2}$, with $\aleph_\omega$ strong limit, is consistent with $2^{\aleph_\omega}$ equal to $\aleph_{\omega+2+n}$ for an arbitrary natural number n. The results in this section are joint with Sy-David Friedman and Radek Honzik, and were submitted as [28].

The exposition is structured as follows.

In Section 7.1 we review the forcings which we will use: in particular, we define a variant of Mitchell forcing which ensures the large value of $2^{\aleph_\omega}$ (see Section 7.1.1), and provide a product analysis of Mitchell forcing followed by Prikry forcing with collapses which is reminiscent of the analysis in [1] and [10] (Section 7.1.2).

In Section 7.2, we argue that it is possible to start with a strong cardinal κ of a suitable degree and prepare the ground model so that a further forcing with the Cohen product at κ (of a prescribed length) does not destroy the measurability of κ.[27]

In Section 7.3, we show that over the prepared ground model, the standard Mitchell forcing followed by Prikry forcing with collapses forces that $\kappa = \aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$.

In Section 7.4 we generalise the construction in Section 7.3 to any finite gap $2 \le n < \omega$.

7.1 Forcing notions

7.1.1 A variant of Mitchell forcing

We will use a variant of the standard Mitchell forcing as presented in [1], see Section 3.4.1.

Let $\kappa < \lambda$ be regular cardinals, and assume λ is inaccessible. Let $\mu > \lambda$ be an ordinal. We define a variant of Mitchell forcing, $\mathbb{M}(\kappa, \lambda, \mu)$, as follows: Conditions are pairs (p, q) such that p is in $\mathrm{Add}(\kappa, \mu)$, and q is a function whose domain is a subset of λ of size at most κ such that for every $\xi \in \mathrm{dom}(q)$, $q(\xi)$ is an $\mathrm{Add}(\kappa, \xi)$-name, and $\emptyset \Vdash_{\mathrm{Add}(\kappa,\xi)} q(\xi) \in \dot{\mathrm{Add}}(\kappa^+, 1)$, where $\dot{\mathrm{Add}}(\kappa^+, 1)$ is the canonical $\mathrm{Add}(\kappa, \xi)$-name forCohen forcing at κ^+. The ordering is as in the standard Mitchell forcing, i.e.: $(p', q') \le (p, q)$ if and only if p' is stronger than p in Cohen forcing, the domain of q' contains the domain of q and if ξ is in the domain of q, then p' restricted to ξ forces $q'(\xi)$ extends $q(\xi)$.

Lemma 7.1. *Assume* GCH.

(i) $\mathbb{M}(\kappa, \lambda, \mu)$ *is* λ*-Knaster.*

(ii) In $V[\mathbb{M}(\kappa, \lambda, \mu)]$, $2^\kappa = |\mu|$, *and the cardinals in the open interval* (κ^+, λ) *are collapsed (and no other cardinals are collapsed).*

Proof. The proof is standard (using a Δ-system argument for Knasterness). □

The following follows as in [1]:

Lemma 7.2. *(i)* $\mathbb{M}(\kappa, \lambda, \mu)$ *is a projection of* $\mathrm{Add}(\kappa, \mu) \times \mathbb{T}$, *where* $\mathbb{T}$ *is a* κ^+*-closed term forcing defined by* $\mathbb{T} = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}(\kappa, \lambda, \mu)\}$.

[27] With more work, we can also preserve the initial degree of strongness; see Remark 7.7.

(ii) $\mathbb{M}(\kappa, \lambda, \mu)$ *is equivalent to* $\mathrm{Add}(\kappa, \mu) * \dot{\mathbb{Q}}_{\mathbb{M}}$*, where* $\dot{\mathbb{Q}}_{\mathbb{M}}$ *is forced to be* κ^+*-distributive.*

PROOF. The proof is as in [1]. □

As will be apparent from the arguments in Section 7.3, it is also the case that if λ is weakly compact, then the tree property holds at $\lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa, \lambda, \mu)]$.

7.1.2 Mitchell followed by Prikry forcing with collapses

Assume $\kappa < \lambda < \mu$ are as above, $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \mu)$ is Mitchell forcing, and $\dot{U}$ and $\dot{G}^g$ are $\mathbb{M}$-names such that the weakest condition in $\mathbb{M}$ forces that $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ is Prikry forcing with collapses defined with respect to $\dot{U}$ and $\dot{G}^g$ (see Section 3.4.3 for more details about Prikry forcing with collapses).

Lemma 7.3. $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ *is* λ*-Knaster.*

PROOF. This follows by a Δ-system argument applied to $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ (the compatibility of the Prikry component is determined by the compatibility of the stems, and there are only κ-many of these). □

Lemma 7.4. *In* $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$*, conditions* $((p, q), r)$*, where* r *in Prikry forcing depends only on the Cohen information of Mitchell forcing and its stem is a checked name, are dense.*

PROOF. This is because all conditions in Prikry forcing exist already in the extension by the Cohen part of Mitchell forcing (in contrast, the definition of $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ itself may require the whole $\mathbb{M}$ in order to refer to $\dot{U}$ and $\dot{G}^g$; this will be the case in our argument in Section 7.3.3). Given $((p, q), r)$ we can extend (p, q) to some (p', q') such that (p', q') forces that r is equal to some r' in the generic extension by the Cohen part of Mitchell forcing with its stem being a ground model object (since Cohen forcing at κ does not add bounded subsets of V_κ). □

Using Lemma 7.4, we can formulate a projection-of-product analysis of the forcing $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ reminiscent of Abraham's analysis of Mitchell forcing in [1] (see Section 3.4.1 for more details). Let us define:

$$\mathbb{C} = \{((p, \emptyset), r) \mid ((p, \emptyset), r) \in \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)\}, \tag{7.1}$$

where we require that r depends only on the Cohen information of Mitchell forcing and its stem is a checked name.[28] Let us also define:

$$\mathbb{T} = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}\}. \tag{7.2}$$

Define a function π from $\mathbb{C} \times \mathbb{T}$ to $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ as follows: π applied to the pair composed of $((p, \emptyset), r)$ and $(\emptyset, q)$ is equal to the condition $((p, q), r)$.

Lemma 7.5. *The following hold:*

(i) π *is a projection from* $\mathbb{C} \times \mathbb{T}$ *onto a dense part of* $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$.

[28] The requirement that the stem of r is a checked name is not important for Lemma 7.5, but will be useful in Section 7.3.3 when a similar analysis is performed.

(ii) $\mathbb{T}$ *is* κ^+*-closed in* V*.*
(iii) $\mathbb{C}$ *is* κ^+*-cc in* V*.*

PROOF. (i). If $((p',\emptyset),r') \leq ((p,\emptyset),r)$ and $(\emptyset,q') \leq (\emptyset,q)$, then $((p,q'),r') \leq ((p,q),r)$, hence π is order-preserving.

Now, suppose we are given $((p',q'),r') \leq ((p,q),r)$, where

$$((p,q),r) = \pi(((p,\emptyset),r),(\emptyset,q)) \tag{7.3}$$

and r' depends only on the Cohen information of Mitchell forcing and its stem is a checked name (by Lemma 7.4 such conditions are dense). We will find q^* and r^* such that

(a) $(\emptyset,q^*) \leq (\emptyset,q)$ in $\mathbb{T}$,
(b) $((p',\emptyset),r^*) \leq ((p,\emptyset),r)$ in $\mathbb{C}$,
(c) $\pi(((p',\emptyset),r^*),(\emptyset,q^*)) = ((p',q^*),r^*) \leq ((p',q'),r')$.

In order to get (a)–(b), first define q^* so that it interprets as q' below p', and as q below conditions incompatible with p' (ensuring (a)). Since we assume that r and r' depend only on Cohen forcing (and have checked names for their stems), we can take $r^* = r'$ (ensuring (b)). (c) is clear by the definition of q^* and r^*.

Items (ii) and (iii) are obvious. □

The existence of the projection π in Lemma 7.5 will be useful (in a quotient setting) in Section 7.3.3.

7.2 Preserving measurability by Mitchell forcing

In [10], the construction which yields the tree property at the double successor of a singular strong limit κ with countable cofinality starts by assuming that κ is supercompact. The reason is that we can then invoke Laver's indestructibility result [51], and assume that adding any number of Cohen subsets of κ will preserve the measurability of κ. Such an assumption tends to simplify the subsequent constructions because one can avoid the work of lifting a weaker embedding using a surgery argument, or some other methods.

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. In this section, we use an idea of Cummings and Woodin (see [7]) to argue that it is possible to have a limited indestructibility for μ-tall cardinals κ, where $\mu > \kappa$ is a regular cardinal.[29]

7.2.1 Stage 1

Assume GCH and suppose that $\mu > \kappa$ is the successor[30] of the least weakly compact cardinal λ above κ and $j : V \to M$ is an $H(\mu)$-strong embedding with the extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \ \& \ \alpha < \mu\}. \tag{7.4}$$

[29] With more work, one can also preserve the strongness of κ; see [40].
[30] Or more generally the n-th successor for some finite $n > 1$.

In particular, $H(\mu)$ is included in M and M is closed under κ-sequences in V. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have width κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has width μ_κ, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain μ_κ, where μ_κ is the successor of the least weakly compact cardinal above κ in N, in particular $(\kappa^{++})^N < \mu_\kappa < i(\kappa) < \kappa^{++}$. See [9] for more details regarding the lifting of embeddings and the notion of width.

Let P denote the forcing $\mathrm{Add}(\kappa, \mu)$ in V, $Q = i(P)$, and let g be a Q-generic filter over V. Then the following hold:

Theorem 7.6. *Assume* GCH. *Forcing with Q preserves cofinalities and the following hold in $V[g]$:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.
(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.
(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.
(iv) g is Q-generic over $N[i^1(g)]$.
(v) There is $\tilde{g}$ in $V[g]$ such that $\tilde{g}$ is $k(Q) = j(P)$-generic over $M[j^1(g)]$.

Proof. We show that Q is κ^+-closed and κ^{++}-cc in V. Closure is obvious by the fact that N is closed under κ-sequences in V. Regarding the chain condition, notice that every element of Q can be identified with the equivalence class of some function $f : \kappa \to \mathrm{Add}(\kappa, \mu)$. For $f, f' : \kappa \to \mathrm{Add}(\kappa, \mu)$, set $f \le f'$ if for all $i < \kappa$, $f(i) \le f'(i)$; it suffices to check that the ordering $\le$ on these f's is κ^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size κ^+ and is closed under κ-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most κ^+.

(i) and (ii). These follow by κ^+-distributivity of Q in V and the fact that j, i have width κ: the pointwise image of g generates a generic for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $\mu_\kappa < i(\kappa^+)$, we use the distributivity of $i(Q)$ and the fact that k has width μ_κ to argue that the pointwise image k"$(i^1(g))$ generates a generic filter which is equal to the generic filter generated by j"g by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. Therefore g and $i^1(g)$ are mutually generic over N by Easton's lemma 3.32.

(v). Q is $i(\kappa)$-closed in $N[i^1(g)]$ since the generic $i^1(g)$ does not add new sequences of length $i(\kappa)$; it follows as in (iii) that k^1"g generates a $j(P)$-generic filter $\tilde{g}$ over $M[j^1(g)]$. □

Remark 7.7. Notice that g is not present in M^1. However, if so desired,[31] we can ensure that κ is still $H(\mu)$-strong after the generic object $\tilde{g}$ is added; see [40] for more details.

[31] This is not required for the present proof, but may be useful if more complicated forcings are to be defined over V^1 (such as the Radin forcing).

7.2.2 Stage 2

Let us work in the model $V[g] = V^1$ and let us use the notation j^1, V^1, M^1 to denote the resulting models and embeddings in Theorem 7.6. Recall that by Remark 7.7, j^1 is just μ-tall (but the initial $H(\mu)$-strongness of j still implies that the cardinals in the interval $[\kappa, \mu]$ coincide between V^1 and M^1). Note that λ is no longer strong limit in V^1, but we will argue in Section 7.3.1 that it retains enough of weak compactness in V^1 for further arguments.

Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is measurable} \rangle, \tag{7.5}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \mu_\alpha)$, where λ_α is the least weakly compact cardinal above α, and $\mu_\alpha = (\lambda_\alpha)^+$.

Theorem 7.8. *The following hold:*

(i) In $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \mu)]$, $\lambda = \kappa^{++}$, $2^\kappa = \kappa^{+3} = \mu$, and κ is measurable.
(ii) The measurability of κ is witnessed by a lifting of j^1, which we call j^2,

$$j^2 : V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \mu)] \to M^2 = M^1[j^2(P_\kappa * \mathbb{M}(\kappa, \lambda, \mu))].$$

Moreover, j^2 is the normal measure embedding derived from j^2, and M^2 satisfies $\lambda = \kappa^{++}$ and $2^\kappa = \kappa^{+3} = \mu$.

Proof. Let $G_\kappa * H$ be $P_\kappa * \mathbb{M}(\kappa, \lambda, \mu)$-generic over V^1.

(i). We follow closely the argument in Cummings [7] but with the important simplification that we use the factoring through k only in stage 1 (Theorem 7.6), and use directly the generic object $\tilde{g}$ (Theorem 7.6) to lift only the embedding j^1 (we do not lift k^1 and i^1).[32]

Using standard methods, lift j^1 to

$$j^2 : V^1[G_\kappa] \to M^1[G_\kappa][H][h], \tag{7.6}$$

where h is constructed using the extender representation of M^1: the dense open sets in the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ can be grouped into κ^+-many groups each of size μ in $M^1[G_\kappa][H]$; these groups are of the form $\{j^1(f)(\alpha) \,|\, \alpha < \mu\}$, where f is a function from κ to $H(\kappa)$. The intersection of each group is a dense set because the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ is μ^+-closed in $M^1[G_\kappa][H]$. Since there are only κ^+-many of these groups, a generic h can be constructed in $V^1[G_\kappa][H]$ which meets them all.

It remains to find a generic filter for the j^2-image of $\mathbb{M}(\kappa, \lambda, \mu)$. Using the fact that Mitchell forcing decomposes into $\mathrm{Add}(\kappa, \mu) * \dot{\mathbb{Q}}_{\mathbb{M}}$ for some $\dot{\mathbb{Q}}_{\mathbb{M}}$ which is forced to be κ^+-distributive by $\mathrm{Add}(\kappa, \mu)$ (see Section 3.4.1), it suffices first to lift $\mathrm{Add}(\kappa, \mu)$, and then (easily) lift the distributive part $\dot{\mathbb{Q}}_{\mathbb{M}}$. Let us write $H = g_\kappa * h_\kappa$ where g_κ is Cohen generic and h_κ is $\dot{\mathbb{Q}}_{\mathbb{M}}$-generic.

In order to lift $\mathrm{Add}(\kappa, \mu)$, we use the generic object $\tilde{g}$ which we prepared in V^1. Notice that $\tilde{g}$ is generic for the wrong forcing: it is $j^1(\mathrm{Add}(\kappa, \mu))$-generic over M^1, but we need a

[32]Lifting through k^1 is problematic at stage κ where we deal with the forcing $\mathbb{M}(\kappa, \lambda, \mu)$ in the sense of the ultrapower (the forcing is non-trivially moved by k^1 – a fact innocuous for Cohen forcing at κ, but problematic for Mitchell forcing).

generic object for $j^2(\mathrm{Add}(\kappa,\mu))$ over $M^1[G_\kappa][H][h]$. We use the following fact to overcome this problem.[33]

Fact 7.9. *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa}=\kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa,\mu)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\mu)$.*

By elementarity, Fact 7.9 implies that in $V^1[G_\kappa][H]$, $\tilde{g}$ yields a generic object g^* over $M^1[G_\kappa][H][h]$ for $j^2(\mathrm{Add}(\kappa,\mu))$ (note that $j^1(P_\kappa)$ has size $j^1(\kappa)$ in M^1 and is $j^1(\kappa)$-cc). g^* is still not good enough to lift j^2 because it may not contain the pointwise image j^2"g_κ. Using the method of surgery (see [7]), we modify g^* to g^{**} which is still $j^2(\mathrm{Add}(\kappa,\mu))$-generic, but in addition contains the pointwise image j^2"g_κ. It follows we can lift to

$$j^2 : V^1[G_\kappa][g_\kappa] \to M^1[G_\kappa][H][h][g^{**}], \tag{7.7}$$

and then finally to $V^1[G_\kappa][g_\kappa][h_\kappa] = V^1[G_\kappa][H]$:

$$j^2 : V^1[G_\kappa][H] \to M^2 = M^1[G_\kappa][H][h][g^{**}][h^*], \tag{7.8}$$

where h^* is generated from j^2"h_κ. The last lifting shows that κ remains measurable as desired.

(ii). It remains to show that j^2 is a measure ultrapower embedding. Let N^* be the normal measure ultrapower via the measure U generated from j^2 with the associated embedding $i_U : V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)] \to N^*$, and let $j^2 = k^* \circ i_U$ be the commutative triangle with $k^* : N^* \to M^2$. First note that k^* is the identity on μ since its critical point must be a regular cardinal in N^* and N^* computes κ^{+3} $(=\mu)$ correctly. Then the claim follows since k^* must be onto (and therefore the identity) using the extender representation of M^2 and elementarity: any element of M^2 is of the form $j^2(f)(\alpha)$ for some $\alpha<\mu$, and if k^* is the identity on α, then $j^2(f)(\alpha) = k^*(i_U(f))(\alpha) = k^*(i_U(f))(k^*(\alpha)) = k^*(i_U(f)(\alpha))$, and thus $j^2(f)(\alpha)$ is in the range of k^*. ☐

Remark 7.10. It can also be shown that the tree property holds at $\kappa^{++}=\lambda$ in the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)]$. This is implicit in the proof of Theorem 7.13.

7.3 The tree property with gap 3

In this section we will prove that it is consistent to have a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$. It is relatively straightforward to generalise this construction to get a finite gap: $2^{\aleph_\omega} = \aleph_{\omega+n}$, $3 \le n < \omega$ (see Section 7.4).

Let us work with the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\lambda^+)]$. As we showed in Theorem 7.8, κ is measurable in here. In order to analyse this model, let us introduce notation for the generic filters: let $G_\kappa * H$ be a generic filter over V^1 for $P_\kappa * \mathbb{M}(\kappa,\lambda,\lambda^+)$. As we showed in Theorem 7.8, the lifted extender embedding j^2 in Theorem 7.8 becomes a measure ultrapower embedding i_U in $V^1[G_\kappa * H]$, generated by the normal measure U derived from j^2. Let us rename j^2 to j for simplicity.

[33] This appears as Fact 2 in [7]. Recall that Q_μ – mentioned in Fact 7.9 – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\mu)$ of $V[S]$. The ordering is $\tau \le \sigma \leftrightarrow 1_S \Vdash \tau \le \sigma$.

In particular, we can define Prikry forcing with collapses $\mathrm{PrkCol}(U, G^g)$ using this U and a suitable guiding generic G^g which we construct in Lemma 7.11.[34]

Let Coll denote the forcing $\mathrm{Coll}((\kappa^{+4}), < j(\kappa))^{M^1[j(G_\kappa * H)]}$.

Lemma 7.11. *In $V^1[G_\kappa * H]$, there exists an $M^1[j(G_\kappa * H)]$-generic filter for* Coll.

Proof. Consider the extender representation $j^1 : V^1 \to M^1$ ensured by the arguments in Section 7.2.1, where

$$M^1 = \{j^1(f)(\alpha) \mid f \in V^1 \ \&\ f : \kappa \to V^1 \ \&\ \alpha < \lambda^+\}. \tag{7.9}$$

Now notice that every maximal antichain of Coll in $M^1[j(G_\kappa * H)]$ has a name of the form $j^1(f)(\alpha)$ for some $f : \kappa \to H(\kappa)^{V^1}$ and $\alpha < \lambda^+$, with the range of f being composed of P_κ-names. There are only κ^+-many such f's, and since Coll is κ^{+4}-closed in $M^1[j(G_\kappa * H)]$, we can built a Coll-generic filter G^g in $V^1[G_\kappa * H]$ over $M^1[j(G_\kappa * H)]$ by the standard method of grouping the antichains into κ^+ many blocks each of size at most λ^+, where λ^+ is equal to κ^{+3} in $M^1[j(G_\kappa * H)]$. □

Let us define in V:

$$\mathbb{P} = Q * P_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{7.10}$$

where Q is the forcing from Theorem 7.6, and $\dot{G}^g$ is a name for a guiding generic which we know exists by Lemma 7.11.

Lemma 7.12. *$\mathbb{P}$ is λ-cc.*

Proof. This is a standard argument using Theorem 7.6 for Q and Lemma 7.3 for the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$. □

We plan to show that $V[\mathbb{P}]$ is the desired model, i.e. that the tree property holds with gap 3.

Theorem 7.13. (GCH). *Assume that κ is $H(\lambda^+)$-strong, where $\lambda > \kappa$ is the least weakly compact above κ. Then the forcing $\mathbb{P}$ in (7.10) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

By standard facts about Mitchell forcing and Prikry forcing with collapses (see Sections 3.4.1 and 3.4.3), it suffices to check that we have the tree property at $\aleph_{\omega+2}$.

The argument starts with an observation (see Section 7.3.1) which allows us to work over $V^1[P_\kappa]$ with a fragment of a weakly compact embedding with critical point λ (but still strong enough for our purposes).[35]

The core argument has two parts and starts over the model $V^1[P_\kappa]$. In Part 1 (Section 7.3.2), we show that if there were in $V^1[P_\kappa]$ an $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-name $\dot{T}$ for an $\aleph_{\omega+2}$-Aronszajn tree, we could find a suitable β, $\lambda < \beta < \lambda^+$, and define a "truncation" $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ of the original forcing which forces that there is an $\aleph_{\omega+2}$-Aronszajn tree (witnessed by $\dot{T}$). Then in Part 2 (Section 7.3.3), we show that in fact this cannot be the case, i.e. we show that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces the tree property at $\aleph_{\omega+2}$. This will yield the final contradiction, finishing the proof of Theorem 7.13.

[34]See Section 3.4.3 for more details about this forcing.

[35]Note that Q destroys the strong limitness of λ by adding many subsets of κ^+.

7.3.1 The fragment of weak compactness of λ in $V^1[P_\kappa]$

Suppose for contradiction that $\mathbb{P}$ forces that there is an $\aleph_{\omega+2}$-Aronszajn tree (assume for simplicity the weakest condition forces this, otherwise work below a suitable condition); let $\dot{W}$ be a $Q * P_\kappa$-name for an $\mathbb{M}(\kappa, \lambda, \lambda^+) * \text{PrkCol}(\dot{U}, \dot{G}^g)$-name $\dot{T}$ such that over $V^1[P_\kappa]$, $\mathbb{M}(\kappa, \lambda, \lambda^+) * \text{PrkCol}(\dot{U}, \dot{G}^g)$ forces that $\dot{T}$ is an $\aleph_{\omega+2}$-Aronszajn tree.

By Lemma 7.12, we can assume that $\dot{W}$ can be expressed as a nice name for a subset of λ, and that $\dot{T}$ itself is a nice name for a subset of λ in $V^1[P_\kappa]$.

Let β^* be an ordinal between λ and λ^+ (the choice of β^* is described in Section 7.3.3) large enough so that $\dot{W}$ only uses coordinates below β^* in the sense that we can fix a weakly compact embedding k with critical point λ,

$$k : \mathscr{M} \to \mathscr{N} \tag{7.11}$$

with the following properties:[36]

(i) $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under $<\lambda$-sequences,
(ii) $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta^* < k(\lambda)$, and
(iii) $\mathscr{M}$ contains all relevant information (in particular, β^*, $\mathbb{P}$ and $\dot{W}$ are elements of $\mathscr{M}$).

Let $g * G_\kappa$ be $Q * P_\kappa$-generic over V and let us consider Q restricted to β^* (let us denote it $Q(\beta^*)$)[37]; note that $Q(\beta^*)$ is an element of $\mathscr{M}$. Let $g(\beta^*)$ be the restriction of g to β^* so that $g(\beta^*) * G_\kappa$ is $Q(\beta^*) * P_\kappa$-generic. Note that $Q(\beta^*) * P_\kappa$ is actually equivalent to $Q(\beta^*) \times P_\kappa$ since $Q(\beta^*)$ does not change V_κ where P_κ lives. By standard arguments, k lifts to $\mathscr{M}[G_\kappa] \to \mathscr{N}[G_\kappa]$ since $k(P_\kappa) = P_\kappa$, and both the models are still closed under $<\lambda$-sequences in $V[G_\kappa]$.

By elementarity, $k(Q(\beta^*))$ is Q restricted to $k(\beta^*)$. Let $b : k(\beta^*) \to k(\beta^*)$ be a bijection which swaps γ and $k(\gamma)$ for every $\lambda \leq \gamma < \beta^*$, and is the identity otherwise. b extends to an automorphism on $k(Q(\beta^*))$ by mapping $p \in k(Q(\beta^*))$ to $b(p)$ where the coordinates in $b(p)$ are swapped by b. Note that $b(p)$ is a valid condition in Q since by the elementarity of k, $k(p) = k"p$ is a condition in $k(Q(\beta^*))$ (and hence in Q) for every p in $Q(\beta^*)$.[38]

Let $g(k(\beta^*))$ be the restriction of g to $k(\beta^*)$. The automorphism b generates from $g(k(\beta^*))$ a generic filter g^* on $k(Q(\beta^*))$ which contains the pointwise image $k"g(\beta^*)$. It follows k lifts to

$$k : \mathscr{M}[G_\kappa][g(\beta^*)] \to \mathscr{N}[G_\kappa][g^*]. \tag{7.12}$$

Since Q is κ^+-distributive over P_κ it holds that both the models are still closed under κ-sequences in $V[g * G_\kappa]$.[39]

Thus for any $\dot{W}$ and β^* as above, we have in $V[g * G_\kappa]$ a fragment of a weakly compact embedding (7.12) such that all the relevant parameters are in $\mathscr{M}$, including the name $\dot{T}$, and the models are closed under κ-sequences in the universe.

[36] See [9] Theorem 16.1. To ensure $\beta^* < k(\lambda)$, define E in the proof of Theorem 16.1 so that it also codes a well-ordering of β^* of type λ: then $\mathscr{N} \models |\beta^*| = \lambda$ and therefore $k(\lambda) > \beta^*$ since by elementarity, $k(\lambda)$ is in $\mathscr{N}$ a limit cardinal greater than λ.

[37] Note that λ^+ is a fixed point of the mapping i so Q is $\text{Add}(i(\kappa), \lambda^+)$ of the measure ultrapower N.

[38] The support of p is some set of size less than $i(\kappa)$ in the measure ultrapower N, but certainly less than λ in V: thus the k-image of the support is just its pointwise image.

[39] They are not closed under κ^+-sequences though.

7.3.2 The proof: Part 1

As we argued in Section 7.3.1, we can work in $V^1[G_\kappa]$ and assume for contradiction that $\dot{T}$ is a name for an $\aleph_{\omega+2}$-Aronszajn tree (we assume that $\dot{T}$ is a nice name for a subset of λ). There is $\bar{\beta}$, $\lambda \le \bar{\beta} < \lambda^+$, such that all the coordinates in the forcing $\mathrm{Add}(\kappa, \lambda^+)$ which appear in $\dot{T}$ are below $\bar{\beta}$ (there are only λ-many of them by the chain condition of the forcing).

We wish to find β, $\bar{\beta} < \beta < \lambda^+$, which allows us to define a suitable truncation of $\mathbb{M}(\kappa,\lambda,\lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ to β which we will analyse in Part 2.

Using standard arguments, construct an elementary submodel $\mathscr{A}$ of $H(\theta)$ for some large enough regular θ so that $\mathscr{A}$ satisfies the following conditions:

(i) $|\mathscr{A}| = \lambda$, and $\mathscr{A}$ is closed under κ-sequences,
(ii) $\bar{\beta} + 1 \subseteq \mathscr{A}$,
(iii) $\mathbb{M}(\kappa,\lambda,\lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, $\dot{U}$, $\dot{G}^g$, $\dot{T}$ are elements of $\mathscr{A}$,
(iv) $\mathscr{A} \cap \lambda^+ = \beta$ for some β of cofinality κ^+, $\bar{\beta} < \beta$,
(v) There is an $\mathbb{M}(\kappa,\lambda,\beta)$-name $\dot{U}_\beta$ which is forced by $\mathbb{M}(\kappa,\lambda,\lambda^+)$ to be a normal measure and a restriction of the measure $\dot{U}$ to $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\beta)]$.

The last item (v) follows as in [10].

Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. The following hold (the proofs are routine):

(i) $c(\lambda^+) = \beta$,
(ii) $c(\mathbb{M}(\kappa,\lambda,\lambda^+)) = \mathbb{M}(\kappa,\lambda,\beta)$,
(iii) $c(\dot{U})$ is forced by $\mathbb{M}(\kappa,\lambda,\beta)$ to be equal to $\dot{U}_\beta$,
(iv) $c(\dot{G}^g)$, which we denote by $\dot{G}^g_\beta$, is forced by $\mathbb{M}(\kappa,\lambda,\beta)$ to be a guiding generic with respect to $\dot{U}_\beta$, and therefore $\mathbb{M}(\kappa,\lambda,\beta)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a Prikry forcing with collapses,
(v) $\mathbb{M}(\kappa,\lambda,\lambda^+)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a regular subforcing of $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$,
(vi) $c(\dot{T}) = \dot{T}$ is forced by $\mathbb{M}(\kappa,\lambda,\beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be an $\aleph_{\omega+2}$-Aronszajn-tree.

By elementarity, $c(\dot{T}) = \dot{T}$ is forced in $\bar{\mathscr{A}}$ by the forcing $\mathbb{M}(\kappa,\lambda,\beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be a λ-Aronszajn tree. This by itself would not be enough to conclude that $\mathbb{M}(\kappa,\lambda,\beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ adds such a tree over the universe $V^1[G_\kappa]$. However, since the collapse $c(\dot{T})$ is equal to $\dot{T}$, and (v) holds, any $\mathbb{M}(\kappa,\lambda,\lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-generic filter $h * x$ yields an $\mathbb{M}(\kappa,\lambda,\beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$-generic filter $h' * x'$ over $V^1[G_\kappa]$ (and therefore over $\bar{\mathscr{A}}$) such that $(\dot{T})^{h*x} = (\dot{T})^{h'*x'}$. It follows that $\mathbb{M}(\kappa,\lambda,\beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces over $V^1[G_\kappa]$ that $\dot{T}$ is a λ-Aronszajn tree. In Part 2 we show this is not possible, and this will be the desired contradiction.

7.3.3 The proof: Part 2

Let $\mathbb{M}$ denote the forcing $\mathbb{M}(\kappa,\lambda,\beta)$, where β is as in Part 1. Let us work in $V^1[G_\kappa]$.

Using the arguments in Section 7.3.1, let us fix

$$k : \mathscr{M} \to \mathscr{N} \tag{7.13}$$

which is the fragment of a weakly compact embedding with critical point λ such that $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under κ-sequences, $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta <$

$k(\lambda)$,[40] and $\mathscr{M}$ contains all relevant information (in particular, β, $\mathbb{M}*\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$, $\dot{U}_\beta$ and $\dot{G}^g_\beta$ are elements of $\mathscr{M}$). Let $\mathbb{M}^*$ denote $k(\mathbb{M}(\kappa,\lambda,\beta))$, which is equal to $\mathbb{M}(\kappa, k(\lambda), k(\beta))$.

Let h^* be $\mathbb{M}^*$-generic over $V^1[G_\kappa]$; use h^* to define h which is $\mathbb{M}$-generic over $V^1[G_\kappa]$ and $k``h \subseteq h^*$. Now lift to

$$k : \mathscr{M}[h] \to \mathscr{N}[h^*]. \tag{7.14}$$

Let us abuse notation a little and write $U = (\dot{U}_\beta)^h$ and $G^g = (\dot{G}^g_\beta)^h$ instead of U_β and G^g_β (to simplify notation).

In $\mathscr{N}[h^*]$, consider $U^* = k(U)$, and $G^{g*} = k(G^g)$, and the forcing $\mathrm{PrkCol}(U^*, G^{g*})$. Note that by elementarity $U \subseteq U^*$ (since $k(X) = X$ for every $X \in U$), and all functions F whose equivalence class is in G^g appear in the forcing $\mathrm{PrkCol}(U^*, G^{g*})$ (since $k(F) = F$ for every $F : \kappa \to V_\kappa^{\mathscr{M}[h]}, F \in \mathscr{M}[h]$), and $k(\mathrm{PrkCol}(U, G^g)) = \mathrm{PrkCol}(U^*, G^{g*})$.[41]

It follows that k is a regular embedding:

$$k : \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \to \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}), \tag{7.15}$$

as by the λ-cc of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, if A is a maximal antichain in $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, then $k(A) = k``A$ is a maximal antichain in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let x^* be $\mathrm{PrkCol}(U^*, G^{g*})$-generic over $V^1[G_\kappa][h^*]$; the pull-back of x^* via k^{-1} is a generic filter x for $\mathrm{PrkCol}(U, G^g)$ such that $k``x \subseteq x^*$. Let us lift k further to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*]. \tag{7.16}$$

By (7.15) and (7.16), we can define in $\mathscr{N}[h^*][x^*]$ a generic filter $h * x$ for the forcing $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \in \mathscr{N}$ using the inverse of k (by our assumptions in (7.13), k is an element of $\mathscr{N}$). By standard arguments for complete Boolean algebras it follows that there is a projection σ,[42]

$$\sigma : \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \to \mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)). \tag{7.17}$$

Notice that if $((p,q),r)$ is a condition in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$, then we can identify $\sigma(p) = \sigma((p,\emptyset), 1_{\mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})})$ with $(k^{-1})``p$, i.e. with

$$p \restriction (\kappa \times \lambda) \cup \{((\gamma,\alpha),i) \mid \gamma < \kappa, \alpha \in [\lambda,\beta), i \in \{0,1\}, ((\gamma, k(\alpha)), i) \in p\}. \tag{7.18}$$

In the analysis of the quotient determined by σ, it will be important to control the names for the conditions in $\mathrm{PrkCol}(U^*, G^{g*})$. Recall that by Lemma 7.4, we can adopt the following convention:

Convention. We now view $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ as consisting of conditions $((p,q),r)$, where r depends only on Cohen information, and its stem is a checked name (such conditions are dense by Lemma 7.4). With this convention in mind, let us denote the quotient determined by σ in (7.17) as $\mathbb{Q}_\sigma$:

$$\mathbb{Q}_\sigma = \{((p,q),r) \in \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \mid \sigma(((p,q),r)) \in h * x\}, \tag{7.19}$$

[40] Choose β^* in Section 7.3.1 high enough to ensure this inequality.

[41] However, note that the equivalence classes of a fixed F with respect to U and U^* may be different objects (after the transitive collapse).

[42] Alternatively, we can use Lemma 3.39 and work with the complete embedding k in (7.15) and use the quotient analysis for complete embeddings as discussed in (3.8).

where we identify $h * x$ with the generic filter for the associated complete Boolean algebra.

The following product analysis reformulates the analysis in Section 7.1.2 in a quotient setting.

Define:

$$\mathbb{C}_\sigma = \{((p, \emptyset), r) \mid ((p, \emptyset), r) \in \mathbb{Q}_\sigma\}, \tag{7.20}$$

The ordering is the one inherited from $\mathbb{Q}_\sigma$.

Define:

$$\mathbb{T}_\sigma = \{(\emptyset, q) \in \mathbb{M}^* \mid (\emptyset, q) \restriction \lambda \in h\}. \tag{7.21}$$

The ordering is the one inherited from $\mathbb{M}^*$.

Define a function π from $\mathbb{C}_\sigma \times \mathbb{T}_\sigma$ to $\mathbb{Q}_\sigma$ as follows: π applied to the pair composed of $((p, \emptyset), r)$ and $(\emptyset, q)$ is equal to the condition $((p, q), r)$. Note that if $((p, \emptyset), r)$ is in $\mathbb{C}_\sigma$ and $(\emptyset, q)$ is in $\mathbb{T}_\sigma$, then $((p, q), r)$ is a condition in the quotient $\mathbb{Q}_\sigma$ since $\sigma((p, q), r)$ is the infimum of $\sigma((p, \emptyset), r)$ and $((\emptyset, q) \restriction \lambda, 1_{\mathrm{PrkCol}(\dot{U}, \dot{G}^g)})$ in $\mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g))$.

Lemma 7.14 is proved exactly as Lemma 7.5(i):

Lemma 7.14. *π is a projection from $\mathbb{C}_\sigma \times \mathbb{T}_\sigma$ onto $\mathbb{Q}_\sigma$.*

The following lemma is obvious:

Lemma 7.15. *$\mathbb{T}_\sigma$ is κ^+-closed in $\mathscr{N}[h]$.*

Finally, we analyse the chain condition of $(\mathbb{C}_\sigma)^2$ in $\mathscr{N}[h][x]$.

Lemma 7.16. *$(\mathbb{C}_\sigma)^2$ is κ^+-cc in $\mathscr{N}[h][x]$.*

Proof. Assume for contradiction that A is an antichain in $(\mathbb{C}_\sigma)^2$ in $\mathscr{N}[h][x]$ of size κ^+. Denote the elements of A by $(1a_i, 2a_i)$ for $i < \kappa^+$. By thinning out the antichain if necessary, we can choose a condition $((p, q), r)$ in $h * x$ which forces that A is an antichain and also forces that stems of all conditions $1a_i$, $i < \kappa^+$, are the same and the stems of all conditions $2a_i$, $i < \kappa^+$, are the same (they may not equal each other, but they are compatible; denote them $1t$, and $2t$). Now choose $((p_i, q_i), r_i)$ in $h * x$ which decide the $1a_i$'s and $2a_i$'s; let us write $1a_i = ((1p_i^*, \emptyset), 1r_i^*)$ and $2a_i = ((2p_i^*, \emptyset), 2r_i^*)$, $i < \kappa^+$.

By further thinning and extending the stems if necessary, we may assume that the stems of $((p, q), r)$ and $((p_i, q_i), r_i)$, $i < \kappa^+$, are all the same; denote this stem s. Note that s extends both $1t$ and $2t$.

Now, we need to handle together $1a_i$ and $2a_i$, for all $i < \kappa^+$, to get mutual compatibility in Claim 7.19 below: Let $((1p_i^{**}, 1q_i^{**}), 1r_i^{**})$ be a lower bound of $((1p_i^*, \emptyset), 1r_i^*)$, $((p_i, q_i), r_i)$, and $((p, q), r)$ with stem s such that $\sigma(1p_i^{**})$[43] is in the Cohen part of the generic $h * x$ (this can be done since such conditions are dense). Analogously, let $((2p_i^{**}, 2q_i^{**}), 2r_i^{**})$ be a lower bound of $((2p_i^*, \emptyset), 2r_i^*)$, $((p_i, q_i), r_i)$, and $((p, q), r)$ with stem s such that $\sigma(2p_i^{**})$ is in the Cohen part of the generic $h * x$. Note that in particular $\sigma(1p_i^{**})$ is compatible with $\sigma(2p_i^{**})$.

[43] See (7.18) for definition.

Using a Δ-system argument, find $i < j$ such that $1p_i^{**}$ is compatible with $1p_j^{**}$ and $2p_i^{**}$ is compatible with $2p_j^{**}$. Let us define:

(7.22) $\quad 1(*)$ is the greatest lower bound (glb) of$((1p_i^{**}, \emptyset), 1r_i^{**})$ and $((1p_j^{**}, \emptyset), 1r_j^{**})$

and

(7.23) $\quad 2(*)$ is the greatest lower bound (glb) of$((2p_i^{**}, \emptyset), 2r_i^{**})$ and $((2p_j^{**}, \emptyset), 2r_j^{**})$.

Note that both $1(*)$ and $2(*)$ have the same stem s.

Denote

$$p' = \sigma(1p_i^{**}) \cup \sigma(1p_j^{**}) \cup \sigma(2p_i^{**}) \cup \sigma(2p_j^{**}). \tag{7.24}$$

Note that p' is correctly defined by the construction of the $1p_i^{**}$'s and $2p_i^{**}$'s. Let $((\bar{p}, \bar{q}), \bar{r})$ denote the glb of the conditions $((p', \emptyset), \emptyset)$, $((p, q), r)$, $((p_i, q_i), r_i)$, $((p_j, q_j), r_j)$. Note that $((\bar{p}, \bar{q}), \bar{r})$ has stem s.

We need the following claims to finish the proof.

Claim 7.17. *Assume $((p, q), r)$ is a condition in $\mathbb{M}^*\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ and $((p^*, \emptyset), r^*)$ is a condition in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ and the following conditions are satisfied:*

- *(i) Stems of r and r^* are checked names,*
- *(ii) $p \leq \sigma(p^*)$,*
- *(iii) The length of the stems of r and r^* is the same and the ordinals on the stems coincide,*
- *(iv) The collapsing information in the stem of r extends the collapsing information in the stem of r^*.*

Then $((p, q), r)$ does not force $((p^*, \emptyset), r^*)$ out of the quotient $\mathbb{C}_\sigma$.

Proof. It suffices to find a generic filter $h' * x'$ for $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ such that $\sigma((p, q), r)$ is in the $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-generic filter obtained by σ from $h' * x'$. Any filter $h' * x'$ containing a lower bound of $((p, q), r)$ and $((p^*, \emptyset), r^*)$ (such a lower bound exists by conditions (i)-(iv)) satisfies this. ☐

Recall that Prikry forcing with collapses satisfies the Prikry condition: any statement in the forcing language is decidable by a direct extension (note that a direct extension does not lengthen the stem, but is allowed to extend the collapsing information).

Claim 7.18. *Let $((p, q), r)$ and $((p^*, \emptyset), r^*)$ are as in Claim 7.17. Then there is a direct extension $((p', q'), r')$ of $((p, q), r)$ which forces $((p^*, \emptyset), r^*)$ into $\mathbb{C}_\sigma$.*

Proof. By the Prikry property, there is a direct extension of $((p, q), r)$ which decides the statement "$((p^*, \emptyset), r^*)$ is in $\mathbb{C}_\sigma$". The negative decision contradicts Claim 7.17 (when applied to the direct extension); it follows that the decision must be positive. ☐

Returning to our proof, we get:

Claim 7.19. *There is a direct extension of $((\bar{p}, \bar{q}), \bar{r})$ which forces $1(*)$ and $2(*)$ into $\mathbb{C}_\sigma$.*

Proof. $((\bar{p}, \bar{q}), \bar{r})$ and $1(*)$ satisfy the conditions in Claim 7.17, and therefore by Claim 7.18, there is a direct extension $a_1 \leq ((\bar{p}, \bar{q}), \bar{r})$ which forces $1(*)$ into $\mathbb{C}_\sigma$. a_1 and $2(*)$ satisfy the conditions in Claim 7.17, and therefore by Claim 7.18 there is a direct extension $a_2 \leq a_1$ as desired. □

This finishes the proof since a_2 forces that $(1(*), 2(*))$ is in $\mathbb{C}_\sigma$ a witness for compatibility of $(1a_i, 2a_i)$ and $(1a_j, 2a_j)$ in the antichain A. As a_2 is below $((p, q), r)$, it also forces that A is an antichain. Contradiction. □

The good chain condition of $(\mathbb{C}_\sigma)^2$ and the closure of $\mathbb{T}_\sigma$ are enough to argue that $\mathbb{C}_\sigma \times \mathbb{T}_\sigma$, and therefore $\mathbb{Q}_\sigma$, do not add branches to λ-trees, finishing the argument in the standard way.

Suppose T is a λ-tree in the model $\mathscr{N}[h][x]$, where $\lambda = \kappa^{++}$. By Lemma 7.16 and Fact 4.5, $\mathbb{C}_\sigma$ does not add new branches to T. As $\mathrm{PrkCol}(U, G^g) * \mathbb{C}_\sigma$ has the κ^+-cc in $\mathscr{N}[h]$, we can apply Fact 4.6 over $\mathscr{N}[h]$ (with Q being $\mathbb{T}_\sigma$), and conclude that $\mathbb{T}_\sigma$ does not add branches to trees in $\mathscr{N}[h][\mathrm{PrkCol}(U, G^g) * \mathbb{C}_\sigma]$, and therefore $\mathbb{C}_\sigma \times \mathbb{T}_\sigma$ does not add new branches to trees in $\mathscr{N}[h][x]$.

This finishes the proof of Theorem 7.13.

7.4 The tree property with a finite gap

We would like to generalise the result of the previous section to a finite gap m, i.e. obtain the tree property at $\aleph_{\omega+2}$ and have $2^{\aleph_\omega} = \aleph_{\omega+m}$ for any $2 < m < \omega$. To this end, we need to do some straightforward modifications to the definitions and lemmas we used to obtain gap 3. To simplify indexing of the forcing notions, we will use the index n, where $m = n + 2$ (thus gap 3 is obtained with $n = 1$).

As in the previous section, let κ be the large cardinal which will be collapsed to $\aleph_\omega$, and λ the least weakly compact cardinal above κ.

We now list the modifications we need to do:

- In Section 7.2.1, we choose $\mu = \lambda^{+n}$ so that the preparation, which we now call Q^n, ensures that κ stays measurable after adding μ-many Cohen subsets of κ. Let us denote the resulting model as V^1.
- The definition of P_κ in (7.5) is to be modified as follows:

  (7.25) $$P^n_\kappa = \langle (P^n_\alpha, \dot{Q}^n_\alpha) \mid \alpha < \kappa \text{ is measurable} \rangle,$$

  where $\dot{Q}^n_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \lambda_\alpha^{+n})$.
- Let $G_\kappa * H$ be a generic filter for $P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n})$, and let $j : V^1[G_\kappa * H] \to M^1(j(G_\kappa * H))$ be the lifting of j as in Theorem 7.8.
- Let Coll^n denote the forcing $\mathrm{Coll}((\kappa^{+3+n}, < j(\kappa)))^{M^1[j(G_\kappa * H)]}$. As in Lemma 7.11, we can fix a guiding generic G^g for Coll^n over $M^1[j(G_\kappa * H)]$.
- The definition of the forcing $\mathbb{P}$ in (7.10) is modified as follows:

  (7.26) $$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g),$$

where $\dot{U}$ is a name for a normal measure and $\dot{G}^g$ is a name for a guiding generic (defined with respect to $\dot{U}$).

Now we get the following generalisation of Theorem 7.13:

Theorem 7.20. (GCH) *Let $1 \le n < \omega$ be fixed and assume that κ is $H(\lambda^{+n})$-strong, where $\lambda > \kappa$ is the least weakly compact cardinal above κ. The forcing $\mathbb{P}^n$ in (7.26) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2+n}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

Proof. The basic strategy of the proof is to reduce the general case to a configuration essentially identical to the argument for the gap 3 (see Remark 7.21).

Recall that the whole forcing in V looks as follows:

$$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{7.27}$$

where Q^n is the preparation $i(\mathrm{Add}(\kappa, \lambda^{+n}))^N = (\mathrm{Add}(i(\kappa), \lambda^{+n})^N$ (i is the normal measure embedding derived from j which witnesses the $H(\lambda^{+n})$-strongness of κ). Let us denote by Q^n_β the natural truncation of Q^n to length $\beta < \lambda^{+n}$. Note that the forcing (7.27) is λ-cc.

Suppose for contradiction that the forcing in (7.27) adds a λ-Aronszajn tree $\dot{T}$ (and assume for simplicity that the weakest condition forces it).

Let $\mathscr{A}$ be an elementary substructure of large enough $H(\theta)^V$ which has size λ^+, is closed under λ-sequences, and contains the name $\dot{T}$ and other relevant data. Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. Then the following hold:

(i) $c(\lambda^{+n})$ is an ordinal between λ^+ and λ^{++}, let us denote this ordinal as β.
(ii) $c(Q^n)$ is isomorphic to Q^n_β.
(iii) The name $c(P^n_\kappa)$ interprets in $V[Q^n_\beta]$ as P^n_κ does in $V[Q^n]$.
(iv) The name $c(\mathbb{M}(\kappa, \lambda, \lambda^{+n}))$ interprets in $V[Q^n_\beta * P^n_\kappa]$ as a forcing equivalent to $\mathbb{M}(\kappa, \lambda, \beta)$ as interpreted in $V[Q^n * P^n_\kappa]$.
(v) The name $c(\dot{U})$ interprets in $V[Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta)]$ as a normal ultrafilter on κ generating some guiding generic $c(\dot{G}^g)$, and therefore $Q^n_\kappa * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ is a Prikry forcing with collapses.
(vi) $c(\dot{T})$ is forced (over $\bar{\mathscr{A}}$) by

$$Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$$

to be a λ-Aronszajn tree.

In contrast to the analogous construction in Section 7.3.2, we cannot claim now that $c(\dot{T})$ is equal to $\dot{T}$. However, since this time the model $\bar{\mathscr{A}}$ has size λ^+ and is closed under λ-sequences, the forcing $Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ (which is λ-cc) adds a λ-Aronszajn tree not only over $\bar{\mathscr{A}}$ (which follows by elementarity), but also over V. The reason is that by λ-closure of $\bar{\mathscr{A}}$, a name for a cofinal branch in $c(\dot{T})$ would appear already in $\bar{\mathscr{A}}$.

Let us work in $V[Q^n_\beta * P^n_\kappa]$ and let f be any bijection between β and λ^+ which is the identity on λ. This bijection extends into an isomorphism between $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ and $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$, where $\dot{U}_{\lambda^+}$ and $\dot{G}^g_{\lambda^+}$ are names obtained naturally from f.

This is a contradiction since we can argue as in Theorem 7.13 that the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$ does not add a λ-Aronszajn tree over $V[Q^n_\beta * P^n_\kappa]$. □

Remark 7.21. Strictly speaking, the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$ in the previous proof is not of the type considered in Theorem 7.13: Instead of P_κ, we now have P^n_κ, and the guiding generic $\dot{G}^g_{\lambda^+}$ is generic for the forcing $\mathrm{Coll}(\kappa^{+3+n}, < j(\kappa))$ of the measure ultrapower generated by $\dot{U}_{\lambda^+}$, and not for $\mathrm{Coll}(\kappa^{+4}, < j(\kappa))$ as in Theorem 7.13 (where j is generated by $\dot{U}_{\lambda^+}$). However, it is easy to check that the argument for the tree property at $\aleph_{\omega+2}$ only uses the chain condition and closure properties of the relevant forcings, and these are not affected by these modifications.

8 Further progress and open questions

By way of conclusion, we discuss topics for future research and mention some open questions.

Let us start by introducing some other principles which are similar to the tree property in the sense that they postulate a variant of compactness at a successor cardinal. We will then formulate open questions and problems in this more general framework.

Let κ be an uncountable cardinal in what follows (unless said otherwise).

We say that κ^+ satisfies *the stationary reflection*, and we write it as $\mathsf{SR}(\kappa^+)$, if every stationary subset of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflects at a point of cofinality κ, i.e. for every stationary $S \subseteq \kappa^+ \cap \mathrm{cof}(<\kappa)$ there is $\gamma < \kappa^+$ of cofinality κ such that $S \cap \gamma$ is stationary in γ. Stationary reflection has been extensively studied in literature, see for instance [3, 12, 45, 61, 11, 13, 14].

If we require that the stationary subsets reflect simultaneously, we get stronger principles introduced in Magidor [53]: We say that κ^+ satisfies *the simultaneous stationary reflection*, and we write it as $\mathsf{SSR}(\kappa^+)$, if every two stationary subsets of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflect at a common point of cofinality κ. An even stronger principle is the following: We say that κ^+ satisfies *the club stationary reflection*, and we write it as $\mathsf{CSR}(\kappa^+)$, if every stationary subset of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflects on a κ-club subset of κ^+ (an unbounded subset of κ^+ closed at limit stages of cofinality κ).

Fact 8.1. *Let κ be an uncountable cardinal.* $\mathsf{CSR}(\kappa^+) \to \mathsf{SSR}(\kappa^+) \to \mathsf{SR}(\kappa^+)$.

Recall the definition of *the approachability ideal* $I[\kappa^+]$. Let $\langle a_\alpha \mid \alpha < \kappa^+ \rangle$ be some sequence of bounded subsets of κ^+. We say that a limit ordinal $\gamma < \kappa^+$ is approachable with respect to the sequence if there is an unbounded subset A of γ of ordertype $\mathrm{cf}(\gamma)$ such that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{a_\beta \mid \beta < \gamma\}$. We define $I[\kappa^+]$ as the collection of all $S \subseteq \kappa^+$ for which there is a sequence $\langle a_\alpha \mid \alpha < \kappa^+ \rangle$ as above and a club subset C of κ such that every $\gamma \in S \cap C$ is approachable with respect to the sequence.

The ideal $I[\kappa^+]$ has proved to be closely connected with many topics in combinatorial set theory, for example PCF theory in Shelah's [62], saturated ideals in Foreman's and Magidor's [20], and the extent of diamond in Rinot's [59] (see also [34] and [36]).

We say that κ^+ has the *approachability property* if $\kappa^+ \in I[\kappa^+]$, and we write it as $\mathsf{AP}(\kappa^+)$. $\mathsf{AP}(\kappa^+)$ is a weak form of the square principle on κ, and therefore we consider $\neg\mathsf{AP}(\kappa^+)$ as a compactness property of κ^+.

We list some fact related to these notions (for more details see [8]).

Fact 8.2. *Let κ be an uncountable cardinal.*

(i) $\square_\kappa \to \neg\mathsf{SR}(\kappa^+)$.
(ii) $\square^*_\kappa \to \mathsf{AP}(\kappa^+)$.
(iii) A Mahlo cardinal suffices to get $\mathsf{SR}(\kappa^+)$ *(and it is necessary). See Harrington and Shelah [39].*
(iv) A weakly compact cardinal suffices to get $\mathsf{SSR}(\kappa^+)$ *and* $\mathsf{CSR}(\kappa^+)$ *(and it is necessary). See Magidor [53].*
(v) A Mahlo cardinal suffices to get $\neg\mathsf{AP}(\kappa^+)$ *(and it is necessary). See Cummings and others [15].*

Let us know mention some open problems and areas of future research.

8.1 The continuum function

Let us first consider direct generalisations of the problems we studied in this thesis.

Q1. Is it possible to show that $\mathsf{TP}(\aleph_n)$ holds for each $1 < n < \omega$ while the continuum function is arbitrary (subject to the condition that GCH must fail below $\aleph_\omega$)?

It seems natural to start with the model constructed in [10] by Cummings and Foreman and use Cohen forcings to control the continuum function.

Q2. Let $\aleph_n$ for $1 < n < \omega$ be fixed. Is it possible show that $\neg\mathsf{AP}(\aleph_n)$ and $\mathsf{SR}(\aleph_n)$ pose no restriction on the continuum function (except for the restriction exerted by $\neg\mathsf{AP}(\aleph_n)$ as we discussed above)?

A challenging extension of this problem adds the requirement to control the value of $2^{\aleph_\omega}$ with $\aleph_\omega$ strong limit (see [35] for more details) while having some compactness principles below $\aleph_\omega$.

Q3. The above two questions can also be studied on $\aleph_{\omega+2}$. In particular, is it possible to show that the compactness principles at $\aleph_{\omega+2}$ are consistent with an arbitrary finite gap at $\aleph_\omega$, i.e. with $2^{\aleph_\omega} = \aleph_{\omega+n}$ for any $2 \leq n < \omega$?

Notice that the results in this thesis show that this is possible for the tree property.

Q4. The previous question can be formulated with an infinite gap, i.e. with $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for some countable α. More specifically, one can ask whether we can get (to start modestly) the tree property at $\aleph_{\omega+2}$ with $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$.

Note that an infinite gap was first shown by Magidor in [52] (for $\alpha = \omega$) and generalised by Shelah in [60]. These methods use supercompact cardinals and collapse cardinals above the large cardinal which gets collapsed to $\aleph_\omega$ so we presume that the proofs would be quite different from those in this thesis.

8.2 Mixing the compactness principles

Let us now mention questions which study the interactions between the various compactness principles.

By the results of [15] by Cummings and others, it is possible to force any Boolean combination of truth and falsity of the principles $\mathsf{TP}(\kappa^+)$, $\mathsf{SR}(\kappa^+)$ and $\mathsf{AP}(\kappa^+)$ for a fixed cardinal κ^+ in the set $\{\aleph_n \,|\, 2 \leq n < \omega\} \cup \{\aleph_{\omega+2}\}$.[44]

With $\aleph_\omega$ strong limit, we can ask the following:

Q5. Is it possible to generalise the results of [15] by Cummings and others to include the variants of the principles $\mathsf{TP}(\kappa^+)$ and $\mathsf{SR}(\kappa^+)$ which we introduced above? As a test case, is it possible to achieve $\mathsf{TP}(\aleph_n) + \mathsf{CSR}(\aleph_n) + \neg\mathsf{AP}(\aleph_n)$ for some $1 < n < \omega$?

[44]In fact, they showed it for any κ^+ such that κ is a successor cardinal; we apply their result here in the context of the cardinals close to $\aleph_\omega$ on which we focus.

Note that the known method to obtain $\mathsf{CSR}(\kappa^+)$ (see Magidor [53]) requires an additional forcing over a model of $\mathsf{SSR}(\kappa^+)$, hence it is not obvious how this combines with the methods to obtain for instance $\mathsf{TP}(\kappa^+)$.

Q6. One might ask whether a Mahlo cardinal is sufficient to obtain certain configurations of the compactness principles at $\aleph_n$ for some $1 < n < \omega$. In particular, is it possible to force $\neg\mathsf{AP}(\aleph_n) + \mathsf{SR}(\aleph_n)$ from a Mahlo cardinal? The latter configuration was achieved in [15] by Cummings and others using a weakly compact cardinal (both with $\neg\mathsf{TP}(\aleph_n)$ and with $\mathsf{TP}(\aleph_n)$).

Note that the fact that $\neg\mathsf{AP}(\aleph_n)$ and $\mathsf{SR}(\aleph_n)$ require by themselves just a Mahlo cardinal does not necessarily imply that their combinations do. However, we conjecture it is the case and that a Mahlo cardinal should be sufficient.

Q7. We may consider compactness principles holding at successive cardinals, or on an interval of regular cardinals.

We reviewed the existing results for the tree property and the weak tree property in Section 4.1. Stationary reflection on multiple cardinals was studied in papers by Jech and Shelah [45] and Shelah [61]. Recently, Unger [77] considers successive failures of AP ($\aleph_\omega$ is not strong limit in his model). It is natural to study this question for other compactness principles and their combinations. In particular, is it consistent to combine the tree property on cardinals below $\aleph_\omega$ with $\mathsf{SR}(\aleph_n)$, $\mathsf{SSR}(\aleph_n)$ and $\mathsf{CSR}(\aleph_n)$ for $1 < n < \omega$, and if so, under which large cardinal assumptions?

This question can be extended to the context where SCH fails at $\aleph_\omega$, a first step to obtaining compactness principles at $\aleph_{\omega+2}$ (which imply the failure of SCH). A paper by Unger [75] shows that this is possible for the tree property; it is worth asking this question for SR and $\neg\mathsf{AP}$.

With $\aleph_{\omega_1}$ strong limit, we may analogously ask:

Q8. Is it possible to force compactness principles at $\aleph_{\omega_1+2}$?

We think it is possible, using a suitable version of the Radin forcing.

8.3 Generalised cardinal invariants

The cardinal invariants of the continuum provide as a finer classification of the properties relevant for the real numbers (identified with $2^{\aleph_0}$). The invariants are an interesting topic of study if CH fails (if CH holds they are typically equal to $2^{\aleph_0}$). Since both wTP and $\neg\mathsf{AP}$ imply the negation of CH it is natural to ask what cardinal invariants patterns can be realised in models where they hold at $\aleph_2$.

There are more forcings available to force $\mathsf{TP}(\aleph_2)$ in addition to Mitchell forcing: for instance Sacks forcing (see [47] or [26]) and its variants (see [43] or [70]) and forcings with side conditions such as [57]. It is also known by the result of Friedman and Torres [29] that MA can hold with $\mathsf{TP}(\aleph_2)$, starting just from a weakly compact cardinal.[45]

[45]In the model constructed in [29] the continuum has size $\aleph_2$. It is open whether the size of the continuum can be larger with MA and $\mathsf{TP}(\aleph_2)$ (note in this context that PFA implies $\mathsf{MA} + \mathsf{TP}(\aleph_2)$ but also $2^{\aleph_0} = \aleph_2$).

Q9. What is the structure of the cardinal invariants in the models with compactness principles at $\aleph_2$, where $2^{\aleph_0} = \aleph_2$?

Cardinal invariants generalise to larger cardinals (see [16], [63] or [5]).

Q10. What is the structure of the generalised cardinal invariants in the models with compactness principles at $\aleph_n$, for $1 < n < \omega$?

8.4 Definability

Finally let us consider the question of definability. It is known that SCH can fail definable at $\aleph_\omega$ in the sense that there is a lightface definable wellorder of the subsets of $\aleph_\omega$ in $H(\aleph_{\omega+1})$ with $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$ (see [25]).

It is natural to ask whether the definability can be combined with the tree property at $\aleph_{\omega+2}$.

Q11. Is it possible for SCH to fail definably at $\aleph_\omega$ (in the above sense) with the tree property at $\aleph_{\omega+2}$?

Notice that in this context Mitchell forcing will probably not work as the coding of the wellorder usually requires an iteration. One may conjecture that the method of the proof in [25] might be modified to yield the desired result since it is based on a coding using a variant of Sacks forcing (which is known to force the tree property, see [47] and [26] for more details).

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