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External Examiner's Report on the Dissertation of Tomáš Lávička

"An Abstract Study of Completeness in Infinitary Logics"

Submitted in year 2018 at the Department of Logic

I. Brief summary of the dissertation

This thesis is a contribution to the field of abstract algebraic logic (AAL for short). Roughly speaking, AAL is the study of arbitrary propositional logics, understood as substitution-invariant consequence relations over the set of formulas (built up with an infinite set of variables) of arbitrary algebraic languages. Despite the general character of AAL, some of its strongest result are restricted to the case of finitary logics, i.e. logics that can be presented by sets of rules with finitely many premises. The aim of this thesis is to contribute to amend this lack in a variety of ways, mainly related to the study of enhanced forms of completeness theorems and of the abstract Lindenbaum lemma, and to the study of logics with certain generalized connectives such as implications, disjunctions and negations.

II. Brief overall evaluation of the dissertation

My impression on the thesis is highly positive. For what concerns the style, it is well written and clear. The proofs are sufficiently detailed, but still concise, and plenty of examples are provided. For what concerns the content, I personally find it an important contribution to AAL, in particular because of the excellent work on protonegational logics. Several passages of the thesis are very creative both when it comes to proofs and to counterexamples. For these reasons, I strongly recommend to approve this doctoral thesis.

III. Detailed evaluation of the dissertation and its individual aspects

As I mentioned, I find the thesis well written and clear, both for what concerns the its style and presentation. The contents are arranged in a coherent way, which simplifies the reading of the thesis. The formal aspects of the thesis are

addressed in a satisfactory way, and the thesis is visually well-presented, and graphically well-formatted. The usage of source material is well planned, and references to the literature are always provided.

In what follows, I proceed to address the content of thesis in detail. The thesis is organized in two parts.

The first part focuses on the study of enhanced forms of completeness theorems. Even if it is well known that every logic is complete with respect to a distinguished class of logical matrices (called the reduced models of the logic), most prominent logics typically enjoy forms of completeness, which are stronger than the abstract completeness theorem with respect to their reduced models. This is the case for instance of all finitary logics, which are complete with respect to their relatively subdirectly irreducible reduced models. However, this result depends on the validity of the so-called abstract Lindenbaum lemma, which is not immediately available in the setting of infinitary logics. For this reason, the overall goal of this part of the thesis is to investigate the abstract Lindenbaum lemma and some enhanced forms of completeness in the setting of infinitary logics (focus on infinitary logics here is crucial, since these properties are easily seen to hold in the finitary setting).

The second part of the thesis focuses on the study of special classes of theories and on their interplay with generalized connectives. More in detail, the relation between linear theories and implications, prime theories and disjunctions, and maximally consistent theories and negations is investigated in detail. As a consequence, some sufficient conditions for the validity of the abstract Lindenbaum lemma are identified in presence of either a disjunction or of a negation.

The thesis opens with the introductory Chapter 1 in which the contributions of the thesis are outlined in a clear and reasonably simple fashion. The subsequent Chapter 2 presents an overview of some basic concepts of AAL and universal algebra, but features as well some new concepts and results related to antitheorems and natural expansions. The first are sets of formulas whose substitution instances are inconsistent, while the second can be viewed as a technical tool for establishing results about the preservation of some properties under the formation of expansions (of logics).

The first part of the thesis consists of Chapters 3 and 4.

It begins by introducing the "hierarchy of infinitary logics", i.e. a hierarchy of increasingly stronger conditions related to enhanced completeness theorems. Roughly speaking, these conditions amount to variants of the requirement on a logic that the collection of all (completely) intersection prime theories forms a base for the closure system consisting of all theories of the logic.

Perhaps the main result of this part consists in individuating the valid implications between the conditions in the hierarchy of infinitary logics. In this respect, the most creative part of the work consists in establishing that RSI-completeness does not imply the IPEP. This is achieved by constructing a highly non-trivial and creative example of an RSI-complete logic without the IPEP (Section 4.4).

Another remarkable passage of this part is the proof that the cardinality of the logic formulated in an arbitrarily large set of variables induced by the standard Lukasiewicz matrix is bounded by Aleph1 (Proposition 3.21). The proof of this fact exploits elegant topological intuitions, which are available because of the continuity of the basic operations of the algebra underlying the matrix. Remarkably, the proof can be seen as a natural generalization of the classical topological argument establishing that logics determined by finite sets of finite matrices are finitary.

The second part of the thesis consists of Chapters 5 and 6.

Chapter 5 deals with the study of semilinear logics, and of logics with a disjunction. In both cases it is shown that intersection prime theories can be characterized in a more "concrete" fashion, as linear and prime theories respectively. Perhaps the most interesting results of this chapter consist in the proof that (infinitary) logics with a countable axiomatization and a strong disjunction satisfy the Lindenbaum lemma, and of a characterization of the latter in terms of pair-extension lemmas (Theorems 5.12 and 5.16). These results are subsequently applied to provide an alternative proof of the axiomatization of infinitary Lukasiewicz logic.

Finally, Chapter 6 is devoted to the study of maximally consistent theories. In my view, this is by far the most creative and valuable part of the thesis, and an excellent contribution to AAL which features deep and unexpected insights in the role of negation in arbitrary logical systems.

The main concept of this part is that of a protonegational logic, which is defined in terms of the order-theoretic behaviour of the Leibniz operator over the set of (simple) theories. In the wide framework of compact logics, protonegational logics can be viewed as those satisfying a local and parametrized version of the inconsistency lemma (Theorem 6.9). This result discloses the unexpected fact that at least a very weak form of the inconsistency lemma is captured by the order-theoretic behaviour of the Leibniz operator. For this reason, it is natural to compare it with the classical result stating that a logic is protoalgebraic exactly when it satisfies a local and parametrized local version of the deduction theorem.

Driven by this analogy, the thesis contains an array of results, showing that a significant amount of the classical theory of protoalgebraic logics can be

recovered in the setting of protonegational logics, including for instance a variant of the Correspondence Theorem, and the characterization of local inconsistency lemmas in terms of simple filter extension properties (Proposition 6.16 and Theorem 6.35), which have analogues in the realm of protoalgebraic logics.

In this respect, it is worth remarking that despite the imaginative power of the analogy between protonegational and protoalgebraic logics, none of the above mentioned results was to be expected in principle, which makes the contributions of this part of the thesis both unexpected and very solid.

Finally, the general machinery of the theory of protonegational logics is applied to the study of a variety of topics, including a variant of structural completeness, and Glivenko-like theorems (where a characterization of substructural logics which are Glivenko-equivalent to classical logics is obtained).

IV. Questions for the author

It is natural to wonder whether a work, similar to the one carried on for protonegational logics in the thesis, could be done in the realm of logics with a disjunction. In particular, it is natural to wonder whether it possible that local and parametrized version of the proof by cases property may correspond to a suitable notion of protodisjunctionality. I would be interested in knowing the opinion of the author on this topic.

V. Conclusion

The work submitted meets the standard customarily required of a doctoral dissertation. I recommend the dissertation for a public defence, and recommend the submitted dissertation with the tentative grade of pass.

16th of September 2018 T. Moraschini