Report on the Ph.D. thesis "An Abstract Study of Completeness in Infinitary Logics" by Tomáš Lávička submitted at the Department of Logic in 2018

The thesis is a contribution to the study of general propositional logical systems in the framework of abstract algebraic logic (AAL). In AAL, logics are studied by means of a semantics using algebra-based logical matrices; there is rich theory built around investigation of semantical properties of logics, and transfer theorems relating their semantical and syntactical properties. In spite of the fact that it strives to be as general as possible, a significant amount of work in AAL is restricted to *finitary* logics.

This is in a way a reasonable compromise: on the one hand, finitary systems are by far the most common among logics that one comes upon in practice (in particular, this includes all logics that are explicitly axiomatized by axioms and finitary derivation rules); on the other hand, the assumption of finitarity is eminently useful—many arguments essentially rely on it, or at least it considerably simplifies them. Nevertheless, there are interesting instances of naturally occurring infinitary logics (often semantically defined, such as logics of specific t-norms, which are taken as a running example in the thesis), and treatment of infinitary logics is in any case important for the sake of completeness of the general theory.

The present thesis aims to partly remedy this gap: it investigates various properties of logics with emphasis on their applicability to infinitary logics, highlighting differences to the familiar case of finitary logics. Broadly speaking, the thesis concentrates on *completeness* properties (or completeness theorems) which assert that the logic is sound and complete with respect to a "nice" class of logical matrices. Due to the nature of algebraic semantics, such completeness properties amount to theory extension properties stating that any theory of the logic is an intersection of a family of "nice" theories, or more general filter extension properties that apply in a similar vein to filters on arbitrary algebras in place of the algebra of formulas. Typically, these filter extension properties are easy enough to prove for finitary logics with the help of Zorn's lemma, but the situation is considerably more complex for infinitary logics.

Part I of the thesis introduces a classification of arbitrary logics (though some results only apply to protoalgebraic logics) by means of a hierarchy of several completeness properties, all implied by finitarity: the hierarchy includes the intersection-prime extension property (IPEP) stating that any theory is an intersection of intersection-prime theories, its strengthening (CIPEP) to completely intersection-prime theories, transferred versions of these properties applying to filters in arbitrary algebras (τ-(C)IPEP), and completeness wrt relatively finitely subdirectly irreducible (RFSI) or relatively subdirectly irreducible (RSI) matrices. The author undertakes a systematic investigation of this hierarchy, including implications between the properties, their alternative characterizations in terms of surjective completeness or subdirect representability, preservation of the properties by extensions of logics, tools useful for placing specific logics in the hierarchy, and examples separating classes of the hierarchy.

Part II of the thesis studies the interaction of particular connectives with completeness properties. Section 5 deals with logics with a semilinear implication, and logics with a disjunction. In both cases, there are natural concepts of extension properties specific to the connectives (the linear extension property, and the prime extension property, respectively), which are seen to be variants of IPEP. The thesis discusses relationships among these and other related properties (e.g., variants of the strong cut property).

Section 6 is devoted to logics with a negation. A central role is played here by simple (i.e., maximal consistent) theories and filters. The author introduces the class of protonegational logics as a weakening of protoalgebraicity, restricting some of its defining conditions to simple theories. The simple extension property (τ-)SEP is a strengthening of (τ-)CIPEP. Section 6.2 discusses at length variants of inconsistency and dual inconsistency lemmas, and their connections to other properties, including SEP, the simple filter extension property, the law of excluded middle, and deduction-detachment theorems. A related notion of antistructural completion is discussed in §6.3. Finally, in §6.4–6.6, the author outlines some topics for further research related to simple theories (Glivenko theorems, infinitary deduction theorems, and protoalgebraic pairs).

All in all, I find the thesis quite impressive. It contains a wealth of results that provide a comprehensive in-depth analysis of the chosen topic, clearly demonstrating the skills of the candidate as a researcher. The material should be valuable to the AAL community, and indeed parts of the thesis have already been published. The thesis is well organized, though I would appreciate an index of all the abbreviations (MCP, PLIL, SFEP, PEP, . . .).

Having said that, there is room for improvement: the author should have been more careful at various places, and there are small mistakes and typos. One recurring problem is that theorems are missing minor (but nontrivial) assumptions, often the existence of an antitheorem (e.g., the proof of Theorem 6.22 needs an antitheorem to apply Proposition 6.21, but this assumption is neither stated in Theorem 6.22 nor it follows from the other assumptions, as exemplified by CL^+). A rather annoying habit is that theorems are applied in a different—usually stronger—form than how they were stated and proved. In particular, Lemma 3.5 is mostly applied assuming only that F is (completely) intersection prime instead of $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{R(F)SI}}\,\mathrm{L}$, and Theorem 6.9 is used with MCP in place of compactness.

Concerning Theorem 6.9, the proof of (i) $\to$ (vii) is highly problematic. First, as the author notes himself, it uses a stronger assumption than just (i). Second, it relies on the claims $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}(\Sigma_{\mathrm{L}}\langle\varphi,\psi\rangle)$ and $\mathrm{Th}_{\mathrm{L}}(\emptyset) \subseteq \Sigma_{\mathrm{L}}\langle\varphi,\psi\rangle$

for which no justification is provided, and that are most likely false. For these reasons, I am only willing to accept it as a proof of (vi) $\to$ (vii). Noting also that compactness was used in the proof of (v) $\to$ (vi), rather than (vi) $\to$ (i) as written by the author, the claim that *every protonegational logic with the MCP has all the properties of the previous theorem* [i.e., 6.9] on p. 112 is unjustified: it has only been proved that they have properties (i)–(v), not (vi) or (vii). What is true is that all logics satisfying MCP and (vi) have all the properties of the theorem. Consequently, several subsequent statements (in particular, 6.10–6.12) should assume (vi) rather than just L being protonegational. Moreover, I suspect that the (omitted) proof of Prop. 6.16 (ii) follows the incorrect part of the proof of 6.9.

The claim on p. 15 that a surjective homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is strict if $F = h^{-1}h[F]$ is wrong: consider e.g. $\boldsymbol{A} = \boldsymbol{B}$, $h = \mathrm{id}$, and $F \subsetneq G$.

The right-to-left implication in Corollary 6.73 is unjustified: $\Gamma \vdash_{\alpha\mathrm{L}} I(\overline{\varphi}, \overline{\delta})$ does not imply $\Gamma \vdash_{\mathrm{L}} I(\overline{\varphi}, \overline{\delta})$.

In §6.4–6.6, proofs are often missing, and those that are included are not always coherent: e.g., the proof of 6.91 does not make sense to me (how does $\sigma[\mathrm{Th}_{\mathrm{L}}(\Delta)]$ being an L-theory help?); I am going to ignore this as these sections are qualified with *we only briefly suggest possible directions for further research* in the introduction, but perhaps this should be stressed more (e.g., at the beginning of each section).

Example 5.14 is a bit pointless: the statement of PEP is meaningless if the logic has no disjunction in the first place. Is there an example of a countably axiomatized logic with a disjunction (but not strong disjunction) which does not enjoy the PEP?

Here is a suggestion concerning Proposition 6.25. It is not entirely clear to me what is the author's intended proof of the second claim, but I believe every semisimple logic L has τ-SEP, with no further assumptions: since $\mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{L} \subseteq \mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L}$, Theorem 3.19 shows that L has τ-CIPEP and is protoalgebraic. Moreover, by definition, RSI models cannot be expressed as subdirect products in a nontrivial way, hence semisimplicity implies $\mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L} \subseteq \mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{L}$. Thus, completely intersection-prime filters are simple, and the τ-CIPEP of L implies τ-SEP.

But let us not get bogged down in details. Despite occasional shortcomings that are to some extent inevitable in any work of this size, I reiterate that this is a very good thesis showing many important insights, exceeding the usual standards for a doctoral dissertation. I endorse the thesis be accepted for public defence, and recommend its grading as "Pass".

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