An Abstract Study of Completeness in Infinitary Logics

Abstraktní Studium Úplnosti pro Infinitární Logiky

Tomáš Lávička

Charles University, Faculty of Arts
Department of Logic

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Supervisor: Carles Noguera

To everyone whom I deem dear to me

Prohlašuji, že jsem dizertační práci vypracoval samostatně, že jsem řádně citoval všechny použité prameny a literaturu a že práce nebyla využita v rámci jiného vysokoškolského studia či k získání jiného nebo stejného titulu.

V Praze dne 11.7. 2018 Tomáš Lávička

Abstract

In this thesis we study completeness properties of infinitary propositional logics from the perspective of abstract algebraic logic. The goal is to understand how the basic tool in proofs of completeness, the so called Lindenbaum lemma, generalizes beyond finitary logics. To this end, we study few properties closely related to the Lindenbaum lemma (and hence to completeness properties). We will see that these properties give rise to a new hierarchy of infinitary propositional logic. We also study these properties in scenarios when a given logic has some (possibly very generally defined) connectives of implication, disjunction, and negation. Among others, we will see that presence of these connectives can ensure provability of the Lindenbaum lemma.

Keywords: abstract algebraic logic, infinitary logics, Lindenbaum lemma, disjunction, implication, negation

Abstrakt

V této dizertační práci se zabýváme studiem vlastností úplnosti infinitárních výrokových logik z pohledu abstraktní algebraické logiky. Cílem práce je pochopit, jak lze základní nástroj v důkazech uplnosti, tzv. Lindenbaumovo lemma, zobecnit za hranici finitárních logik. Za tímto účelem studujeme vlastnosti úzce související s Lindenbaumovým lemmatem (a v důsledku také s vlastnostmi úplnosti). Uvidíme, že na základě těchto vlastností lze vystavět novou hierarchii infinitárních výrokových logik. Také se zabýváme studiem těchto vlastností v případě, kdy naše logika má nějaké (případně hodně obecně definované) spojky implikace, disjunkce a negace. Mimo jiné uvidíme, že přítomnost daných spojek může zajist platnost Lindenbaumova lemmatu.

Keywords: abstraktní algebraická logika, infinitární logiky, Lindenbaumovo lemma, disjunkce, implikace, negace

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While I am writing these last (and at the same time first) words of my dissertation, I am stricken with an interesting mixture of melancholy and joy. It has been some journey since I started studying logic almost eight years ago. Going back I realize how fortunate I was to have found this path in my life and I feel indebted to many for it. It comes with a great pleasure to have this possibility to express all my sincere gratitude to them on the following lines.

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Contents

1 Introduction

The goal of this dissertation is to contribute to the theory of abstract algebraic logic with a focus on infinitary logics. Abstract algebraic logic studies non-classical propositional logics, seen as structural consequence relations, via their connections to algebraic semantics. It is a usual practice in the field to consider mostly logics that can be axiomatized by means of rules with finitely many premises (thus every proof in these logics is finite). A logic with this property is called *finitary*; otherwise it is called *infinitary*.[1] Finitarity seems to be a well-motivated restriction especially regarding the argument that reasoning is a process performed by human minds or other finite machines thus, in principle, it should not be infinite. On the other hand, at least in mathematical practice, we deal with implicitly infinite arguments all the time. The best example lies behind the ω-rule: to prove that all natural numbers have some property $\varphi(n)$, that is to prove $\forall n \varphi(n)$, the rule asserts that we need to prove all the instances $\varphi(0), \varphi(1), \varphi(2), \ldots$ What makes this rules feasible in everyday mathematical practice is the fact that we do not have to produce all the particular instances. Indeed, it suffices to have a proof of $\varphi(0)$ and know that this proof can be rewritten to a proof of $\varphi(n)$ for every natural number n. In this sense the ω-rule reduces to the well-known induction principle. Thus, arguably, infinitary rules are, from a philosophical point of view important concepts, and, in many cases, they can be used in actual reasoning, because they can be represented by finite means.

Another motivation for a systematic study of infinitary logics is that they can be used to understand some important algebraic structures such as algebras defined over the unit interval [0,1] of reals that play a prominent role in the field of mathematical fuzzy logic. The main examples that we consider are the standard Łukasiewicz, product, or Gödel algebras, where the first induces the infinitary Łukasiewicz logic $Ł_\infty$, the second one the infinitary product logic Π_∞, while the last one actually induces a finitary logic.

[1] This notion should not be mistaken with with the one regarding logics with infinitary languages. In our setting all connectives have finite arities.

In the thesis we intend to remedy the lack of a systematic study of infinitary propositional logics. Along the way we deal with various, more or less natural, examples of infinitary logics which illustrate the general theory. Some of these examples have a rather unique properties: e.g. we consider a logic semantically defined by an algebra which contains a part of the absolutely free algebra (Section 4.4), or a logic presented in a language with only finitely many variables, which ensures some strong properties (Section 5.2). However, the main examples of infinitary logics are the above mentioned Łukasiewicz and product logic, which are intensively studied throughout the text.

In Part I we focus on the most basic logical property: completeness. It is well known that every logic has a semantics based on logical matrices and that this semantics in the usual cases corresponds to the expected algebraic semantics: e.g. the classical logic is complete w.r.t. matrices based on Boolean algebras, and intuitionistic logic w.r.t. matrices based on Heyting algebras. However, we are often interested in completeness with a more refined semantics: in classical logic that is completeness w.r.t. the two-element Boolean algebra, or in case of fuzzy (semilinear) logics it is completeness w.r.t. chains—abstractly, in both cases we are looking for completeness w.r.t. *relatively (finitely) subdirectly irreducible algebras*. There is a key component to obtain this completeness result, the so called *Lindenbaum lemma*: for example in classical logic, using its finitarity, we can show that for every set of formulas, if a formula is not provable from the set, then it can be separated by a maximally consistent theory. The latter then induces an ultrafilter on the free algebra and thus its quotient is the two-element Boolean algebra, as desired.

In non-classical logics, the role of maximally consistent theories is played by other (weaker) notions of theories. In particular, there are two abstract notions: since the collection of all theories is a closure system (it is closed under intersections and the set of all formulas is a theory) we can speak of *(completely) intersection-prime* theories. The abstract Lindenbaum lemma reads as follows, this time in symbols:

> If $\Gamma \nvdash \varphi$, then there is a (completely) intersection-prime theory T such that $T \nvdash \varphi$ and $\Gamma \subseteq T$,

which is true for every finitary logic. Therefore, every finitary logic is complete w.r.t. its relatively (finitely) subdirectly irreducible models. The main objective of the first part is to understand how far the Lindenbaum lemma and completeness can be extended in the realm of infinitary logics.

On the other hand, some special kinds of theories are omnipresent in the literature on non-classical logics:

- *Prime* theories (either $\varphi \in T$ or $\psi \in T$ whenever $\varphi \vee \psi \in T$) in logics with a *disjunction*.
- *Linear* theories (either $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$) in logics with a *implication*.
- *Maximally consistent* theories which are usually interesting in logics with a *negation* that has some strong properties (see below).

Since the prime and linear theories are intersection-prime and maximally consistent theories are completely intersection-prime, the first part can also be seen as an abstract study on the role of these theories in the proof of completeness. Thus in Part II we study these theories one by one with the aim of understanding how the presence of certain connectives (that is, of disjunction, implication, and negation) interacts with the general study of Part I. It should be noted that in the literature there are plenty of results regarding completeness for infinitary logics, which can be separated into two groups:

- Results for infinitary modal expansions of classical logic [38, 56, 57, 86, 88], where the proofs of the particular instances of the Lindenbaum lemma exploit the strong properties of classical negation.
- Results for infinitary expansions of prominent fuzzy logics [18, 65, 70, 90], where the Lindenbaum lemma is mostly obtained using properties of a disjunction.

Among others we identify two general conditions that ensure provability of the Lindenbaum lemma in infinitary logics. These results subsume most of the above results. The Lindenbaum lemma is provable for

- countably axiomatizable logics with a *strong disjunction*, and
- countably axiomatizable logics that enjoy some generalized version of the *law of the excluded* middle of classical logic.

The main objective of Part II is to present a general study of the mentioned three kinds of theories (linear, prime, and maximally consistent) and their corresponding connectives (implication, disjunction, and negation).

Outline

In the preliminary chapter (Chapter 2), we first present the basic notions and notations of universal algebra and abstract algebraic logic (Sections 2.1–2.5). While these first five sections contain no new material and are in general well known (at least to algebraic logicians), the remaining ones are more or less new (especially the Sections 2.6 and 2.8). For that reason we decided to call the chapter "Basic concepts" rather than more usual "Preliminaries". In these sections we are mostly occupied with concepts related to infinitarity. In particular, we speak about *compactness* and its tranferability to all algebras. Moreover, we introduce a new notion of an *antitheorem* (a set of formulas that cannot be jointly designated in nontrivial matrix) and study its properties (both compactness and antitheorems are studied in a yet unpublished manuscript [68]). Finally, in the last section we introduce a dual notion to natural extensions, the *natural expansions* (presented in [66]), which can be useful to prove preservation of properties under expansions (we use it later in Section 3.3).

Part I: Hierarchy of infinitary logics

First, in Chapter 3, we present the main classes of logics investigated in the dissertation. In particular, two classes are defined in terms of completeness properties:

- RSI-*complete* logics, that is logics complete w.r.t. relatively subdirectly irreducible models.
- RFSI-*complete* logics, that is logics complete w.r.t. relatively **finitely** subdirectly irreducible models.

Furthermore, two other classes are defined in terms of the two variants of the abstract Lindenbaum lemma (which we call *extension properties*, following [27]):

- the CIPEP class of logics with the ***completely*** *intersection-prime extension property*, and
- the IPEP class of logics with the *intersection-prime extension property*.

and, finally, two classes based on the transferred (semantical) counterparts of these properties:

- the τ-CIPEP class, and
- the τ-IPEP class.

We then explain their basic relations (as described above)—see Figure 1.1.

For protoalgebraic logics, we provide semantical characterizations of the extension properties (C)IPEP by means of *surjective completeness* (Subsection 3.1.1). Further, we define a notion of *(finitely) subdirectly representable* logic, that is, a logic where the class of all models coincides with class of models subdirectly representable by the relatively (finitely) subdirectly

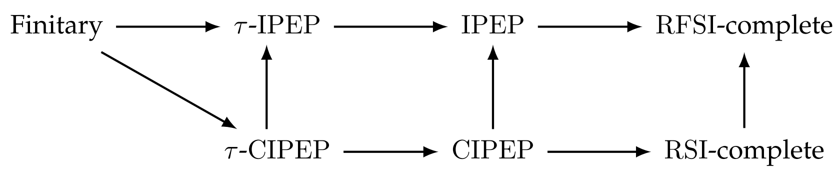


Figure 1.1: The classes and their basic relations

irreducible ones. We show that this property precisely corresponds to the fact that a logic is protoalgebraic and has the transferred extension property (τ-(C)IPEP). The last section of the chapter deals with preservation of these properties under extension and expansions.

In Chapter 4, we answer the questions postulated above: all of them negatively. That is, we present separating examples witnessing that all the above mentioned classes of logics are pairwise different. In particular, there are infinitary logics that do not enjoy the extension properties (the Lindenbaum lemma is not provable) and moreover some of these logics are still RFSI-complete. Therefore, Figure 1.1 in fact describes a new hierarchy of infinitary logics.

Main results of Part I

Most of the results presented in this part is a joint work with Carles Noguera published in [66, 67],[2] we present a short list of the main ones:

- A new hierarchy of infinitary logics is proposed.
- τ-(C)IPEP + protoalgebraicity $\iff$ (finite) subdirect representation.
- The best behaved natural example of an infinitary logic (regarding the position in the hierarchy) is the infinitary Łukasiewicz logic Ł_∞. It has the τ-CIPEP and, in particular, it is subdirectly representable.
- We present a topological result that postulates bounds on the cardinality of logic given by a class of matrices, thus generalizing the well known result saying that strongly finite logics, i.e. those that are complete with respect to finite class of finite matrices, are finite.
- Birkhoff's subdirect representation theorem does not extend beyond quasi-varieties: this is consequence of the fact that the infinitary prod-

[2] We mention that the least trivial separating example (Section 4.4) was already introduced in the author's master thesis, although in a completely different form.

uct logic Π_∞ does not enjoy the τ-IPEP (although it is still in the CIPEP class). In particular, it is not (finitely) subdirectly representable (it is a fuzzy logic not representable by chains). The same is true about its equivalent algebraic semantics. Moreover, this class of algebras is natural and almost a quasi-variety.

- We tend to think about the extension properties (C)IPEP as a natural generalizations of finitary (for example it substitutes the role of finitarity in the completeness proofs). However, these properties are not in general preserved under extensions by finitary rules. Though they are preserved under axiomatic extensions and in many cases also to axiomatic expansions.

Part II: Theories and connectives

In the second part we investigate the role of connectives regarding the general theory presented in Part I. Such connectives can be either primitive symbols of the language or definable by sets of formulas (possibly infinite with parameters). The basic connection is established via meta-rules that are typical for each connective. These meta-rules generalize some well-known properties of classical logic. The *semilinearity property* for implication

$$\text{SLP} \quad \Gamma, \varphi \to \psi \vdash_{\mathrm{CL}} \chi \quad \text{and} \quad \Gamma, \psi \to \varphi \vdash_{\mathrm{CL}} \chi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \chi.$$

The *proof by cases property* for disjunction:

$$\text{PCP} \quad \Gamma, \varphi \vdash_{\mathrm{CL}} \chi \quad \text{and} \quad \Gamma, \psi \vdash_{\mathrm{CL}} \chi \quad \Longrightarrow \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{CL}} \chi.$$

And finally, the *law of excluded middle* for negation:

$$\text{LEM} \quad \Gamma, \varphi \vdash_{\mathrm{CL}} \psi \quad \text{and} \quad \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \psi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \psi.$$

The validity of the general forms of these meta-rules is equivalent to the fact that the (completely) intersection-prime theories coincide with the particular kinds of theories corresponding to each connective, that is linear, prime, and maximally consistent. Similarly for each connective we define a corresponding notion of extension property (Lindenbaum lemma for linear, prime, and maximally consistent theories), which are again equivalent to the abstract ones in presence of the corresponding meta-rules.

In Chapter 5, we start with *semilinear* logics and linear theories. Semilinear logics were introduced [25] as a mathematical definition of the informal notion of fuzzy logic. They were also studied in the follow-up papers [29, 30]. We contribute to theory of semilinear logics mainly by studying particular examples of infinitary semilinear logics, most notably the infinitary Łukasiewicz and product logic, but we consider many others: for example, we define and study an infinitary version of truth degree preserving logics with truth constants based on (i) Łukasiewicz and (ii) Gödel logic. Regarding our theory, these logics always have the linear extension property (that is, they enjoy some form of the Lindebaum lemma) and, in fact, we see that the IPEP class is the smallest one in the hierarchy which contains all of them.

Then, we focus on logics with a disjunction. Disjunction connectives, of course, were the subjects of many contributions in abstract algebraic logic (e.g. [32, 35, 48, 51, 89]) or more recently also from the perspective of infinitary logics [27]. Most importantly, we prove that countably axiomatizable logics with disjunction enjoy the prime extension property and we demonstrate the applicability of this result by presenting (simple) proofs of completeness for some infinitary logics.

In Chapter 6 we investigate maximally consistent consistent theories which we decided to call *simple*. This is motivated by the notion of simplicity from universal algebra. Unlike in the previous cases (of linear and prime theories), simple theories as far as we know were not systematically studied in the literature.

The starting point of the new theory that we present in this chapter is the recent paper by Raftery [82], where he introduces the notion of the so called *inconsistency lemma*. This is again a generalization of a well known property of classical logic:

$$\varGamma \cup \{\varphi\} \text{ is inconsistent} \quad \Longleftrightarrow \quad \varGamma \vdash \neg\varphi.$$

Observe that the inconsistency lemma in some sense can be seen as a restriction of the classical deduction-detachment theorem:

$$\varGamma \cup \{\varphi\} \vdash \psi \quad \Longleftrightarrow \quad \varGamma \vdash \varphi \to \psi.$$

The study of deduction-detachment theorems and their rich hierarchy (including global, local, and parametrized local versions) is one of the important parts of the field of abstract algebraic logic. It generalizes the deduction-detachment theorem of intuitionistic and classical logic to e.g. substructural and modal logics. To compare: the initial paper [82] on inconsistency lemmas was about global version of this property. We aim to continue where the paper left off and extend the theory, analogously to deduction-detachment theorems, to local and parametrized local versions.

Arguably, the right framework to study deduction-detachment theorems is the class of finitary protoalgebraic logics. Indeed, deduction-detachment theorems imply protoalgebraicity and their properties (characterizations, algebraic equivalents, transferability etc.) are always proved for logics in this class. In comparison, even though in [82] inconsistency lemmas were studied in the same class (finitary protoalgebraic logics), we claim it is not the best possible framework. For this purpose we generalize the class of protoalgebraic logics to a richer class of logics that we call *protonegational* (they bear a stronger connection with negation rather than implication). Interestingly enough, in the case of inconsistency lemmas even the assumption of finitarity can weakened to compactness. Schematically on one side we have

- deduction-detachment theorems, protoalgebraic logics, implication, finitarity, and rules,

while on the other side we have

- inconsistency lemmas, protonegational logics, negation, compactness, and inconsistent sets.

We thus start this chapter presenting the protonegational logics, which in essence are precisely logics enjoying the same properties as protoalgebraic logics but restricted to simple theories. In the second section, we deal with the hierarchy of inconsistency lemmas. We also investigate a natural dual notion of this property; in classical logic:

$$\mathit{\Gamma} \cup \{\neg\varphi\} \text{ is inconsistent} \quad \Longleftrightarrow \quad \mathit{\Gamma} \vdash \varphi,$$

which is nothing else than the law of excluded middle in disguise. Interestingly enough, the most general form of this property, that is the dual parametrized local inconsistency lemma is a syntactical counterpart of *semisimplicity*. Moreover, countably axiomatizable logics with this property enjoy the simple extension property (which is again a version of the Lindenbaum lemma in infinitary setting). Then, in Section 6.3 we define yet another closely related notion, *antistructural completion*, which is, dually to structural completion, the strongest extension with the same simple theories (resp. the same inconsistent sets of formulas/antitheorems). In the remaining three sections, we only briefly suggest possible directions for further research.

- We explain the utility of inconsistency lemmas and antistructural completions in the study of Glivenko-like theorems [4, 14, 15, 16, 17, 54, 55].
- We generalize suitably the standard form of deduction-detachment theorems [33, 34, 36, 81] and inconsistency lemmas [82] to remedy the following shortcoming of the definition in case of infinitary logics. Namely, the idea behind deduction-detachment theorems is to turn rules into theorems, but with the standard definition this cannot in general be achieved for infinitary logics (because we can move only finitely many premises to right-hand side of the turnstile). In fact, it turns out, rather surprisingly, that every logic with some deduction-detachment theorem always enjoys at least the parametrized local one in the stronger setting (every rule corresponds to the provability of some theorems).

- Finally, we provide a possible alternative presentation of protonegational logics. This time generalizing the notion of protoalgebraicity by splitting its defining properties into pair of logics (protoalgebraic pairs).

Main results of Part II

Again, we briefly summarize the main results of this part. The part about disjunction is a joint work with Marta Bílková and Petr Cintula published in [5]. The part about simple theories is a joint work with Adam Přenosil contained in an unpublished manuscript [68].

- The Lindenbaum lemma is proved for logics with a disjunction and countable axiomatic system.
- We demonstrate the applicability of the previous abstract result to prove completeness for some infinitary logics and summarize the already well known (but nowhere published) axiomatizations of these logics.
- We investigate various cut properties and relate them to the pair extension lemma.
- We introduce and study a new hierarchy of inconsistency lemmas. We show that in some setting the full pair extension lemma is equivalent to finitarity.
- A new class of protonegational logics is defined as a framework to study these properties.
- We provide a syntactical characterization for logics such that their simple models are closed under submatrices.
- The dual inconsistency lemma are proved to be a syntactical characterization of semisimplicity.
- Antistructural completions are introduced and studied.
- We describe the local deduction-detachment theorem of the infinitary Łukasiewicz logic, which is so far the only one known in the literature which necessarily uses a family of deduction-detachment terms consisting of infinite sets.
- We fully characterize substructural logics which are Glivenko equivalent to classical logic using only syntactical means (inconsistency lemmas and antistructural completions are used).

2 Basic concepts

In this preliminary chapter we will mostly review the basic notions and notations of abstract algebraic logic. For more information we refer the reader to [7, 8, 35, 49, 50], for a comprehensive treatment we recommend the recent monograph [46]. We remark that the last three sections are largely based on new results.

2.1 Closure operators and logics

A mapping $C\colon P(A) \to P(A)$ is called a *closure operator* on a set A if it is

- *extensive*, i.e. $X \subseteq C(X)$ for every $X \subseteq A$,
- *idempotent*, i.e. $C(X) = C(C(X))$ for every $X \subseteq A$, and
- *monotone*, i.e. $C(Y) \subseteq C(X)$ whenever $Y \subseteq X \subseteq A$.

A family $\mathcal{C} \subseteq P(A)$ is called a *closure system* on a set A provided that it is closed under arbitrary intersection and it contains the set A. It is well known that a closure system $\mathcal{C}$ on A gives rise to a closure operator on A, defining $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ for every $X \subseteq A$. And *vice versa* the collection of all *closed sets* of the closure operator C on A, i.e. the fixed points of C (sets $X \subseteq A$ such that $X = C(X)$), is a closure system on A. Additionally, every closure system $\mathcal{C}$ can be given a complete lattice lattice structure by taking intersections as meets and $C(X \cup Y)$ as joins of $X, Y \subseteq A$.

In fact, closure operators (systems) provide the fundamental connection allowing the general study of logical systems with tools of (universal) algebra: indeed, we can define the abstract notion of logic as a special kind of closure operator on the set (algebra) of formulas of a given type.

A *propositional language* $\boldsymbol{\mathcal{L}}$ is a pair $\langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}}} \rangle$, where $\mathcal{L}$ is an algebraic type[1] and $\mathit{Var}_{\boldsymbol{\mathcal{L}}}$ an infinite set of variables. For simplicity we always assume that

[1] We stress out that all connectives have finite arities.

the size of $Var_{\mathcal{L}}$ is an infinite regular cardinal. As a notational convention we use boldface calligraphic variables for languages $\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{L}'} \ldots$ and, unless said otherwise, we assume them to be pairs of the corresponding form, that is $\langle \mathcal{L}, Var_{\boldsymbol{\mathcal{L}}}\rangle, \langle \mathcal{L}', Var_{\boldsymbol{\mathcal{L}'}}\rangle \ldots$

By $\boldsymbol{Fm}_{\mathcal{L}}(X)$ (resp. $Fm_{\mathcal{L}}(X)$) we denote the absolutely free term algebra of type $\mathcal{L}$ with the set X as generators (resp. its universe). We call $\boldsymbol{Fm}_{\mathcal{L}}(Var_{\boldsymbol{\mathcal{L}}})$ the *algebra of $\boldsymbol{\mathcal{L}}$-formulas* and we denote it simply by $\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}$ (we write $Fm_{\boldsymbol{\mathcal{L}}}$ for its universe, i.e. for the set of all $\boldsymbol{\mathcal{L}}$-formulas).

An endomorphism of $\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}$ is called an *$\boldsymbol{\mathcal{L}}$-substitution*. An *$\boldsymbol{\mathcal{L}}$-consecution* is a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$. We write simply φ instead of $\emptyset \rhd \varphi$. We use small Greek letters $\varphi, \psi, \chi \ldots$ as variables for formulas and capital Greek letters $\Gamma, \Delta, \ldots$ as variables for sets of formulas. Given a set of $\boldsymbol{\mathcal{L}}$-consecutions L, we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$.

A *logic* L in the language $\boldsymbol{\mathcal{L}}$ is a set of $\boldsymbol{\mathcal{L}}$-consecutions (that is $\mathrm{L} \subseteq P(Fm_{\boldsymbol{\mathcal{L}}}) \times Fm_{\boldsymbol{\mathcal{L}}}$) satisfying:

- $\varphi \vdash_{\mathrm{L}} \varphi$ (Reflexivity)
- $\Gamma \vdash_{\mathrm{L}} \varphi \quad\Longrightarrow\quad \Gamma, \Delta \vdash_{\mathrm{L}} \varphi$ (Monotonicity)
- $\Gamma \vdash_{\mathrm{L}} \Delta$ and $\Delta \vdash_{\mathrm{L}} \varphi \quad\Longrightarrow\quad \Gamma \vdash_{\mathrm{L}} \varphi$. (Cut)
- $\Gamma \vdash_{\mathrm{L}} \varphi \quad\Longrightarrow\quad \sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\boldsymbol{\mathcal{L}}$-substitution σ (Structurality)

Intuitively by $\Gamma \vdash_{\mathrm{L}} \varphi$ we mean that in the logic L the formula (the statement) φ is a deductive consequence of the set of premises Γ. We also write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$.

For every logic L there is a natural mapping $\mathrm{Th}_{\mathrm{L}} \colon P(Fm_{\boldsymbol{\mathcal{L}}}) \to P(Fm_{\boldsymbol{\mathcal{L}}})$ defined by $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \in Fm_{\boldsymbol{\mathcal{L}}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$—i.e. the mapping that assigns to every set of formulas its deductive closure. It is easy to see that Th_{L} is a closure operator on $Fm_{\boldsymbol{\mathcal{L}}}$, which satisfies an additional property, the *structurality*: for every substitution σ and $\Gamma \subseteq Fm_{\boldsymbol{\mathcal{L}}}$ it holds that $\sigma[\mathrm{Th}_{\mathrm{L}}(\Gamma)] \subseteq \mathrm{Th}_{\mathrm{L}}(\sigma[\Gamma])$. On the other hand, any structural closure operator C on $Fm_{\boldsymbol{\mathcal{L}}}$ induces a logic by $\Gamma \vdash \varphi$ iff $\varphi \in C(\Gamma)$.

A formula φ derivable from empty set of premises, i.e. $\emptyset \vdash_{\mathrm{L}} \varphi$, is called a L-*theorem*. If $\Gamma \vdash_{\mathrm{L}} Fm_{\boldsymbol{\mathcal{L}}}$, we call Γ an *inconsistent* set of formulas. An L-*theory* T is a deductively closed set of formulas, that is $\varphi \in T$ whenever $T \vdash_{\mathrm{L}} \varphi$. The set of all L-theories is denoted as $\mathrm{Th}\,\mathrm{L}$ and, in fact, it is the closure system associated to Th_{L}. Thus, $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ is the L-theory generated by Γ. A logic L is called *trivial* if $Fm_{\boldsymbol{\mathcal{L}}}$ is the only L-theory.

There is a natural relation between logics:

$$\mathrm{L} \leq \mathrm{L}' \quad\Longleftrightarrow\quad \Gamma \vdash_{\mathrm{L}} \varphi \text{ implies } \Gamma \vdash_{\mathrm{L}'} \varphi \text{ for every } \Gamma, \varphi \text{ in the language of L.}$$

We say that L′ is an *expansion* (resp. extension) of L if $\mathrm{L} \leq \mathrm{L}'$ (and they share the same language).

We will often drop writing L-, $\boldsymbol{\mathcal{L}}$-, resp. $\mathcal{L}$- when the logic, language, resp. type is known from the context. Similar convention will be adopted for other later defined notions. Also often when proving general properties about logics or algebras we will not be mentioning the language or the type at all.

2.2 Universal algebra

For a detailed treatment of universal algebra we address the reader to the monographs [3, 9]. Given a type $\mathcal{L}$ and a set X by $Eq_{\mathcal{L}}(X)$, we denote the set of all *equations* in variables X, that is $Eq_{\mathcal{L}}(X) = Fm_{\mathcal{L}}(X) \times Fm_{\mathcal{L}}(X)$. Similarly, as in the case of formulas, we usually work with a chosen set of variables; thus, for a language $\boldsymbol{\mathcal{L}}$, we write again $Eq_{\boldsymbol{\mathcal{L}}}$ instead of $Fm_{\mathcal{L}}(Var_{\boldsymbol{\mathcal{L}}})$. A *generalized quasi-equation* in $\boldsymbol{\mathcal{L}}$ is an expression of the form $\Theta \rhd \alpha \approx \beta$, where $\Theta \cup \{\alpha \approx \beta\} \subseteq Eq_{\boldsymbol{\mathcal{L}}}$. It can be seen as an equational analog to the notion of consecution. We drop the word "generalized" when Θ is a finite set of equations. We use variables $\boldsymbol{A}, \boldsymbol{B}, \dots$ for algebras. Now, given a a class of algebras K, the corresponding equational consequence relation $\models_{\mathsf{K}} \subseteq P(Eq_{\boldsymbol{\mathcal{L}}}) \times Eq_{\boldsymbol{\mathcal{L}}}$ is defined by

$$\Theta \models_{\mathsf{K}} \alpha \approx \beta \iff \text{for each } \boldsymbol{A} \in \mathsf{K} \text{ and each homomorphism } e\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{A}$$
$$e(\alpha) = e(\beta) \text{ whenever } e(\delta) = e(\epsilon) \text{ for every } \delta \approx \epsilon \in \Theta.$$

By $\mathbf{I}, \mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P_{SD}}, \mathbf{P_U}$, and $\mathbf{P_{R_\kappa}}$ respectively we denote the usual class operators of isomorphisms, homomorphisms, subalgebras, products, isomorphic copies of subdirect products, ultraproducts, and reduced products over κ-complete filters (i.e. filters closed under intersection of families of size $< \kappa$). A class of algebras is called a *generalized quasi-variety (resp. quasi-variety, variety)* if it is axiomatized by a collection of generalized quasi-equations (resp. quasi-equations, equations) in some language $\boldsymbol{\mathcal{L}}$ or equivalently if the class is closed under $\mathbf{I}, \mathbf{S}, \mathbf{P}$, and $\mathbf{U}_\kappa$ for $\kappa = |Var_{\boldsymbol{\mathcal{L}}}|$ (resp. under $\mathbf{I}, \mathbf{S}, \mathbf{P}, \mathbf{P_U}$, resp. $\mathbf{H}, \mathbf{S}, \mathbf{P}$), where $\mathbf{U}_\kappa$, studied in [6], stands for

$$\mathbf{U}_\kappa(\mathsf{K}) = \{\boldsymbol{A} \mid \text{every } \kappa\text{-generated subalgebra of } \boldsymbol{A} \text{ belongs to } \mathsf{K}\}.$$

If $\kappa = \omega$ we write simply $\mathbf{U}$.

Given a class of algebras K and an algebra $\boldsymbol{A}$, by $\mathrm{Con}\,\boldsymbol{A}$ we denote the lattice of all congruences on $\boldsymbol{A}$, and by $\mathrm{Con}_{\mathsf{K}}\,\boldsymbol{A}$ we denote the lattice of all K-*congruences*, i.e. the collection of congruences Θ on $\boldsymbol{A}$ such that $\boldsymbol{A}/\Theta \in \mathsf{K}$. When K is a quasi-variety, the lattice $\mathrm{Con}_{\mathsf{K}}\,\boldsymbol{A}$ is algebraic.

An algebra $\boldsymbol{A}$ is *subdirectly representable* by algebras $\{\boldsymbol{B}_i\}_{i\in I}$, if there is a *subdirect embedding* $e\colon \boldsymbol{A} \hookrightarrow_{\mathbf{SD}} \prod_{i\in I} \boldsymbol{B}_i$, that is, an embedding such that every homomorphism $\pi_i \circ e$ is surjective (where $\pi_i \colon \prod_{i \in I} \boldsymbol{B}_i \to \boldsymbol{B}_i$ is the projection map). An algebra $\boldsymbol{A}$ is *(finitely) subdirectly irreducible relative* to a class K, in symbols $\boldsymbol{A} \in \mathsf{K}_{\mathrm{R(F)SI}}$, if it cannot be non-trivially subdirectly represented by a (finite) family of algebras from K (that is, one of the maps $\pi_i \circ e$ is an isomorphism). Equivalently,

- $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RSI}} \iff \mathrm{Con}_{\mathsf{K}}\, \boldsymbol{A} \setminus \{\mathrm{Id}_{\boldsymbol{A}}\}$ has a minimum (w.r.t. $\subseteq$).
- $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RFSI}} \iff \theta \cap \varphi = \mathrm{Id}_{\boldsymbol{A}}$ implies that either $\theta = \mathrm{Id}_{\boldsymbol{A}}$ or $\varphi = \mathrm{Id}_{\boldsymbol{A}}$ for every $\theta, \varphi \in \mathrm{Con}_{\mathsf{K}}\, \boldsymbol{A}$.

2.3 Matrix semantics

Given a type $\mathcal{L}$, an $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra (the *algebraic reduct* of the matrix) and $F \subseteq A$ is called a *filter* of the matrix. A matrix $\langle \boldsymbol{A}, F \rangle$ is called *trivial* provided that $A = F$. Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices and a language $\boldsymbol{\mathcal{L}}$, the corresponding *semantical consequence relation* is defined as:

$$\Gamma \models_{\mathbb{K}} \varphi \iff \text{for each } \langle \boldsymbol{A}, F \rangle \in \mathbb{K} \text{ and each } \boldsymbol{A}\text{-evaluation } e \text{ (i.e. a homomorphism } e \colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{A}) \text{ such that } e[\Gamma] \subseteq F, \text{ we have } e(\varphi) \in F.$$

It is easy to see that the relation $\models_{\mathbb{K}}$ is a logic in $\boldsymbol{\mathcal{L}}$. Note that we often write $\models_{\mathbf{A}}$ instead of $\models_{\{\mathbf{A}\}}$.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, we say that a congruence θ of $\boldsymbol{A}$ is *compatible* with F iff for each $a, b \in A$:

$$\langle a, b \rangle \in \theta \text{ and } a \in F \implies b \in F.$$

Compatible congruences with F form a complete sublattice of the lattice of all congruences of $\boldsymbol{A}$, and thus there exists the largest such congruence, which is called the *Leibniz congruence* of $\mathbf{A}$ and is denoted as $\boldsymbol{\Omega}^{\boldsymbol{A}} F$. Equivalently, $\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ if and only if

$$\varphi^{\boldsymbol{A}}(a, c_1, \ldots, c_n) \in F \iff \varphi^{\boldsymbol{A}}(b, c_1, \ldots, c_n) \in F$$

for every formula $\varphi(p, r_1, \ldots, r_n)$ and every $c_1, \ldots, c_n \in A$. We can see $\boldsymbol{\Omega}^{\boldsymbol{A}}$ as a mapping from L-filters on $\boldsymbol{A}$ into $\mathrm{Con}(A)$, this mapping is called the *Leibniz operator*. In case $\boldsymbol{A} = \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}$ we write simply $\boldsymbol{\Omega}$. We say that $\mathbf{A}$ is a *reduced matrix* if $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}$. A (reduced) matrix $\mathbf{A}$ is a *(reduced) model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\{\mathbf{A}\}}$. The class of all models (resp. reduced models) of a logic L is denoted as $\mathbf{Mod}\,\mathrm{L}$ (resp. $\mathbf{Mod}^*\,\mathrm{L}$).

Let us briefly describe the *reduction process*. A map $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is called a matrix *homomorphism* if it is a homomorphism between the underlying algebras and $h(a) \in G$ whenever $a \in F$. Moreover, it is called *strict* if also the other implication holds, that is, $h(a) \in G$ implies $a \in F$. Note that a surjective $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is

$$\text{strict} \iff F = h^{-1}h[F] \iff \operatorname{Ker} h \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F.$$

An *embedding* is a strict and injective homomorphism, moreover, if it is also surjective, it is an *isomorphism*. For every $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ its *reduction* is defined as

$$\langle \boldsymbol{A}, F\rangle^* = \langle \boldsymbol{A}^*, F^*\rangle = \langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F, F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle,$$

where $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the quotient of $\boldsymbol{A}$ by the congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ and $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the collection of $\boldsymbol{\Omega}^{\boldsymbol{A}}F$-equivalence classes $[a]$ such that $a \in F$. There is a natural reduction map $r\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{A}, F\rangle^*$ assigning to every $a \in A$ the equivalence class $[a]$. This map is a strict and surjective homomorphism.

It is well known that, for any logic L, the class of its (reduced) models gives a complete semantics (meaning $\vdash_{\mathrm{L}} = \models_{\mathbf{Mod}\,\mathrm{L}} = \models_{\mathbf{Mod}^*\,\mathrm{L}}$); however, it is common to consider meaningful subclasses of reduced models, such as the relatively (finitely) subdirectly irreducible matrices, which may provide stronger completeness theorems.

A matrix $\mathbf{A} \in \mathbb{K}$ is *relatively (finitely) subdirectly irreducible in the class of matrices* $\mathbb{K}$, if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from $\mathbb{K}$. We write $\mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L}$ (resp. $\mathbf{Mod}^*_{\mathrm{RFSI}}\,\mathrm{L}$) for the classes of models that are relatively (resp. finitely) subdirectly irreducible in $\mathbf{Mod}^*\,\mathrm{L}$. The models in $\mathbf{Mod}^*_{\mathrm{R(F)SI}}\,\mathrm{L}$ are sometimes simply called R(F)SI-*models*. A logic L is R(F)SI-*complete* if $\vdash_{\mathrm{L}} = \models_{\mathbf{Mod}^*_{\mathrm{R(F)SI}}\,\mathrm{L}}$.

The class of algebraic reducts of $\mathbf{Mod}^*\,\mathrm{L}$ is denoted as $\mathbf{Alg}^*\mathrm{L}$. $\mathbf{Alg}^*\mathrm{L}$ is also called the *algebraic counterpart* of L. Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, we say that F is an L-*filter* provided that $\mathbf{A}$ is a model of L. By $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ we denote the set of all L-filters over $\boldsymbol{A}$; $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ is also a closure system (and, consequently, a complete lattice) and hence it also induces a closure operator which we denote as $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}$ (note that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ is the least L-filter containing X). The L-filters on the algebra of formulas are precisely the theories of L, that is, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}) = \mathrm{Th}\,\mathrm{L}$.

Example 2.1. *Classical logic* CL in language $\boldsymbol{\mathcal{CL}} = \langle\{\vee, \wedge, \to, \neg, \overline{0}, \overline{1}\}, \mathit{Var}\rangle$ can be introduced as the logic semantically given by the matrix $\langle \mathbf{2}, \{1\}\rangle$, i.e. $\mathrm{CL} = \models_{\langle \mathbf{2}, \{1\}\rangle}$, where $\mathbf{2}$ is the two-element Boolean algebra. As expected, the class of all algebraic models of CL, i.e. $\mathbf{Alg}^*\mathrm{CL}$, is the class of all Boolean algebras denoted as BA. Moreover, for every Boolean algebra $\boldsymbol{B}$, the collection of all CL-filters, $\mathcal{F}i_{\mathrm{CL}}\,\boldsymbol{B}$, precisely coincides with the collection of all lattice filters (upward closed subsets which are closed under meets)—which motivates the terminology for L-filters. The (finitely) subdirectly irreducible models of CL have a particularly nice description, in fact, the only one, which is non-trivial is the two-element Boolean algebra based matrix $\langle \mathbf{2}, \{1\}\rangle$.

2.4 Lebniz hierarchy and implicational logics

One of the basic research interests in abstract algebraic logic is to provide general classifications of logics by means of their properties. One of the most fruitful approaches of this kind is the investigation of the so called Leibniz hierarchy, which classifies logics via properties of the Leibniz operator $\boldsymbol{\Omega}$. The hierarchy was proposed and first developed by Blok, Pigozzi, and Czelakowski and others [7, 8, 31]. The original hierarchy, which is more important for the present text, classifies logics based on the formula definability of the Lebniz congruence/operator. However, let us mention that the hierarchy was extended by Raftery in [80] and more recently by Moraschini, see [71, 72], by classes of logics based on definability of truth.

Since in the present text we are more interested in the study of implication, we are following the implication-based approach to introduce the Leibniz hierarchy as in [25] (where the author generalizes the implication connectives in sense of Rasiowa [83]) rather than the standard equivalence (congruence) based approach.

Let $\Delta(p, q, \bar{r}) \subseteq Fm_{\boldsymbol{\mathcal{L}}}$ be a set of formulas in two variables and, possibly, parameters $\bar{r}$. Then, for an algebra $\boldsymbol{A}$ and $a, b \in A$, we define

$$\Delta^{\boldsymbol{A}}\langle a, b\rangle = \{\chi^{\boldsymbol{A}}(a, b, \bar{c}) \in A \mid \chi(p, q, \bar{r}) \in \Delta \text{ and } \bar{c} \in \bar{A}\}.$$

In case of formulas φ, ψ we write simply $\Delta\langle\varphi, \psi\rangle$. In many cases we speak about a set of formulas $\Rightarrow(p, q, \bar{r})$, then we use rather the infix notation, that is we write $a \Rightarrow^{\boldsymbol{A}} b$ instead of $\Rightarrow^{\boldsymbol{A}}\langle a, b\rangle$. Moreover, we denote by $\Leftrightarrow(p, q, \bar{r})$ the set $\Rightarrow(p, q, \bar{r}) \cup \Rightarrow(q, p, \bar{r})$, the *symmetrization* of $\Rightarrow$. We say that $\Rightarrow$ is a *weak p-implication* (or just *weak implication* if there are no parameters $\bar{r}$) in L if the following conditions are satisfied:

(R) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$,
(MP) $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$,
(T) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$,
(sCng) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \Rightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and each $i < n$.

A logic is called *weakly (p-)implicational* if it has a weak (p-)implication. $\Rightarrow$ induces a natural preorder on every model $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ given by

$$a \leq^{\mathbf{A}}_{\Rightarrow} b \iff a \Rightarrow^{\boldsymbol{A}} b \subseteq F.$$

We will often write simply $\leq^{\mathbf{A}}$ when $\Rightarrow$ can be inferred from the context. In case of semantically defined logics this relation can be used to determine that the logic is weakly p-implicational:

Observation 2.2. If $\Rightarrow$ is a weak p-implication in L, then $\leq^{\mathbf{A}}_{\Rightarrow}$ on every reduced model $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ which makes F an upset. Conversely, if $\mathrm{L} = \models_{\mathbb{K}}$ for a class of matrices $\mathbb{K}$ and $\leq^{\mathbf{A}}_{\Rightarrow}$ is an order relation which makes F an upset on every $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ then $\Rightarrow$ is a weak p-implication in L and, consequently, $\mathbb{K} \subseteq \mathbf{Mod}^* \mathrm{L}$.

Weakly p-implicational logics are actually an alternative presentation of *protoalgebraic logics*—analogously all classes of implicational logics correspond to classes in Leibniz hierarchy; cf. Figure 2.1, where the black ones form the original hierarchy.

A set of formulas $\Delta(p, q, \bar{r})$ in two variables (and with parameters $\bar{r}$) is a called a set of *congruence formulas (with parameters)* for L if for every model $\langle \boldsymbol{A}, F\rangle$ and $a, b \in A$ we have

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}\langle a, b\rangle \subseteq F. \tag{2.1}$$

If $\Rightarrow$ is a weak (p-)implication, then its symmetrization is a set of congruence formulas (with parameters) for L and, conversely, every set of congruence formulas (with parameters) is a weak (p-)implication. A set of formulas $\Rightarrow(p, q)$ in two variables satisfying (R) and (MP) is called a *protoimplication*.

Proposition 2.3. *The following are equivalent for every logic* L*:*

(i) L *is weakly p-implicational (protoalgebraic).*
(ii) L *has a set of congruence formulas with parameters.*
(iii) L *has a protoimplication.*
(iv) L *enjoys the* correspondence theorem*: for every strict surjective homomorphism f from a model $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$ the function $\boldsymbol{f}\colon [F, A] \to [G, B]$ defined by $\boldsymbol{f}(H) = f[H]$ for $H \in [F, A]$ is a lattice isomorphism.*
(v) $\boldsymbol{\Omega}$ *is monotone over* $\mathrm{Th}\,\mathrm{L}$.
(vi) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is monotone over* $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ *for every algebra* $\boldsymbol{A}$.

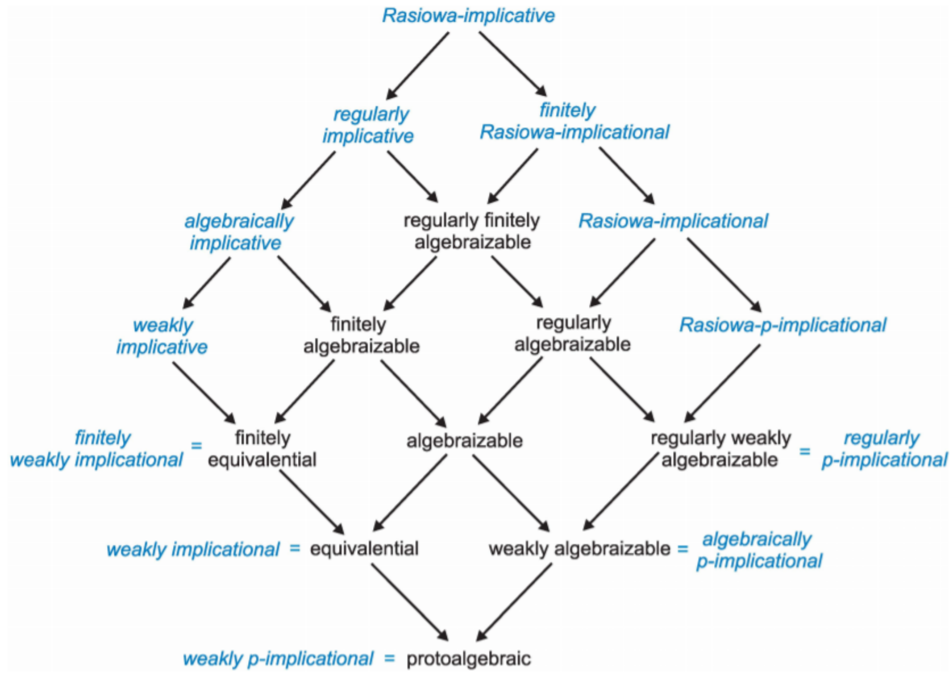


Figure 2.1: Hierarchy on implicational logics and Leibniz hierarchy compared.

A logic L with a weak (p-)implication $\Rightarrow$ is called *algebraically (p-)implicational*, resp. (*weakly*) *algebraizable*, provided that there is a set of equations in one variable $\mathcal{E}(p)$ such that

$$p \dashv\vdash_{\mathrm{L}} \{\alpha(p) \Leftrightarrow \beta(p) \mid \alpha \approx \beta \in \mathcal{E}\} \tag{Alg}$$

or, equivalently, L is algebraically (p-)implicational if there is a set of formulas $\Delta(p, q, \bar{r})$ in two variables p, q (and with parameters $\bar{r}$), a set of equations in one variable $\mathcal{E}(p)$, and a class of algebras K such that

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \mathcal{E}[\Gamma] \models_{\mathsf{K}} \mathcal{E}(\varphi) \tag{Alg$_1$}$$

$$\alpha \approx \beta \;=\!\mid\models_{\mathsf{K}} \mathcal{E}[\Delta\langle\alpha, \beta\rangle]. \tag{Alg$_2$}$$

An *equivalent algebraic semantics* for L is the largest K satisfying (Alg$_1$) and (Alg$_2$), it coincides with the class $\mathbf{Alg}^*\mathrm{L}$. If K satisfies (Alg$_1$) and (Alg$_2$), then $\mathbf{Alg}^*\mathrm{L}$ is the generalized quasi-variety in $\boldsymbol{\mathcal{L}}$ generated by K:

$$\mathbf{Alg}^*\mathrm{L} = \mathbf{GQ}(\mathsf{K}) = \mathbf{U}_{|Var_{\boldsymbol{\mathcal{L}}}|}\mathbf{ISP}(\mathsf{K}).$$

If L is algebraically p-implicational, then the Leibniz operator is a complete lattice isomorphism $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A} \cong \mathrm{Con}_{\mathbf{Alg}^*\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$. This implies that on every algebra $\boldsymbol{A} \in \mathbf{Alg}^*\mathrm{L}$, there is a unique $F \subseteq A$ such that $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^*\,\mathrm{L}$ (F is the smallest element in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$). Thus, if $\Rightarrow$ satisfies (Alg), then every algebra $\boldsymbol{A} \in \mathbf{Alg}^*\mathrm{L}$ can be equipped with a unique order relation $\leq^{\boldsymbol{A}}$ given by $\leq^{\mathbf{A}}_{\Rightarrow}$ as defined above (for the unique filter which makes the algebra a reduced model).

A logic L with a weak (p-)implication $\Rightarrow$ is called *Rasiowa-(p-)implicational* if it satisfies

$$\psi \vdash_{\mathrm{L}} \varphi \Rightarrow \psi, \tag{Ras}$$

and *regularly (p-)implicational* if it satisfies

$$\varphi, \psi \vdash_{\mathrm{L}} \varphi \Rightarrow \psi. \tag{Reg}$$

In regularly (p-)implicational logics the class $\mathbf{Mod}^*\,\mathrm{L}$ is *unital*, that is, if $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^*\,\mathrm{L}$, then F is a singleton, i.e. $F = \{a\}$ for some $a \in A$. Moreover, if L is Rasiowa-(p-)implicational, then this element a is the largest w.r.t. $\leq^{\boldsymbol{A}}$.

Finally, we speak about *implicative* logics if $\Rightarrow$ can be taken as one formula without parameters. We usually write $\to$ instead of $\Rightarrow$ and $\leftrightarrow$ for its symmetrization.

2.5 Deduction-detachment theorems

In this section, we will briefly describe the hierarchy of deduction-detachment theorems, which we generalize later in Chapter 6.

A family of sets of formulas $\Psi(p, q, \bar{r})$ with two designated variables p, q and possibly parameters $\bar{r}$ is called a *family of deduction-detachment (DD) sets* for L, provided that for every $\Gamma \cup \{\varphi, \psi\}$, we have

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} I(\varphi, \psi, \bar{\delta}) \text{ for some } I(p, q, \bar{r}) \in \Psi \text{ and some } \bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}}.$$

A logic L is said to enjoy the

- *parametrized local deduction-detachment theorem*, PLDDT, if it has a family of DD sets.
- *local deduction-detachment theorem*, LDDT, if it has a family of DD sets **without parameters**.
- *global deduction-detachment theorem*, GDDT, if it has a family of DD sets **without parameters consisting of a single set**.

It is well known that every logic enjoys the PLDDT if and only if it is weakly p-implicational (resp. protoalgebraic) [36]). In other words, there is a obvious strong connection between generalized implications and deduction-detachment theorems.

In the remaining part of this section we recall the main semantical counterparts of local and global DDT. A matrix $\langle \boldsymbol{A}, F\rangle$ is a *submatrix* of $\langle \boldsymbol{B}, G\rangle$, in symbols $\langle \boldsymbol{A}, F\rangle \leq \langle \boldsymbol{B}, G\rangle$, if $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$ and $F = G \cap A$. A logic L is said to enjoy the *filter extension property*, FEP, if for every two models $\langle \boldsymbol{A}, F\rangle$, $\langle \boldsymbol{B}, G\rangle$: if $\langle \boldsymbol{A}, F\rangle \leq \langle \boldsymbol{B}, G\rangle$ and there is $F \subseteq F' \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$, then there is $G \subseteq G' \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{B}$ such that $\langle \boldsymbol{A}, F'\rangle \leq \langle \boldsymbol{B}, G'\rangle$. It was proved in [34], that at least for finitary protoalgebraic logics the FEP is equivalent to the LDDT.

An algebra $\boldsymbol{A} = \langle A, \star, \vee, \top\rangle$ is called *dually Brouwerian semilattice* if $\langle A, \vee, \top\rangle$ is a join semilattice with top element $\top$ and for every $a, b, c \in A$:

$$a \star b \leq c \iff a \leq b \vee c.$$

For a finitary protoalgebraic logic L, it was proved in [33], that the GDDT is equivalent to the fact the join semilattice of finitely generated (compact) theories is dually Brouwerian (or, equivalently, the join semilattice of finitely generated filters on every algebra $\boldsymbol{A}$ is dually Brouwerian).

There are other notions of deduction-detachment theorems, e.g. the contextual DDTs [81], which we, however, do not study in the thesis.

2.6 Finitarity and beyond

In this section, we will investigate basic concepts that are related to the notion of infinitarity such as cardinality and compactness.

A closure operator C on A is said to have *cardinality* κ for κ an infinite cardinal, we write $\mathrm{card}\,\mathrm{C} = \kappa$, if κ is the least infinite cardinal such that for every $\{a\} \cup X \subseteq A$, it is the case that $a \in C(X)$ implies that there is $Y \subseteq X$ of cardinality $< \kappa$ such that $a \in C(Y)$. Moreover, C is called *finitary* provided $\mathrm{card}\,\mathrm{C} = \omega$. It is well known that a closure operator C is finitary if and only if the associated closure system $\mathcal{C}$ is *inductive*, meaning that the union every directed family $\mathcal{D} \subseteq \mathcal{C}$ is a closed set, i.e. $\bigcup \mathcal{D} \in \mathcal{C}$. Additionally, the lattice of closed elements of a finitary closure operator is algebraic, meaning that every closed set can be expressed as a join of some family of compact elements, which are precisely the closed sets of the form $C(X)$ for X finite—for more details see [9, §5].

Translating this terminology to logics, we say that L is *finitary* (has cardinality κ, we write $\mathrm{card}\,\mathrm{L} = \kappa$), when the closure operator Th_{L} is finitary (has cardinality κ). In other words, in finitary logics every consequence derivable from a given set of premises, already follows from finitely many of them—that is, if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$. Finally, logics, which are not finitary, are called *infinitary*.

Often, when we investigate infinitary logics, it happens that we need to presuppose some cardinality restrictions in order to prove the desired results. The most common ones are the following:

Definition 2.4. We say that L has

(i) *enough variables* if $\operatorname{card} \mathrm{L} \leq |Var_{\mathcal{L}}|^+$,
(ii) a *small type* if $|\mathcal{L}| \leq |Var_{\mathcal{L}}|$.

We also say that a set (or a matrix) is κ-*small* if it has cardinality $\leq \kappa$ (the universe of the algebra is κ-small).

Observe that L has a small type if and only if $|Fm_{\mathcal{L}}| = |Var_{\mathcal{L}}|$, and every such logic has enough variables. In particular, both of these restrictions are rather weak.

There is another notion closely related to cardinality of logics and closure operators, which was studied for certain classes of logics e.g. in [19, 24] and more recently in [68]—it is the notion of compactness. A closure operator C on A is said to be κ-*compact*, we write $\operatorname{card}^- \mathrm{C} = \kappa$, if κ is the least infinite cardinal such that for every $X \subseteq A$, if $A = C(X)$, then we can infer that there is a set $Y \subseteq X$ of cardinality $< \kappa$ such that $A = C(Y)$. An ω-compact closure system is simply called *compact*. Similarly for logics, that is, L is *compact* if every inconsistent set has a finite inconsistent subset.

Example 2.5. In context of logics compactness has the well-known natural reading saying that every finitely satisfiable set of formulas is satisfiable.

Next, we shall investigate some properties of cardinality and compactness and how they are related together. In fact, if our finitary logic satisfies some natural precondition, we can deduce it is compact—Proposition 2.14. The necessary assumption is the following notion:

Definition 2.6. We say that a set of formulas $\mathcal{A}$ is an *antitheorem*[2] in L, we write $\mathcal{A} \vdash_{\mathrm{L}} \emptyset$, provided it cannot be designated by an evaluation on a non-trivial matrix—meaning there is no $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ with $F \neq A$ and an $\boldsymbol{A}$-evaluation e such that $e[\mathcal{A}] \subseteq F$.

[2] Antitheorems are introduced and studied in a joint paper with A. Přenosil [68]. Some of the results were already published in Přenosil's Ph.D. thesis [79].

To motivate the terminology: an antitheorem is a natural dual notion to theorem, which, in contrast, is a formula that is designated in every model by every evaluation. Next example shows that a finitary logic without an antitheorem need not be compact.

Example 2.7. Let CL^+ be the positive fragment of CL, that is, the logic in the language of classical logic without negation $\neg$ and the constant $\bot$ such that $\Gamma \vdash_{\mathrm{CL}^+} \varphi$ iff $\Gamma \vdash_{\mathrm{CL}} \varphi$ for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{CL}^+}$. First, we observe that it does not have an antitheorem: since there is the evaluation $e\colon \boldsymbol{Fm}_{\mathcal{CL}^+} \to \mathbf{2}$ which assigns 1 to every formula, every set of formulas can be designated in the model $\langle \mathbf{2}, \{1\}\rangle \in \mathbf{Mod}^* \, \mathrm{CL}^+$. Similarly, one can see that $\Gamma \nvdash_{\mathrm{CL}^+} p$, whenever the variable p does not occur in Γ, which clearly implies that the logic is not compact.

Before going any further, we shall describe some basic properties of antitheorems. First, observe that while, clearly, every antitheorem is an inconsistent set of formulas, the converse is not in general true (e.g. the set $Fm_{\mathcal{CL}^+}$ from the previous example is inconsistent but not an antitheorem).

Proposition 2.8. *The following are equivalent for every logic* L*:*

(i) *Γ is an antitheorem.*
(ii) *$\sigma[\Gamma] \vdash_{\mathrm{L}} Fm_{\mathcal{L}}$ for every substitution σ.*

Proof. (i)→(ii): $\sigma[\Gamma]$ is also an antitheorem: if some evaluation e would non-trivially designate $\sigma[\Gamma]$, then the evaluation $e \circ \sigma$ would designate Γ.

(ii)→(i): Let $\{p_i\}_{i<\omega}$ be an infinite sequence of variables and consider a substitution σ given by $\sigma(p_i) = p_{i+1}$ and $\sigma(q) = q$ for any remaining variable q (note that p_0 is not in the range of σ). We know that $\sigma[\Gamma] \vdash_{\mathrm{L}} p_0$. Consider a matrix $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\, \mathrm{L}$ and an evaluation v such that $v[\Gamma] \subseteq F$. For every $a \in A$, define an evaluation v_a by $v_a(p_{i+1}) = v(p_i)$, $v_a(p_0) = a$, and $v_a(q) = v(q)$ for the remaining variables. Therefore, since $v_a[\sigma[\Gamma]] = v[\Gamma] \subseteq F$, we obtain that $a = v_a(p_0) \in F$. In other words, $F = A$ and $\langle \boldsymbol{A}, F\rangle$ is a trivial matrix. □

Corollary 2.9. *Antitheorems are closed under substitutions.*

Corollary 2.10. *For every logic it is the case that every inconsistent set of formulas Γ which does not use every variable is an antitheorem.*

The second corollary is often very useful, because it, in particular, implies that every finite inconsistent sets of formulas is an antitheorem. Moreover, once a logic have some antitheorem, the two notion coincide:

Proposition 2.11. *Let* L *be a logic with an antitheorem. Then, every inconsistent set of formulas in an antitheorem. That is, for every* $\varGamma \subseteq Fm_{\mathcal{L}}$*:*

$$\varGamma \vdash \emptyset \quad \iff \quad \varGamma \vdash Fm_{\mathcal{L}}.$$

Proof. Suppose $\varGamma \vdash Fm_{\mathcal{L}}$, then in particular $\varGamma \vdash \varDelta$ for some antitheorem $\varDelta$. The result follows, since if $\varGamma$ would be non-trivially designated, so would be $\varDelta$, which is not possible. □

The previous two propositions allow us to prove characterizations of the fact that a given logic has an antitheorem.

Corollary 2.12. *A logic* L *has an antitheorem if and only if* $\sigma[Fm_{\mathcal{L}}] \vdash_{\mathrm{L}} Fm_{\mathcal{L}}$ *for every substitution* σ*.*

Theorem 2.13. *For every logic* L*, the following are equivalent:*

(i) L *has an antitheorem.*
(ii) *Non-trivial models of* L *are closed under submatrices.*

Proof. (i)→(ii): Suppose $\mathcal{A}$ is an antitheorem and take a trivial submatrix of a model, $\langle \boldsymbol{B}, B\rangle \leq \langle \boldsymbol{A}, F\rangle$. Then, $\mathcal{A}$ can be designated in both $\langle \boldsymbol{B}, B\rangle$ and $\langle \boldsymbol{A}, F\rangle$. Thus, by the definition of an antitheorem, $F = A$.

(ii)→(i): We will argue by the previous corollary. Observe that

$$\langle \sigma[\boldsymbol{Fm}_{\mathcal{L}}], \sigma[Fm_{\mathcal{L}}]\rangle \leq \langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Th}_{\mathrm{L}}(\sigma[Fm_{\mathcal{L}}])\rangle$$

for every substitution σ. Hence, by (ii), the second matrix is trivial, which amounts to $\sigma[Fm_{\mathcal{L}}] \vdash Fm_{\mathcal{L}}$. □

Finally, we relate finitarity, compactness, and antitheorems. For the proof recall that **Mod** L is closed under ultraproducts for every finitary logic L–cf. [46, Proposition 4.66].

Proposition 2.14. *A finitary logic* L *has an antitheorem if and only if it is compact.*

Proof. The left-to-right direction: suppose that every finite subset of $\varGamma$ is consistent. In particular, they are not antitheorems. Thus, each of these finite subsets are non-trivially designated by an evaluation in some model. Using a standard ultraproduct construction (considering an ultrafilter on finite subsets of $\varGamma$ generated by the sets $I_{\varDelta} = \{\varDelta' \mid \varDelta \subseteq \varDelta' \subseteq \varGamma\}$), we can easily collect these evaluations and models into one large non-trivial model witnessing the fact that $\varGamma$ is not an antitheorem. Consequently, $\varGamma$ is not an inconsistent set (Proposition 2.11).

On the other hand every compact logic necessarily has a finite inconsistent set of formulas, which, by Corollary 2.10, is an antitheorem. □

We can also prove a general version of the previous proposition this time by syntactical means.

Proposition 2.15. *If* L *has an antitheorem, then* $\operatorname{card}^{-}\mathrm{L} \leq \operatorname{card}\mathrm{L}$.

Proof. Assume $\Gamma \vdash \emptyset$ and fix a substitution σ with variable p not in its range which has a left inverse substitution δ, i.e. $\delta \circ \sigma = \mathrm{Id}$. Proposition 2.8 implies that $\sigma[\Gamma] \vdash p$. Then, there is $\Delta \subseteq \sigma[\Gamma]$ of cardinality $< \operatorname{card}\mathrm{L}$ such that $\Delta \vdash p$. Since p does not occur in Δ by structurality, we can conclude that $\Delta \vdash Fm_{\mathcal{L}}$ and by Proposition 2.11 we know $\Delta \vdash \emptyset$. Finally, by Corollary 2.9, $\delta[\Delta]$ is an antitheorem of a small size and we are done, because $\delta[\Delta] \subseteq \Gamma$. □

It is well known that once our logic L is finitary, then so is $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$. This transfer result can be easily proved for arbitrary regular cardinality (see Corollary 2.20). On the other hand the transfer of compactness turns out to be a more intricate issue and it remains an open question whether it can proved in complete generality. However, at least we know that in case of protoalgebraic logics it transfers (later in Section 6.1 we show the result for a different class of logics). In the proof we utilize the result characterizing filter generation in protoalgebraic logics [46, Proposition 6.12]. We present a slightly more general version that extends beyond finitary logics. It can be proved in the same way.

Proposition 2.16. *Let* L *be a protoalgebraic logic with enough variables and* $\boldsymbol{A}$ *be an algebra. For any* $X \cup \{a\} \subseteq A$, $a \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ *if and only if there is a* $|\mathit{Var}_{\mathcal{L}}|$*-small* $\Gamma \cup \{\varphi\}$ *such that* $\Gamma \vdash_{\mathrm{L}} \varphi$ *and there is an* $\boldsymbol{A}$*-evaluation* h *such that* $h[\Gamma] \subseteq X \cup \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\emptyset)$ *and* $h(\varphi) = a$.

An analogous result can be proved for antitheorems:

Proposition 2.17. *Let* L *be a protoalgebraic with enough variables and with an antitheorem and let* $\boldsymbol{A}$ *be an algebra. Then, for every* $X \subseteq A$, $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X) = A$ *if and only if there is* $|\mathit{Var}_{\mathcal{L}}|$*-small antitheorem* $\mathcal{A}$ *and an* $\boldsymbol{A}$*-evaluation* h *such that* $h[\mathcal{A}] \subseteq X \cup \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\emptyset)$.

Proof. The direction from right to left should be obvious. For the other one suppose $A = \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ and let $\mathcal{A}(p)$ be an antitheorem in one variable p (it exists by Corollary 2.9). By the assumptions and by Proposition 2.15 we can assume it is $|\mathit{Var}_{\mathcal{L}}|$-small. Let us take an arbitrary $\boldsymbol{A}$-evaluation f. By the previous proposition, for every $\alpha \in \mathcal{A}$, there is a $|\mathit{Var}_{\mathcal{L}}|$-small $\Gamma_\alpha \vdash \varphi_\alpha$ and a suitable evaluation h_α witnessing $f(\alpha) \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$, i.e. we have $h_\alpha(\varphi_\alpha) = f(\alpha)$. Note that we can assume that $\Gamma_\alpha \cup \{\varphi\}$ is written using unique set of variables for every α and none of them uses p. Let $\Rightarrow$ be a protoimplication in L and define

$$\mathcal{A}' = \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha \cup \bigcup_{\alpha \in \mathcal{A}} \varphi_\alpha \Rightarrow \alpha.$$

Clearly, $\mathcal{A}'$ is an antitheorem. Moreover, every evaluation h, which agrees on variables from $\Gamma_\alpha \cup \{\varphi_\alpha\}$ with h_α and on p with f, satisfies the condition of the proposition—note that, since $h(\varphi_\alpha) = f(\alpha) = h(\alpha)$, we obtain $h[\varphi_a \Rightarrow \alpha] \subseteq \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\emptyset)$. Finally, it is easy to compute that $\mathcal{A}'$ is indeed $|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$-small (since L has enough variables, we can assume that $|{\Rightarrow}| \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$). □

Finally, we use this proposition to show that at least in some situations (κ-)compactness transfers to all algebras.

Theorem 2.18. *Let* L *be a protoalgebraic logic with enough variables and with an antitheorem. If* L *is at most* κ*-compact, then so is* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *for every algebra* $\boldsymbol{A}$*.*

Proof. Suppose $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = A$. Then, by the previous proposition, there is an antitheorem $\mathcal{A}(p)$ and a suitable evaluation h. By κ-compactness, the antitheorem can be assumed to have cardinality $<\kappa$ and, clearly, the set $Y = h[\mathcal{A}] \cap X$ generates the trivial filter; in symbols $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(Y) = A$. □

Finally, we shall describe the notion of proof and axiomatic system for (in)finitary logics: an *axiomatic system* $\mathcal{AS}$ in the language $\boldsymbol{\mathcal{L}}$ is a set $\mathcal{AS}$ of consecutions closed under arbitrary substitutions. The elements of $\mathcal{AS}$ of the form $\Gamma \rhd \varphi$ are called *axioms* if $\Gamma = \emptyset$, *finitary deduction rules* if Γ is a finite set, and *infinitary deduction rules* otherwise. An axiomatic system is said to be *finitary* if all its deduction rules are finitary. Note that we require axiomatic systems to be closed under substitutions, which is convenient for example to deal with countably axiomatizable logics.

Since in this text we are mostly interested in infinitary logics, we need to introduce and study the notion of infinitely long proofs. We present a natural generalization of finite Hilbert style proofs, that is, of proofs as finite sequence of formulas. Proofs for us are represented by infinite well-founded trees (i.e. trees with no infinite branches), where the well-foundedness is the key feature allowing us to retain some basic properties of the standard finite proofs, e.g. we can still use inductive arguments to argue about proofs. Formally, a *proof* of a formula φ from a set of formulas Γ in $\mathcal{AS}$ is a well-founded tree labeled by formulas such that

- its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$. It is easy to see that $\vdash_{\mathcal{AS}}$ is a logic and, moreover, if $\mathcal{AS}$ is finitary, then so is $\vdash_{\mathcal{AS}}$ (an easy consequence of the well-known König's lemma about finitely branching trees)—similarly, proof systems with rules with cardinality $< \kappa$ generate logics of cardinality at most κ (provided κ is a regular cardinal). We say that $\mathcal{AS}$ is an *axiomatization* or a *presentation* of L if $\vdash_{\mathrm{L}} = \vdash_{\mathcal{AS}}$.

The notion of a proof also transfers to all algebras: the elements of a filter generated by a set are characterized in the next proposition by means of the notion of a *proof in algebra*.

Proposition 2.19 ([27, Proposition 2.4]). *Suppose* L *is a logic,* $\mathcal{AS}$ *one of its presentations,* $\boldsymbol{A}$ *an algebra, and* $X \cup \{a\} \subseteq A$. *Let us define a set*

$$V_{\mathcal{AS}} = \{\langle e[\Gamma], e(\psi)\rangle \mid e \text{ is an } \boldsymbol{A}\text{-evaluation and } \Gamma \rhd \psi \in \mathcal{AS}\}.$$

Then, $a \in \mathrm{Fi}^{\boldsymbol{A}}(X)$ *if and only if there is a well-founded tree (called a* proof of a from X*) labeled by elements of* A *such that*

- *its root is labeled by* a*, and leaves are labeled by elements* x *such that* $x \in X$ *or* $\langle \emptyset, x\rangle \in V_{\mathcal{AS}}$ *and*
- *if a node is labeled by* x *and* $Z \neq \emptyset$ *is the set of labels of its preceding nodes, then* $\langle Z, x\rangle \in V_{\mathcal{AS}}$.

As an easy consequence of the previous characterization, we can prove:

Corollary 2.20. *If* L *has cardinality* $\leq\kappa$ *then so does* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *on every algebra* $\boldsymbol{A}$.

2.7 Prominent examples of (in)finitary logics

In this section, we introduce the class of substructural logics and some of their prominent infinitary extensions. The weakest substructural logic we consider is SL, the non-associative version of bounded full Lambek calculus, introduced in [22, Chapter 2]. This logic has a language $\boldsymbol{\mathcal{SL}}$ with a type

$$\mathcal{SL} = \{\wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top\}$$

with five binary and four nullary connectives and additional two defined negations $\sim\varphi = \varphi \rightsquigarrow \overline{0}$ and $\neg\varphi = \varphi \to \overline{0}$. Moreover, we define inductively $\varphi^1 = \varphi$ and $\varphi^{n+1} = \varphi \,\&\, \varphi^n$. Then, SL is the weakest logic, which satisfies the consecutions from Table 2.1 and $\to$ is its weak implication.

Consecution	Symbol	Name
$\varphi \to (\psi \to \chi) \lhd\rhd \psi \,\&\, \varphi \to \chi$	(Res)	residuation
$\varphi \to (\psi \to \chi) \lhd\rhd \psi \to (\varphi \rightsquigarrow \chi)$	($E_{\rightsquigarrow}$)	$\rightsquigarrow$-exchange
$\varphi \to \psi \lhd\rhd \varphi \rightsquigarrow \psi$	(symm)	symmetry
$\varphi \wedge \psi \to \varphi$	($\wedge$1)	lower bound
$\varphi \wedge \psi \to \psi$	($\wedge$2)	lower bound
$\chi \to \varphi, \chi \to \psi \rhd \chi \to \varphi \wedge \psi$	($\wedge$3)	infimality
$\varphi \to \varphi \vee \psi$	($\vee$1)	upper bound
$\psi \to \varphi \vee \psi$	($\vee$2)	upper bound
$\varphi \to \chi, \psi \to \chi \rhd \varphi \vee \psi \to \chi$	($\vee$3)	supremality
$\varphi \rhd \overline{1} \to \varphi$	(Push)	push
$\overline{1} \to \varphi \rhd \varphi$	(Pop)	pop
$\varphi \to \top$	(Veq)	*verum ex quolibet*
$\bot \to \varphi$	(Efq)	*ex falso quodlibet*

Table 2.1: Consecutions for SL

For any $X \subseteq \{a_1, a_2, e, c, i, o\}$ we write SL_X for the extension of SL by the corresponding rules/axioms from Table 2.2. We replace the pair 'i, o' by 'w' and the pair 'a_1, a_2' by 'a'. Moreover, if $\{a\} \subseteq X$, then SL_X is an extension of the associative SL, which we identify with the bounded full Lambek calculus FL, in particular, we denote this extension as $\mathrm{FL}_{X\setminus\{a\}}$.

Consecution	Symbol	Name
$\varphi \,\&\, (\psi \,\&\, \chi) \to (\varphi \,\&\, \psi) \,\&\, \chi$	a_1	re-associate to the left
$(\varphi \,\&\, \psi) \,\&\, \chi \to \varphi \,\&\, (\psi \,\&\, \chi)$	a_2	re-associate to the left
$\varphi \to (\psi \to \chi) \rhd \psi \to (\varphi \to \chi)$	e	exchange
$\varphi \to (\varphi \to \psi) \rhd \varphi \to \psi$	c	contraction
$\varphi \to (\psi \to \varphi)$	i	left weakening
$\overline{0} \to \varphi$	o	right weakening

Table 2.2: Structural rules

The logic SL (as well as the considered extensions) is algebraizable with $\mathcal{E} = \{p \wedge \overline{1} \approx p\}$. Its equivalent algebraic semantics is the variety SL of all *unital residuated lattice ordered grupoids*, SL-algebras for short, i.e. structures

$$\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \rightsquigarrow, \overline{0}, \overline{1}, \bot, \top \rangle,$$

where $\&$ is a residuated grupoid operation with residuals $\to$ and $\rightsquigarrow$, i.e.

$$a \,\&\, b \leq c \quad \iff \quad b \leq a \to c \quad \iff \quad a \leq b \rightsquigarrow c,$$

unit $\overline{1}$ ($a \,\&\, \overline{1} = \overline{1} \,\&\, a = a$), and $\langle A, \wedge, \vee \bot, \top \rangle$ is a bounded lattice with top element $\top$ and bottom element $\bot$. The equivalent algebraic semantics for the extensions are denoted SL_{X} (resp. FL_{X}), and they are subvarieties of SL which respectively satisfy

a	$p \,\&\, (q \,\&\, r) \approx (p \,\&\, q) \,\&\, r.$
e	$p \,\&\, q \approx q \,\&\, p.$
c	$p \approx p \,\&\, p$
i	$\top \approx \overline{1}.$
o	$\bot \approx \overline{0}.$

In some extensions of SL the type simplifies, that is we can identify:

- $\overline{1}$ and $\top$ in SL_i (that is we have $\vdash_{\mathrm{SL}_i} \overline{1} \leftrightarrow \top$), and $\overline{0}$ and $\bot$ in SL_o,
- the two implications $\to$ and $\rightsquigarrow$ in SL_e,
- and the two conjunctions $\wedge$ and $\&$ in SL_{ci}.

The logic $\mathrm{FL}_{\mathrm{cw}} = \mathrm{FL}_{\mathrm{ecw}}$ is the intuitionistic logic, denoted as IL.. Its equivalent algebraic semantics is the class HA, of Heyting algebras, that is, bounded relatively pseudo-complemented distributive lattices. For a comprehensive study on the algebraic aspects of FL and its extensions see [53].

Next, we will introduce the most prominent examples of infinitary logics, which we study in this text. All of them are extensions of the well-known basic fuzzy logic BL introduced by Hájek (see the monograph [59]). BL is an axiomatic extension of FL_{ew} with two additional axioms:

- $\varphi \to \psi \vee \psi \to \varphi$ (prelinearity)
- $\varphi \wedge \psi \to \varphi \,\&\, (\varphi \to \psi)$ (divisibility)

To present BL it suffices to consider a type $\{\&, \to, \overline{0}\}$. The remaining connectives can be defined:

$$\varphi \wedge \psi = \varphi \,\&\, (\varphi \to \psi) \qquad \varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$
$$\neg\varphi = \varphi \to \overline{0} \qquad \overline{1} = \neg\overline{0} \qquad \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Hájek conjectured that BL is the logic of all continuous t-norms in [58], he proved the conjecture for BL with two additional axioms, which were later discovered redundant and thus the conjecture was confirmed—see [12].

A *continuous t-norm* is a continuous function $*\colon [0,1]^2 \to [0,1]$ which is associative, commutative, monotone and 1 is its unit. Every continuous t-norm $*$ has its unique residuum $\to_*$ defined as

$$a \to_* b = \max\{c \in [0,1] \mid c * a \leq b\}.$$

Thus, for every continuous t-norm $*$, we have an algebra

$$[0,1]_* = \langle [0,1], *, \to_*, \overline{0} \rangle.$$

We denote the collection of all these algebras as CT. The above mentioned completeness results says that for every **finite** set of formulas $\Gamma \cup \{\varphi\}$ we have

$$\Gamma \vdash_{\mathrm{BL}} \varphi \quad \iff \quad \Gamma \models_{\mathbb{CT}} \varphi,$$

where $\mathbb{CT}$ is the class of all matrices with algebraic reduct from CT and a filter $\{\overline{1}\}$. The main examples of continuous t-norms are

- the Łukasiewicz t-norm: $a *_{Ł} b = \max\{0, a + b - 1\}$,
- the minimum (Gödel) t-norm: $a *_{\mathrm{G}} b = \min\{a, b\}$, and
- the product t-norm: $a *_{\Pi} b = a \cdot b$,

with the corresponding residua

$$a \to_{Ł} b = \begin{cases} 1 \\ 1 - a + b \end{cases} \qquad a \to_{\mathrm{G}} b = \begin{cases} 1 \\ b \end{cases} \qquad a \to_{\Pi} b = \begin{cases} 1 & a \leq b \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

We denote the corresponding algebras simply as $[0,1]_{Ł}$, $[0,1]_{\mathrm{G}}$, and $[0,1]_{\Pi}$.

The full logic of all continuous t-norms $\models_{\mathbb{CT}}$ is infinitary and we call it the *infinitary basic fuzzy logic*, BL_∞, indeed

$$\{\varphi \to \psi^n \mid n \in \omega\} \vdash_{\mathrm{BL}_\infty} \varphi \to \varphi \,\&\, \psi, \tag{R}$$

but it is not the case for any finite subset of the premises (counterexamples can easily be found already over $[0,1]_{Ł}$ or $[0,1]_{\Pi}$). In fact, in Subsection 5.2.2, we will see that a variant of this infinitary rule can be use to axiomatize BL_∞ relative to BL (a recent result proved in [65]).

To see that the rule is indeed valid in BL_∞, we can use the famous decomposition theorem for t-norms [22]: clearly, every continuous t-norm $*$ can be continuously rescaled to any interval $[a,b]$ yielding a function $*_{[a,b]}$. Let $(a_i, b_i)\colon i \in I$ with $0 \leq a_i \leq b_i \leq 1$ be a family of disjoint open intervals and assume $*^i$ is either Łukasiewicz or product t-norm, then

$$a * b = \begin{cases} a *^i_{[a_i,b_i]} b & \text{if } a, b \in (a_i, b_i) \\ \min\{a, b\} & \text{otherwise} \end{cases}$$

is a continuous t-norm, it is called the *ordinal sum* of the t-norms $\{*^i \mid i \in I\}$ with respect to the intervals $\{[a_i, b_i] \mid i \in I\}$. On the other hand a famous result by Mostert and Shields [73] asserts that the convers is also true:

Theorem 2.21. *Every continuous t-norm is an ordinal sum of Łukasiewicz and product t-norms.*

Further, we introduce two infinitary axiomatic extensions of BL_∞:

- *The infinitary Łukasiewicz logic* $Ł_\infty$ [69] is the logic of $*_{Ł}$, that is,

$$Ł_\infty = \models_{\langle [0,1]_{Ł}, \{1\}\rangle}.$$

  Its *finitary companion*, i.e. the weakest logic with the same finitary rules, is the infinitely-valued Łukasiewicz logic, denoted as Ł. Ł (resp. $Ł_\infty$) can be axiomatized relative to BL (resp. BL_∞) by an axiom $\neg\neg\varphi \to \varphi$ (for the infinitary version see [65]). In both logics, we define an additional binary connective $\varphi \oplus \psi = \neg(\neg\varphi \mathbin{\&} \neg\psi)$ and put $1\varphi = \varphi$ and $(n+1)\varphi = n\varphi \oplus \varphi$.

- *The infinitary product logic* Π_∞ [60] is the logic of $*_\Pi$, that is,

$$\Pi_\infty = \models_{\langle [0,1]_{\Pi}, \{1\}\rangle}.$$

  Its finitary companion is the product logic Π. Analogously, Π and Π_∞ can be axiomatized by adding an axiom $\neg\varphi \vee ((\varphi \to \varphi \mathbin{\&} \psi) \to \psi)$ to any presentation of BL and BL_∞.

- Note that the logic of the minimum t-norm, usually called the Gödel logic G [39], is finitary.

Both $Ł_\infty$ and Π_∞ are infinitary: as argued above, the rule (R) is a proper infinitary rule in both logics. On the other hand, they are both compact—see [24, Theorems 4.1 and 5.1].

All the introduced infinitary logics are Rasiowa-implicative with implication $\to$. In particular, they are algebraizable with $\mathcal{E}(p) = \{p \approx 1\}$ and with equivalent algebraic semantics $\mathbf{Alg}^*\mathrm{BL}_\infty$, $\mathbf{Alg}^*Ł_\infty$, and $\mathbf{Alg}^*\Pi_\infty$ being certain subclasses (they are not quasi-varieties) of, respectively, BL-algebras, MV-algebras, and product algebras (see e.g. [20]). In fact, since these logics are Rasiowa-implicative and have countable languages and sets of variables, we obtain the following descriptions of their classes of algebras:

- $\mathbf{Alg}^*\mathrm{BL}_\infty = \mathbf{ISP}_{\mathbf{R}_{\aleph 1}}(\mathsf{CT}) = \mathbf{UISP}(\mathsf{CT})$,
- $\mathbf{Alg}^*Ł_\infty = \mathbf{ISP}_{\mathbf{R}_{\aleph 1}}([0,1]_{Ł}) = \mathbf{UISP}([0,1]_{Ł})$,
- $\mathbf{Alg}^*\Pi_\infty = \mathbf{ISP}_{\mathbf{R}_{\aleph 1}}([0,1]_{\Pi}) = \mathbf{UISP}([0,1]_{\Pi})$.

Of course, the order induced by $\to$ is the usual lattice order on these algebras. Moreover, linear models,[3] which are precisely the RFSI-models, of $Ł_\infty$ and Π_∞ are particularly well-behaved. First, observe that both logics validate another proper infinitary rule:

[3] The precise definition is given later in Section 5.1.

$$\{\varphi \to \psi^n \mid n \in \omega\} \vdash \neg\varphi \vee \psi, \tag{A}$$

Proposition 2.22. *The linear models of* $Ł_\infty$ *(resp.* Π_∞*) are embeddable into* $[0,1]_Ł$ *(resp. into* $[0,1]_\Pi$*). That is,*

$$\mathbf{Mod}^*_{\mathrm{RSI}}\, Ł_\infty \subseteq \mathbf{Mod}^*_{\mathrm{RFSI}}\, Ł_\infty = \mathbf{Mod}^\ell\, Ł_\infty \subseteq \mathbf{S}(\langle [0,1]_Ł, \{1\}\rangle),$$

and analogously for Π_∞.

Proof. A linear MV- (resp. product) algebra $\boldsymbol{A}$ is called *Archimedean*, if for every $a, b \in A$ such that $0 < a < b < 1$, there is a natural number n such that $b^n < a$. It is easy to observe, that the rule (A) implies that every linear model of both logics is, in fact, Archimedean. Every Archimedean MV- (resp. product) algebra is well known to be embeddable into the standard one over [0,1] interval. In both cases it is proved using relations between the categories of corresponding algebras and lattice-ordered Abelian groups (for MV-algebras see e.g. [74] or [20, Chapter 5], and for product algebras see e.g. [13]). The result follows by Hölder's theorem which asserts that every Archimedean linear lattice-ordered Abelian group can be embedded into the additive group of reals. □

2.8 Natural extensions and expansions

Natural extensions are a standard tool, in abstract algebraic logic, to prove *transfer theorems*, that is, to show, for a given logic L, that a property of $\mathrm{Th}\,\mathrm{L}$ remains true in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ for any algebra $\boldsymbol{A}$. They are often obtained by enlarging the set of variables while essentially keeping all the properties of the logic untouched (see [28, 35, 78]). We first recall the precise definition of natural extension and then introduce a dual notion that we call *natural expansion*. The main motivation for us to introduce natural expansions lies in the fact that they will prove useful when arguing about expansions in general.

We say that a language $\boldsymbol{\mathcal{L}}'$ is an *extension* of $\boldsymbol{\mathcal{L}}$ (and denote it as $\boldsymbol{\mathcal{L}} \preceq \boldsymbol{\mathcal{L}}'$) whenever $\mathcal{L} \subseteq \mathcal{L}'$ and $Var_{\boldsymbol{\mathcal{L}}} \subseteq Var_{\boldsymbol{\mathcal{L}}'}$. Furthermore, we say $\boldsymbol{\mathcal{L}}'$ is a *variable (resp. type) extension* of $\boldsymbol{\mathcal{L}}$ whenever $\boldsymbol{\mathcal{L}} \preceq \boldsymbol{\mathcal{L}}'$ and $\mathcal{L} = \mathcal{L}'$ (resp. $Var_{\boldsymbol{\mathcal{L}}} = Var_{\boldsymbol{\mathcal{L}}'}$).

2.8.1 Extensions

Let L be a logic in $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}'}$ its variable extension. Then, the *natural extension* of L to variables $Var_{\boldsymbol{\mathcal{L}'}}$ is a logic in language $\boldsymbol{\mathcal{L}'}$, denoted as $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$, which can be defined in a syntactical way by using any axiomatization of L or, alternatively, semantically by means of the class $\mathbf{Mod}\,\mathrm{L}$, i.e. $\mathrm{L}^{\boldsymbol{\mathcal{L}'}} = \models_{\mathbf{Mod}\,\mathrm{L}}$. It was shown in [28] that under the assumption that L

$$\text{has enough variables or } |Var_{\boldsymbol{\mathcal{L}}}| = |Var_{\boldsymbol{\mathcal{L}'}}|, \tag{As1}$$

we obtain the following useful characterization (proved in [87]):

$$\begin{aligned} \Gamma \vdash_{\mathrm{L}^{\boldsymbol{\mathcal{L}'}}} \varphi \quad \iff \quad & \text{there is a homomorphism } \sigma\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{Fm_{\mathcal{L}'}}, \\ & \text{and a set of formulas } \Delta \cup \{\psi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}} \text{ such that} \\ & \Delta \vdash_{\mathrm{L}} \psi, \sigma[\Delta] \subseteq \Gamma, \text{and } \sigma(\psi) = \varphi. \end{aligned} \tag{N}$$

Moreover, again in [28], the authors show that the same assumption in fact guarantees that $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ is the unique extension of L to $\boldsymbol{\mathcal{L}'}$, which is conservative and has the same cardinality. In symbols, it is the only logic with the following properties:

- $\mathrm{L} \leq \mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ and $\mathrm{L} = \mathrm{L}^{\boldsymbol{\mathcal{L}'}} \restriction Fm_{\boldsymbol{\mathcal{L}}}$
- $\operatorname{card} \mathrm{L} = \operatorname{card} \mathrm{L}^{\boldsymbol{\mathcal{L}'}}$

Observation 2.23. L and $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ have the same matrix models.

We are usually only interested in the cardinality of the set of variables. Thus, we also define *the natural extension of* L *to* κ*-many variables*, denoted as L^{κ}, to be an arbitrary natural extension $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$, where $Var_{\boldsymbol{\mathcal{L}'}}$ is of size κ.

The characterization (N) allows us to apply a general method to prove that syntactical properties are preserved for natural extensions. We demonstrate the method on compactness

Proposition 2.24. *Let* L *be a compact logic with enough variables. Then, every natural extension* L^{κ} *is also compact.*

Proof. Let $\mathcal{A}$ be a finite antitheorem in L. Clearly, the previous observation implies that it is also an antitheorem of L^{κ}. Let $\Gamma \subseteq Fm_{\mathcal{L}}(\kappa)$ be an inconsistent set in L^{κ}. In particular, $\Gamma \vdash_{\mathrm{L}^{\kappa}} \mathcal{A}$. Thus, using the cardinality assumption, we obtain $\Gamma' \vdash_{\mathrm{L}^{\kappa}} \mathcal{A}$, where Γ' is a $|Var_{\boldsymbol{\mathcal{L}}}|$-small subset of Γ. Denote V the set of all variables occurring in $\Gamma' \cup \mathcal{A}$. This set is $|Var_{\boldsymbol{\mathcal{L}}}|$-small. This allows us to find a substitution σ on $Fm_{\mathcal{L}}(\kappa)$, which embeds V into $Var_{\boldsymbol{\mathcal{L}}}$. Then, $\sigma[\Gamma'] \vdash_{\mathrm{L}^{\kappa}} \sigma[\mathcal{A}]$ and conservativity of the extension imply $\sigma[\Gamma'] \vdash_{\mathrm{L}} \sigma[\mathcal{A}]$.

Since antitheorems are closed under substitution, we can apply compactness of L and obtain a finite $\Delta \subseteq \Gamma'$ such that $\sigma[\Delta]$ is an antitheorem in Land, consequently, also in L^{κ}. We can find a substitution σ' such that $\sigma' \circ \sigma$ is identity on V. In particular, $\sigma' \circ \sigma[\Delta]$ is a finite subset of Γ and again the closure of antitheorem under substitutions ensures it is an antitheorem itself—just as we wanted. □

In general, assumption of having enough variables allows to restrict ourselves to rules which are using at most as many variables as the starting logic. Thus, by a suitable substitution, they are essentially the rules of the starting logic.

2.8.2 Expansions

Now, instead of adding variables, we consider logics with additional connectives. Let us fix a logic L in a language $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}}} \rangle$ and its type extension $\boldsymbol{\mathcal{L}'} = \langle \mathcal{L}', \mathit{Var}_{\boldsymbol{\mathcal{L}}} \rangle$.

Definition 2.25. The *natural expansion*[4] *of* L *to* $\boldsymbol{\mathcal{L}'}$, denoted as $\mathrm{L}_{\boldsymbol{\mathcal{L}'}}$, is the logic axiomatized by taking all $\boldsymbol{\mathcal{L}'}$-substitutions of an arbitrary presentation of L.

Arguing as in [78, Proposition 7], we can describe some of the fundamental properties of $\mathrm{L}_{\boldsymbol{\mathcal{L}'}}$ by means of its semantical characterization (*):

Proposition 2.26. $\mathrm{L}_{\boldsymbol{\mathcal{L}'}}$ *is the smallest conservative expansion of* L *to the language* $\boldsymbol{\mathcal{L}'}$ *with the same cardinality.*

Proof. Define S in the language $\boldsymbol{\mathcal{L}'}$ semantically as the logic of the following class of matrices

$$\{\langle \boldsymbol{A}, F \rangle \mid \boldsymbol{A} \text{ an } \mathcal{L}'\text{-algebra and } \langle \boldsymbol{A} \upharpoonright \mathcal{L}, F \rangle \in \mathbf{Mod}\,\mathrm{L}\}. \tag{*}$$

We now show that S has all the properties mentioned in the statement of the proposition. By definition, S is a conservative expansion of L to $\boldsymbol{\mathcal{L}'}$. Moreover, it is the smallest expansion: To this end first observe that $\mathbf{Mod}\,\mathrm{L} = \mathbf{Mod}\,\mathrm{S} \upharpoonright \mathcal{L}$; the inclusion from left to right is by definition and the converse one is true because $\mathrm{L} \subseteq \mathrm{S}$. Thus, if L' is any expansion of L to $\boldsymbol{\mathcal{L}}'$, then $\mathbf{Mod}\,\mathrm{L}' \upharpoonright \mathcal{L} \subseteq \mathbf{Mod}\,\mathrm{L} = \mathbf{Mod}\,\mathrm{S} \upharpoonright \mathcal{L}$. It easily follows that $\mathrm{S} \subseteq \mathrm{L}'$. Let S' denote the restriction of S to consecutions with less than $\operatorname{card}\mathrm{L}$-many

[4] We choose the terminology "natural expansion" because it aptly captures the meaning of the notion and its resemblance to natural extensions, despite the fact that it was already used in the literature for different purposes (cf. [16]).

premises. Then, since it is obviously an expansion of L, we obtain $\mathrm{S} \subseteq \mathrm{S}'$; the other direction is clear. In particular, L and S have the same cardinality.

Finally, since $\mathrm{L}_{\mathcal{L}'}$ is clearly the smallest expansion of L to $\boldsymbol{\mathcal{L}'}$, we obtain $\mathrm{S} = \mathrm{L}_{\mathcal{L}'}$. In particular, $\mathrm{L}_{\mathcal{L}'}$ has all the desired properties. □

Now we aim at developing a link between natural extensions and expansions (Proposition 2.29). Recall that the assumption (As1) on L entails a useful characterization for its natural extensions by means of (N); we will also prove an analogous characterization for natural expansions (Proposition 2.30). To this end, define the following cardinal

$$\mathfrak{e} = \begin{cases} |Fm_{\mathcal{L}'}| = \max\{|\mathit{Var}_{\mathcal{L}}|, |\mathcal{L}'|\} & \text{if } \mathcal{L}' \setminus \mathcal{L} \text{ has a non-nullary connective,} \\ |\mathcal{L}' \setminus \mathcal{L}| & \text{otherwise.} \end{cases}$$

and define the following set of $\boldsymbol{\mathcal{L}'}$-formulas:

$$X_{\mathcal{L}}^{\mathcal{L}'} = \{c(\varphi_1, \dots, \varphi_n) \in Fm_{\mathcal{L}'} \mid \varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}'} \text{ and } c \in \mathcal{L}' \setminus \mathcal{L}\}$$

and observe that $\mathfrak{e} = |X_{\mathcal{L}}^{\mathcal{L}'}|$. Finally, define the following set of variables:

$$\mathit{Var}_{\mathcal{S}} = \mathit{Var}_{\mathcal{L}} \cup \{x_\varphi \mid \varphi \in X_{\mathcal{L}}^{\mathcal{L}'}\} \tag{2.2}$$

Thus, $\mathit{Var}_{\mathcal{S}}$ has a new variable for every formula of the new language starting with a new connective. Now we show that $\mathrm{L}_{\mathcal{L}'}$, the natural expansion of L to $\mathcal{L}'$, and $\mathrm{L}^{\mathcal{S}}$, the natural extension of L to variables $\mathit{Var}_{\mathcal{S}}$ (i.e. to the language $\boldsymbol{\mathcal{S}} = \langle \mathcal{L}, \mathit{Var}_{\mathcal{S}} \rangle$) are actually the same logics modulo a certain translation $\boldsymbol{\tau}$.

A map $h : \boldsymbol{A} \to \boldsymbol{B}$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $\boldsymbol{B}$ an $\mathcal{L}'$-algebra, is called an *$\mathcal{L}$-homomorphism*, if it is a homomorphism between $\boldsymbol{A}$ and the $\mathcal{L}$-reduct of $\boldsymbol{B}$. Then, the translation $\boldsymbol{\tau}$ is defined as an $\mathcal{L}$-homomorphism $\boldsymbol{\tau} \colon \boldsymbol{Fm}_{\mathcal{S}} \to \boldsymbol{Fm}_{\mathcal{L}'}$ by

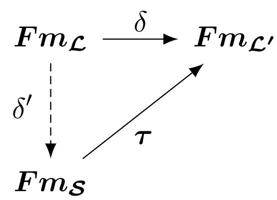
$$\boldsymbol{\tau}(x) = \begin{cases} x & \text{if } x \in \mathit{Var}_{\mathcal{L}} \\ \varphi & \text{if } x = x_\varphi \in X_{\mathcal{L}}^{\mathcal{L}'} \end{cases} \tag{2.3}$$

Moreover, define recursively a map $\boldsymbol{\tau'} \colon Fm_{\mathcal{L}'} \to Fm_{\mathcal{S}}$ as follows: $\boldsymbol{\tau'}(x) = x$, $\boldsymbol{\tau'}(c) = c$ for each constant of $\mathcal{L}$ and $\boldsymbol{\tau'}(c) = x_c$ for each new constant. If c is an n-ary connective of $\mathcal{L}$ and $\varphi = c(\varphi_1, \dots, \varphi_n)$, then $\boldsymbol{\tau'}(\varphi) = c(\boldsymbol{\tau'}(\varphi_1), \dots, \boldsymbol{\tau'}(\varphi_n))$. If c is a new n-ary connective and $\varphi = c(\varphi_1, \dots, \varphi_n)$, then $\boldsymbol{\tau'}(\varphi) = x_\varphi$. Using induction it is easy to prove:

Lemma 2.27. *$\boldsymbol{\tau}$ is a bijection from $Fm_{\mathcal{S}}$ onto $Fm_{\mathcal{L}'}$ with inverse $\boldsymbol{\tau'}$.*

Therefore, the formulas of $\mathrm{L}^{\mathcal{S}}$ and $\mathrm{L}_{\mathcal{L}'}$ are in a bijective correspondence.

Lemma 2.28. *For every $\mathcal{L}$-homomorphism $\delta\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'}$, there is a homomorphism $\delta'\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}}$ such that $\delta = \boldsymbol{\tau} \circ \delta'$, that is, the following diagram commutes:*



Proof. By the previous lemma it is enough to set $\delta'(x) = \boldsymbol{\tau'}\delta(x)$. □

Proposition 2.29. *For any fomulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{\mathcal{S}}}$, we have*

$$\Gamma \vdash_{\mathrm{L}^{\boldsymbol{\mathcal{S}}}} \varphi \iff \boldsymbol{\tau}[\Gamma] \vdash_{\mathrm{L}_{\mathcal{L}'}} \boldsymbol{\tau}(\varphi). \tag{2.4}$$

Proof. By Lemma 2.27, it is enough to show that the translations $\boldsymbol{\tau}$ and $\boldsymbol{\tau'}$ preserve proofs. First, suppose $\Delta \rhd \psi$ is a rule of $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$. By definition of the logic $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$, there is a rule $\Delta' \rhd \psi'$ of L and a homomorphism $h\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}}$ such that $h[\Delta'] = \Delta$ and $h(\psi') = \psi$. Then, $\boldsymbol{\tau}h$ witnesses that $\boldsymbol{\tau}[\Delta] \rhd \boldsymbol{\tau}(\psi)$ is a rule of $\mathrm{L}_{\mathcal{L}'}$. Conversely, let $\Delta \rhd \psi$ be a rule of $\mathrm{L}_{\mathcal{L}'}$. By definition of the logic $\mathrm{L}_{\mathcal{L}'}$, there is a rule $\Delta' \rhd \psi'$ of L and $\mathcal{L}$-homomorphism $\delta\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'}$ such that $\delta[\Delta'] = \Delta$ and $\delta(\psi') = \psi$. Let δ' be as in Lemma 2.28; then obviously $\delta'[\Delta'] \rhd \delta'(\psi')$ is a rule of $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ equal to $\boldsymbol{\tau'}[\Delta] \rhd \boldsymbol{\tau'}(\psi)$. □

Proposition 2.30. *Suppose that either* $\operatorname{card}\mathrm{L} \leq |\,Var_{\mathcal{L}}|^{+}$ *or* $\mathfrak{e} \leq |\,Var_{\mathcal{L}}|$. *Then, the natural expansion of* L *to the language* $\boldsymbol{\mathcal{L}'}$ *can be characterized as:*

$$\Gamma \vdash_{\mathrm{L}_{\mathcal{L}'}} \varphi \iff \begin{array}{l}\textit{there is an } \boldsymbol{\mathcal{L}}\textit{-homomorphism } \sigma\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'},\\ \textit{and a set of formulas } \Delta \cup \{\psi\} \subseteq Fm_{\mathcal{L}} \textit{ such that}\\ \Delta \vdash_{\mathrm{L}} \psi, \sigma[\Delta] \subseteq \Gamma, \textit{and } \sigma(\psi) = \varphi.\end{array} \tag{M}$$

Proof. Take $Var_{\boldsymbol{\mathcal{S}}}$ as in (2.2). Then, the logic $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ satisfies the assumptions (As1), since if $\mathfrak{e} \leq |\,Var_{\mathcal{L}}|$ then $|\,Var_{\boldsymbol{\mathcal{S}}}| = |\,Var_{\mathcal{L}}|$. Then, using the fact that $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ is characterized by (N), one can, similarly as in Proposition 2.29, obtain the desired characterization of $\mathrm{L}_{\mathcal{L}'}$. □

Note that, as in the case of natural extensions, the conditions of the previous proposition are there to ensure that the relation defined by the right side of (M) satisfies (Cut). The conditions are necessary: indeed, thanks to Proposition 2.29 (extending by variables is basically the same as expanding by constants), we can use the same counterexample as in [28]. On the other hand, not even under the assumptions of the previous proposition, we can guarantee that $\mathrm{L}_{\mathcal{L}'}$ is *the unique* conservative natural expansion with the same cardinality. Indeed, let L be the least logic in $\boldsymbol{\mathcal{L}}$. Then, L with an additional new constant c which is also added as an axiom c has all the properties mentioned above (and it is different from $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$).

We can capture the translatability between natural extensions and expansions by means of the following notion.

Definition 2.31. Let L and L' be logics in languages with $\mathcal{L} \subseteq \mathcal{L}'$ and with variables $Var_{\boldsymbol{\mathcal{L}}}$ and $Var_{\boldsymbol{\mathcal{L}'}}$, respectively. We say that L *isomorphically embeds* into L', in symbols $\mathrm{L} \precsim \mathrm{L}'$, if there is an isomorphism $\boldsymbol{\tau}\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}'}} {\restriction} \mathcal{L}$ and for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \boldsymbol{\tau}[\Gamma] \vdash_{\mathrm{L}'} \boldsymbol{\tau}(\varphi). \tag{2.5}$$

In the conditions of the previous definition, we denote $V = \boldsymbol{\tau}^{-1}[Var_{\boldsymbol{\mathcal{L}'}}]$; obviously $V \subseteq Var_{\boldsymbol{\mathcal{L}}}$. It is easy to see that L' is an expansion of $\mathrm{S} = \mathrm{L}' {\restriction} \mathcal{L}$ and L is a conservative extension of S, obtained by extending the set of variables V to $Var_{\boldsymbol{\mathcal{L}}}$. Moreover, if L and S have the same cardinality and S satisfies (As1), then L' is the natural expansion of S to $\boldsymbol{\mathcal{L}'}$.

In particular, $\mathrm{L}^{\boldsymbol{S}}$ isomorphically embeds into $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$, that is $\mathrm{L}^{\boldsymbol{S}} \precsim \mathrm{L}_{\boldsymbol{\mathcal{L}}'}$, where $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ is the natural expansion of L to $\boldsymbol{\mathcal{L}'}$ and $\mathrm{L}^{\boldsymbol{S}}$ the corresponding natural extension to variables $Var_{\boldsymbol{S}}$ described in this subsection.

Proposition 2.32. *If* $\mathrm{L} \precsim \mathrm{L}'$, *then* $\mathrm{Th}\,\mathrm{L}$ *and* $\mathrm{Th}\,\mathrm{L}'$ *are isomorphic lattices.*

Proof. Let τ witness $\mathrm{L} \precsim \mathrm{L}'$ and let τ' be its inverse. Lift these functions in the obvious way to $\boldsymbol{\tau} : P(Fm_{\boldsymbol{\mathcal{L}}}) \leftrightarrows P(Fm_{\boldsymbol{\mathcal{L}}'}) : \boldsymbol{\tau'}$. These lifted mappings are as well inverse to each other and monotonous. Moreover, (2.5) ensures that $\boldsymbol{\tau}[\mathrm{Th}\,\mathrm{L}] \subseteq \mathrm{Th}\,\mathrm{L}'$ and $\boldsymbol{\tau'}[\mathrm{Th}\,\mathrm{L}'] \subseteq \mathrm{Th}\,\mathrm{L}$. Thus, $\boldsymbol{\tau}$ restricted to theories is the desired lattice isomorphism. $\square$

Consequently, if two logics are in the relation $\mathrm{L} \precsim \mathrm{L}'$, then they share lattice theoretical properties.

Part I:

Hierarchy of Infinitary Logics

3 | The Hierarchy

In this chapter we first investigate two basic kinds of closed sets, which in logical setting were defined and studied in [35], completely intersection-prime and intersection-prime closed sets. We associate to each of these kinds a corresponding extension property (recall the formulation of the abstract Lindenbaum lemma). The completely intersection-prime extension property (CIPEP) and the intersection-prime extension property (IPEP), respectively. The IPEP was introduced in [27] and later frequently used also in [29, 30], though not yet systematically studied. Its main merit is that it allows to export to infinitary logics typical properties of finitary logics such as the completeness w.r.t. to finitely subdirectly irreducible models (RFSI-completeness). The CIPEP is a natural strengthening of the IPEP ensuring RSI-completeness. The goal of this chapter is to initiate a systematical study of these properties as well as two additional ones, the τ-IPEP and τ-CIPEP, which are the semantical (transferred) counterparts of the extension properties. The basic relations of these properties are depicted in Figure 3.1.

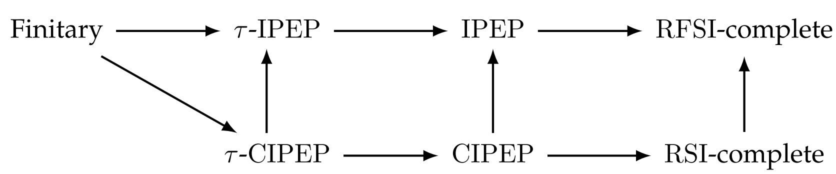


Figure 3.1: Basic relations of the classes

After a general presentation, we provide a semantical characterization for the extensions properties via *surjective completeness*. Then, we explain the relation of the transferred extension properties to subdirect representation. Finally, we study the preservation of these properties under expansions.

It is a non-trivial problem to determine whether the implications in Figure 3.1 can be reversed or not. This issue is solved in Chapter 4, where we see that all the properties are actually pairwise different. Hence, Figure 3.1 depicts a new hierarchy of (in)finitary propositional logics.

3.1 The classes

We start with the formal definitions of the classes in the hierarchy. In general, given a closure system $\mathcal{C}$ on a set A, a set $X \in \mathcal{C}$ is called *intersection-prime in* $\mathcal{C}$ if it is finitely $\cap$-irreducible, i.e. there are no closed sets $X_1, X_2 \in \mathcal{C}$ such that $X = X_1 \cap X_2$ and $X \subsetneq X_1, X_2$. Similarly, X is *completely intersection-prime in* $\mathcal{C}$ if it is $\cap$-irreducible, i.e. whenever $X = \bigcap_{i \in I} X_i$ for a family of closed sets $\{X_i \mid i \in I\} \subseteq \mathcal{C}$, there is $i_0 \in I$ such that $X = X_{i_0}$. Given a logic L, an algebra $\boldsymbol{A}$, and a filter F, we say that F is *(completely) intersection-prime*[1] if it is (completely) intersection-prime in $\mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$; it is analogously defined for theories and $\mathrm{Th}\,\mathrm{L}$. We say $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ is a RSI-model (resp. RFSI-model) if F is completely intersection-prime (resp. intersection-prime)—the terminology is motivated by Lemmas 3.4 and 3.5 below.

Given a closure system $\mathcal{C}$ on A, a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalently formulated as an extension property: for every $X \in \mathcal{C}$ and every $a \in A \setminus X$ there is $Y \in \mathcal{B}$ such that $X \subseteq Y$ and $a \notin Y$). Using these notions one can define the following properties for closure systems and for logics.

We say that a closure system $\mathcal{C}$ has the *(completely) intersection-prime extension property*, (C)IPEP for short, if the (completely) intersection-prime closed sets form a basis of $\mathcal{C}$. A logic L has the (C)IPEP if $\mathrm{Th}\,\mathrm{L}$ does. And finally, L has *transferred*-(C)IPEP, τ-(C)IPEP for short, if for every algebra $\boldsymbol{A}$ the closure system $\mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$ has the (C)IPEP.

We are now going to explain the basic and easy to prove relations between the above defined classes of logics as depicted in figure 3.1. Clearly, (C)IPEP is just a special case of τ-(C)IPEP where the algebra $\boldsymbol{A}$ is the algebra of formulas $\boldsymbol{Fm}_{\mathcal{L}}$. To understand the upward implications it is enough to realize that completely intersection-prime closed sets are always intersection-prime. To prove the rest, we first demonstrate some basic properties of the (completely) intersection-prime closed sets.

Lemma 3.1. *For a closure system $\mathcal{C}$ on A, the following are equivalent for every closed set X*

(i) *X is completely intersection-prime.*
(ii) *X is* saturated*, that is there is $a \in A$ such that X is a maximal closed set not containing a. Any such element a is said to* saturate X.

Lemma 3.2. *Every inductive closure system $\mathcal{C}$ on A has the* CIPEP.

Proof. Suppose $X \in \mathcal{C}$ and $a \notin X$ for some $a \in A$ and define

[1] We follow the terminology used in [35, p. 147].

$$\mathcal{D} = \{Y \in \mathcal{C} \mid X \subseteq Y \text{ and } a \notin Y\}.$$

We argue by maximality principle over $(\mathcal{D}, \subseteq)$. To this end, let $\mathcal{E} \subseteq \mathcal{D}$ be a chain, but, since $\mathcal{E}$ is directed and $\mathcal{C}$ is inductive, we can conclude that $\bigcup \mathcal{E} \in \mathcal{D}$. Then, by the previous lemma, the maximal element $Y_0 \in \mathcal{D}$ is completely intersection-prime. □

Proposition 3.3. *Every finitary logic* L *has the* τ-CIPEP.

Proof. Since finitarity transfers to all models (cf. Corollary 2.20), the closure system $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ is inductive on every algebra $\boldsymbol{A}$. Thus, Lemma 3.2 concludes the proof. □

Reduced R(F)SI-models are nothing else than models with (completely) intersection-prime filters:

Lemma 3.4. *For every* $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}\,\mathrm{L}$*:*

- $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L} \iff$ *F is intersection-prime in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$,*
- $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}_{\mathrm{RSI}}\,\mathrm{L} \iff$ *F is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$.*

Proof. Right to left: suppose $i\colon \langle \boldsymbol{A}, F\rangle \hookrightarrow_{\mathbf{SD}} \prod \langle \boldsymbol{B}_i, G_i\rangle$ is a subdirect representation and let $\pi_i\colon \prod\langle \boldsymbol{B}_i, G_i\rangle \to \langle \boldsymbol{B}_i, G_i\rangle$ be the ith projection. It is easy to check that $F = \bigcap (\pi_i \circ i)^{-1}[G_i]$. Thus, by the assumption $F = (\pi_{i_0} \circ i_0)^{-1}[G_{i_0}]$ for some i_0. In particular, $\pi_{i_0} \circ i_0 \colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}_{i_0}, G_{i_0}\rangle$ is strict and surjective and, consequently, since $\langle \boldsymbol{A}, F\rangle$ is reduced, it is injective and, hence, an isomorphism (note that $\operatorname{Ker} \pi_{i_0} \circ i_0 \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}$).

Conversely, suppose $F = \bigcap F_i$. We can check easily that $i\colon \langle \boldsymbol{A}, F\rangle \to \prod \langle \boldsymbol{A}, F_i\rangle^{*}$, where $i(a)$ is the sequence of equivalence classes $\langle [a]/\boldsymbol{\Omega}^{\boldsymbol{A}} F_i\rangle$, is a subdirect representation. Thus, $\pi_{i_0} \circ i_0$ is an isomorphism for some i_0, hence $F = (\pi_{i_0} \circ i_0)^{-1}[F/\boldsymbol{\Omega}^{\boldsymbol{A}} F_i] = F_i$. □

Lemma 3.5. *If* $\mathbf{A} \in \mathbf{Mod}_{\mathrm{R(F)SI}}\,\mathrm{L}$*, then* $\mathbf{A}^{*} \in \mathbf{Mod}^{*}_{\mathrm{R(F)SI}}\,\mathrm{L}$*. Moreover, if* L *is protoalgebraic, then also the reverse implication holds.*

Proof. We check $F/\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} F$. Suppose $F/\boldsymbol{\Omega}^{\boldsymbol{A}} F = \bigcap F_i$. Then, clearly $F = \bigcap r^{-1}[F_i]$, where r is the reduction map. Therefore, since F is by assumption completely intersection-prime, we get $F = r^{-1}[F_{i_0}]$ for some i_0. Hence $F/\boldsymbol{\Omega}^{\boldsymbol{A}} F = r[F] = F_{i_0}$. □

Proposition 3.6. *For every logic* L*, the* CIPEP *implies* RSI*-completeness and the* IPEP *implies* RFSI*-completeness.*

Proof. We prove only the case of the IPEP (the other can be proved analogously). The soundness is trivial, since by the definition, $\mathbf{Mod}^{*}_{\mathrm{RSI}}\,\mathrm{L} \subseteq \mathbf{Mod}\,\mathrm{L}$. Suppose $\Gamma \nvdash_{\mathrm{L}} \varphi$, by the IPEP, there is intersection-prime theory $T \supseteq \Gamma$ such that $T \nvdash_{\mathrm{L}} \varphi$. By the previous $\langle \mathit{Fm}_{\boldsymbol{\mathcal{L}}}, T\rangle^{*} \in \mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L}$ and since every matrix and its reduction define the same logic, we are done. □

3.1.1 Surjective completeness

In this subsection we provide a rather straightforward but useful semantical characterization of CIPEP and IPEP for protoalgebraic logics via a notion of *surjective completeness*. This notion already appeared in [37] in the context of equational consequence. We use this characterization to show that there exist infinitary logics with the CIPEP (and thus also with the IPEP).

Definition 3.7. A formula φ is a *surjective semantical consequence* of a set of formulas Γ w.r.t. a class $\mathbb{K}$ of $\boldsymbol{\mathcal{L}}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each surjective $\boldsymbol{A}$-evaluation e (surjective as a function $e : Fm_{\boldsymbol{\mathcal{L}}} \to A$), we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models^s_{\mathbb{K}} \varphi$.

First notice that $\models_{\mathbb{K}} \subseteq \models^s_{\mathbb{K}}$. Moreover, it is easy to show that $\models^s_{\mathbb{K}}$ is a consequence relation. However, it is not necessarily structural, and hence not necessarily a logic, as shown by the following example.

Example 3.8. Let $\mathbf{A}$ be the matrix $\langle \mathbf{Ł}_3^{\to}, \{1\}\rangle$, where $\mathbf{Ł}_3^{\to}$ stands for the implication fragment of the standard 3-element Łukasiewicz algebra. Let $\Gamma = \{p \to q, q \to p\} \cup \{r \in Var \mid r \neq q$ and $r \neq p\}$. It can easily be seen that $\Gamma \models^s_{\{\mathbf{A}\}} p$, since there is no surjective evaluation satisfying Γ. On the other hand $\{p \to p, q\} \not\models^s_{\{\mathbf{A}\}} p$. Thus, $\models^s_{\{\mathbf{A}\}}$ is not structural.

We use the notion of cardinality of L to characterize sufficient conditions under which $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$, hence conditions under which $\models^s_{\mathbb{K}}$ is indeed a logic.

Proposition 3.9. *Let κ be an infinite cardinal and $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Assume that $|Var| = \kappa$, $\mathrm{card} \models^s_{\mathbb{K}} \leq \kappa$, and $|A| \leq \kappa$ for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$. Then, $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$ and, in particular, $\models^s_{\mathbb{K}}$ is structural.*

Proof. The inclusion $\supseteq$ trivially holds always. Suppose that $\Gamma \models^s_{\mathbb{K}} \varphi$. Then, we obtain a set $\Gamma' \subseteq \Gamma$ of cardinality less than κ such that $\Gamma' \models^s_{\mathbb{K}} \varphi$. We claim that $\Gamma' \models_{\mathbb{K}} \varphi$ and consequently also $\Gamma \models_{\mathbb{K}} \varphi$. Consider any $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and any evaluation e on $\boldsymbol{A}$ such that $e[\Gamma'] \subseteq F$. Since $\Gamma' \cup \{\varphi\}$ contains less than κ variables, we can easily find a surjective evaluation e' which coincides with e on all variables occurring in $\Gamma' \cup \{\varphi\}$. Obviously, we have $e'[\Gamma'] \subseteq F$ and thus also $e(\varphi) = e'(\varphi) \in F$. $\square$

In the next proposition, we refine the usual completeness results using surjective consequence relations.

Proposition 3.10. *Let* L *be a logic. Then:*

$$\mathrm{L} = \models^{s}_{\mathbf{Mod}\,\mathrm{L}} = \models^{s}_{\mathbf{Mod}^{*}\,\mathrm{L}}.$$

Moreover,

(i) *if* L *has the* IPEP, *then* $\mathrm{L} = \models^{s}_{\mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L}}$,
(ii) *if* L *has the* CIPEP, *then* $\mathrm{L} = \models^{s}_{\mathbf{Mod}^{*}_{\mathrm{RSI}}\,\mathrm{L}}$.

Proof. It is enough to observe that the evaluations (reduction maps) used in the proof of completeness w.r.t. reduced models are in fact surjective—see the proof of Proposition 3.6. □

Next, we prove the characterization result for the CIPEP and the IPEP via surjective completeness.

Proposition 3.11. *Let* L *be a protoalgebraic logic. Then:*

(i) L *has the* IPEP *if and only if* $\mathrm{L} = \models^{s}_{\mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L}} = \models^{s}_{\mathbf{Mod}_{\mathrm{RFSI}}\,\mathrm{L}}$.
(ii) L *has the* CIPEP *if and only if* $\mathrm{L} = \models^{s}_{\mathbf{Mod}^{*}_{\mathrm{RSI}}\,\mathrm{L}} = \models^{s}_{\mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}}$.

Proof. We prove only the second part of the theorem (the first one is identical). $\Rightarrow$: This implication is given by Proposition 3.10.
$\Leftarrow$: Suppose $\Gamma \nvdash_{\mathrm{L}} \varphi$. There is $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}$ and a surjective evaluation $e\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{A}$ such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. We know that $T = e^{-1}[F]$ is an L-theory. Then, by the correspondence theorem T is completely intersection-prime (because F is by Lemma 3.4). And since $\Gamma \subseteq T \not\ni \varphi$, we are done. □

In the next proposition we show that every usual consequence relation given by a class of matrices is actually a surjective consequence relation.

Proposition 3.12. *For every class of matrices* $\mathbb{K}$ *we have:* $\models_{\mathbb{K}} = \models^{s}_{\mathbf{S}(\mathbb{K})}$.

Proof. The direction $\subseteq$ holds always. The other is clear due to the fact that for every evaluation $e\colon \langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle \to \langle \boldsymbol{A}, F\rangle$ the image $\langle e[\boldsymbol{Fm_{\mathcal{L}}}], e[T]\rangle$ is a submatrix of $\mathbf{A}$. □

As an easy consequence of the previous proposition and Proposition 3.11, we obtain a useful sufficient condition for a logic to have the CIPEP (resp. IPEP).

Corollary 3.13. *Let* L *be a protoalgebraic logic and suppose that* $\mathbb{K}$ *is a class of matrices such that* $\mathrm{L} = \models_{\mathbb{K}}$. *Then:*

- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{Mod}_{\mathrm{RFSI}}\,\mathrm{L}$, *then* L *has the* IPEP,
- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}$, *then* L *has the* CIPEP.

Example 3.14. Both $Ł_\infty$ and Π_∞ have the CIPEP. It is not difficult to see that every subalgebra $\boldsymbol{A}$ of $[0,1]_Ł$ has at most two filters, namely $\{1\}$ and A, and every subalgebra $\boldsymbol{A}$ of $[0,1]_\Pi$ has at most 3 filters, namely $\{1\}, A \setminus \{\overline{0}^{\boldsymbol{A}}\}$, and A. Thus, the previous corollary applies.

3.2 Subdirect representation

In this section we consider the notion of subdirect representation, one of the cornerstones of universal algebra, in the framework of abstract algebraic logic. Namely, we say that a logic L is *(finitely) subdirectly representable* if the reduced models are subdirectly representable by the class of (finitely) subdirectly irreducibles models, in symbols: $\mathbf{Mod}^*\,\mathrm{L} = \mathbf{P_{SD}}(\mathbf{Mod}^*_{\mathrm{R(F)SI}}\,\mathrm{L})$. We provide a useful characterization for subdirect representability via natural extensions and we prove it is equivalent to τ-(C)IPEP and protoalgebraicity. As an example, we will show that $Ł_\infty$, and each of its axiomatic expansions, is subdirectly representable (equivalently, it has the τ-CIPEP). On the other hand, later in Section 4.2, we will see that the same cannot be said about Π_∞.

3.2.1 Subdirect representation in abstract algebraic logic

Let us prove first that, in protoalgebraic logics, surjective homomorphisms preserve the (C)IPEP.

Proposition 3.15. *Let* L *be a protoalgebraic logic in a language* $\boldsymbol{\mathcal{L}}$. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* $\mathcal{L}$*-algebras and* $h : \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ *be a surjective homomorphism. Then,* $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$ *has the* (C)IPEP *whenever* $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ *does.*

Proof. Assume $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ has the (C)IPEP and take $F \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$. Consider h as a strict surjective homomorphism between matrices

$$h : \langle \boldsymbol{A}, h^{-1}[F] \rangle \twoheadrightarrow \langle \boldsymbol{B}, F \rangle$$

The correspondence theorem of protoalgebraic logics then ensures that if $h^{-1}[F]$ can be decomposed as an intersection of (completely) intersection-prime filters, so can F. $\square$

Corollary 3.16. *Let* L *be a protoalgebraic logic in a language* $\mathcal{L}$, $|Var_{\mathcal{L}}| \leq \kappa$ *an infinite cardinal, and suppose* L^{κ} *has the* (C)IPEP*. Then,* $\mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$ *has the* (C)IPEP *for every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *with* $|A| \leq \kappa$*.*

Proof. Observation 2.23 clearly implies that $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}(\kappa))$ has the (C)IPEP if and only if $\mathrm{Th}\,\mathrm{L}^{\kappa}$ does (i.e. if and only if L^{κ} has the (C)IPEP). But there is a surjective $h : \boldsymbol{Fm}_{\mathcal{L}}(\kappa) \twoheadrightarrow \boldsymbol{A}$. Hence previous proposition concludes the proof. $\square$

As an easy consequence, we can obtain a useful characterization of the τ-IPEP and the τ-CIPEP in terms of natural extensions:

Corollary 3.17. *Let* L *be a protoalgebraic logic. Then, the following are equivalent:*

(i) L^{κ} *has the* (C)IPEP *for every* $\kappa \geq |Var_{\mathcal{L}}|$*,*
(ii) L *has* τ-(C)IPEP*.*

Moreover, the implication from bottom to top holds for each logic L*.*

Proof. The implication from (i) to (ii) simply follows from Corollary 3.16. The other one: τ-(C)IPEP implies that $\mathcal{F}i_{\mathrm{L}}(Fm_{\mathcal{L}}(\kappa))$ has the (C)IPEP, but then so does $\mathrm{Th}\,\mathrm{L}^{\kappa}$ (by Observation 2.23 they are in fact the same lattices). $\square$

We still need another auxiliary result connecting the transferred extension properties with a decomposition of filters in reduced models:

Proposition 3.18. *A protoalgebraic logic* L *has* τ-(C)IPEP *if, and only if, for each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}\,\mathrm{L}$*,* F *is an intersection of (completely) intersection-prime filters.*

Proof. The direction from left to right is obvious. For the other one consider $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ and let h be the reduction map:

$$h\colon \langle \boldsymbol{A}, F\rangle \twoheadrightarrow \langle \boldsymbol{A}, F\rangle^{*} = \langle \boldsymbol{A}^{*}, F^{*}\rangle.$$

Since L is protoalgebraic, h is strict and surjective, and $F^{*} = \bigcap_{i\in I} G_i$, where every G_i is intersection-prime, we obtain $F = \bigcap_{i\in I} h^{-1}[G_i]$, as we wanted. The correspondence theorem ensures that $h^{-1}[G_i]$ are (completely) intersection-prime. $\square$

Using all these elements now we are ready to prove the main result of this subsection.

Theorem 3.19. *For any logic* L*, the following are equivalent:*

(i) L *has the* τ-IPEP *and is protoalgebraic.*
(ii) $\mathbf{Mod^*}\,\mathrm{L} = \mathbf{P_{SD}}(\mathbf{Mod^*_{\mathrm{RFSI}}}\,\mathrm{L})$.

The same is true for τ-CIPEP *and* $\mathbf{Mod^*_{\mathrm{RSI}}}\,\mathrm{L}$.

Proof. We prove the case of the τ-IPEP (for the τ-CIPEP it is analogous).

(i) implies (ii): Take $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod^*}\,\mathrm{L}$. Then, by the τ-IPEP we have $F = \bigcap_{i\in I} F_i$, where each F_i is an intersection-prime filter. Therefore there is a natural subdirect representation:

$$h : \langle \boldsymbol{A}, F\rangle \hookrightarrow_{\mathbf{SD}} \prod_{i\in I} \langle \boldsymbol{A}, F_i\rangle^*.$$

By Lemma 3.5 we get that $\langle A, F_i\rangle^* \in \mathbf{Mod^*_{\mathrm{RFSI}}}\,\mathrm{L}$, which concludes the proof of the first inclusion. For the other one we have

$$\mathbf{P_{SD}}(\mathbf{Mod^*_{\mathrm{RFSI}}}\,\mathrm{L}) \subseteq \mathbf{P_{SD}}(\mathbf{Mod^*}\,\mathrm{L}) \subseteq \mathbf{Mod^*}\,\mathrm{L},$$

where the second inclusion is due to protoalgebraicity (in fact, it equivalent to it—see [46, Theorem 6.17]).

(ii) implies (i): Clearly $\mathbf{Mod^*}\,\mathrm{L}$ is closed under subdirect product, thus the logic is protoalgebraic. By Proposition 3.18, it is sufficient to prove that every filter F from a reduced model $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod^*}\,\mathrm{L}$ can be decomposed as an intersection of intersection-prime filters. By the assumption, there is a set of RFSI reduced models $\{\langle \boldsymbol{B}_i, G_i\rangle\}_{i\in I}$ and an embedding

$$h\colon \langle \boldsymbol{A}, F\rangle \hookrightarrow_{\mathbf{SD}} \prod_{i\in I} \langle \boldsymbol{B}_i, G_i\rangle.$$

Is is easy to verify that F is the intersection of all filters $F_i = (\pi_i \circ h)^{-1}[G_i]$, where π_i is the i-th projection of the product $\prod_{i\in I}\langle \boldsymbol{B}_i, G_i\rangle$. Moreover every filter F_i is intersection-prime, because G_i is intersection-prime, L is protoalgebraic, and $\pi_i \circ h$ is strict and surjective. □

Note that in the proof of (i)→(ii) protoalgebraicity is not necessary to obtain the inclusion $\mathbf{Mod^*}\,\mathrm{L} \subseteq \mathbf{P_{SD}}(\mathbf{Mod^*_{\mathrm{R(F)SI}}}\,\mathrm{L})$, thus in logics with the τ-(C)IPEP, the reduced models are always representable by subdirect products of R(F)SI-models.

3.2.2 Subdirect representation in universal algebra

In universal algebra, it is well known (Birkhoff's theorem; see e.g. [9]) that every variety V and quasi-variety Q of algebras can be described in terms of subdirect products of their (relatively) subdirectly irreducible members:

$$\mathsf{V} = \mathbf{P_{SD}}(\mathsf{V}_{\mathrm{SI}}) \quad \text{and} \quad \mathsf{Q} = \mathbf{P_{SD}}(\mathsf{Q}_{\mathrm{RSI}}).$$

Notice that there is a clear formal similarity with subdirect representation for models of logics. Finitary logics are (finitely) subdirectly representable. A natural question is whether the subdirect representations can be extended to infinitary logics and to more general classes of algebras. The example in Subsection 4.2 will give a negative answer to this question. On the other, hand in the next subsection we will meet a natural class of algebras, which is not a quasi-variety and is subdirectly representable.

In analogy with the development of the previous subsection, we can characterize when a given generalized quasi-variety is *(finitely) subdirectly representable*, that is, when $\mathsf{K} = \mathbf{P_{SD}}(\mathsf{K}_{\mathrm{RSI}})$ (resp. $\mathsf{K} = \mathbf{P_{SD}}(\mathsf{K}_{\mathrm{RFSI}})$).

Theorem 3.20. *For every generalized quasi-variety* K*, the following are equivalent:*

(i) K *is (finitely) subdirectly representable.*
(ii) $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ *has the* CIPEP *(resp.* IPEP*) for every algebra* $\boldsymbol{A}$*.*
(iii) $\mathrm{Con}_{\mathsf{K}}(\boldsymbol{Fm}_{\mathcal{L}}(\kappa))$ *has the* CIPEP *(resp.* IPEP*) for every cardinal* κ*.*

Note that, similarly to Proposition 3.18, the condition (ii) is equivalent to

(iv) For every $\boldsymbol{A} \in \mathsf{K}$, the identity congruence is an intersection of (completely) intersection-prime K-congruences.

3.2.3 Subdirect representation and topological continuity

In this subsection, we describe a particular method to prove the transferred versions of the extension properties. We first demonstrate the method on the infinitary Łukasiewicz logic. This logic is complete w.r.t. the matrix $\langle [0,1]_{Ł}, \{1\}\rangle$, which haas the following properties:

(i) There is a compact topology on [0,1] (the open interval topology).
(ii) The connectives of $Ł_\infty$ induce continuous functions w.r.t. that topology.

We will see that these properties allow to obtain a bound on the cardinality of the logic of $\langle [0,1]_{Ł}, \{1\}\rangle$ in the language with κ-many variables. We denote it $Ł_{\infty,\kappa}$. From this we will conclude that $Ł_\infty$ enjoys the τ-CIPEP.

Later, we will abstract the properties (i) and (ii) to obtain a general result.

Proposition 3.21. *Let κ be an infinite cardinal. Then, $Ł_{\infty,\kappa}$ has cardinality $\aleph_1$. Consequently, $Ł_{\infty,\kappa}$ is the natural extension of $Ł_\infty$ to κ-many variables, that is, $Ł_{\infty,\kappa} = Ł_\infty^\kappa$.*

Proof. Define for each formula φ and each rational $q \in (0,1)$ the following sets of evaluations:

$$\mathrm{NSAT}(\varphi) = \{v\colon \kappa \to [0,1] \mid v(\varphi) \neq 1\}$$

$$\mathrm{SAT}(\varphi) = \{v\colon \kappa \to [0,1] \mid v(\varphi) = 1\}$$

$$\mathrm{SAT}_q(\varphi) = \{v\colon \kappa \to [0,1] \mid v(\varphi) > q\}$$

Since the operations of Łukasiewicz logic are all continuous w.r.t. the standard interval topology on $[0,1]$, we obtain that for each φ and q the sets $\mathrm{NSAT}(\varphi)$ and $\mathrm{SAT}_q(\varphi)$ are open in $[0,1]^\kappa$, the topological product of κ-many copies of $[0,1]$. This follows from the fact that we can see every formula φ as a *continuous* mapping $\boldsymbol{\varphi} : [0,1]^\kappa \to [0,1]$ such that $\boldsymbol{\varphi}(v) = v(\varphi)$, thus for example $\mathrm{SAT}_q(\varphi) = \boldsymbol{\varphi}^{-1}[\Uparrow q]$, where $\Uparrow q = \{r \in [0,1] \mid q < r\}$, which is, of course, an open set.

Moreover, for every set of formulas Δ and every formula χ, we have the following equivalence:

$$\Delta \vdash_{Ł_{\infty,\kappa}} \chi \iff \bigcup_{\psi \in \Delta} \mathrm{NSAT}(\psi) \cup \mathrm{SAT}(\chi) = [0,1]^\kappa \tag{3.1}$$

Furthermore, since the filter $\{1\}$ can be obtained as the intersection of countably many sets of the form $\Uparrow q$, for each rational $q \in (0,1)$, it follows that $\mathrm{SAT}(\varphi)$ is an intersection of countably many open sets:

$$\mathrm{SAT}(\varphi) = \bigcap_{q \in (0,1)} \mathrm{SAT}_q(\varphi) \tag{3.2}$$

Clearly, since $Ł_{\infty,\kappa}$ is a conservative extension of $Ł_\infty$, it has cardinality at least $\aleph_1$. To prove the other inequality assume $\Gamma \vdash_{Ł_{\infty,\kappa}} \varphi$. We need to show that there is a countable $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{Ł_\infty^\kappa} \varphi$. From (3.1) we obtain

$$\bigcup_{\gamma \in \Gamma} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}(\varphi) = [0,1]^\kappa.$$

Then, for any rational q, since obviously $\mathrm{SAT}(\varphi) \subseteq \mathrm{SAT}_q(\varphi)$, we obtain

$$\bigcup_{\gamma \in \Gamma} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}_q(\varphi) = [0,1]^\kappa.$$

Thus, we have an open cover of $[0,1]^\kappa$. Therefore, by compactness, we obtain a finite $\Gamma_q \subseteq \Gamma$ that generates a subcover. Consequently, we obtain a countable set $\Gamma' = \bigcup_{q\in[0,1]\cap\mathbb{Q}} \Gamma_q$. Then, using (3.2), it is easy to see that

$$\bigcup_{\gamma\in\Gamma'} \mathrm{NSAT}(\gamma) \cup \mathrm{SAT}(\varphi) = [0,1]^\kappa,$$

which, by (3.1), implies $\Gamma' \vdash_{Ł_{\infty,\kappa}} \varphi$. It follows that $Ł_{\infty,\kappa}$ is the unique natural extension of $Ł_\infty$ (cf. Subsection 2.8.1). □

Theorem 3.22. $Ł_\infty$ *has the* τ-CIPEP *and so does each of its axiomatic expansions.*

Proof. By Corollary 3.17 it is enough to show that $Ł_\infty^\kappa$ has the CIPEP for every infinite κ. By Proposition 3.21 we have $Ł_\infty^\kappa = \models_{\langle[0,1]_Ł,\{1\}\rangle}$ thus we can use Corollary 3.13 as in Example 3.14. The part about axiomatic expansions is due to Theorem 3.30. □

From the previous theorem and Theorem 3.19 we obtain:

Corollary 3.23. *Every axiomatic expansion* L *of* $Ł_\infty$ *is (finitely) subdirectly representable, that is:*

$$\mathbf{Mod}^*\,\mathrm{L} = \mathbf{P_{SD}}(\mathbf{Mod}^*_{\mathrm{R(F)SI}}\,\mathrm{L}).$$

Consequently, we obtain an analogous result for the equivalent algebraic semantics of $Ł_\infty$:

$$\mathbf{Alg}^*Ł_\infty = \mathbf{P_{SD}}(\mathbf{Alg}^*Ł_{\infty\mathrm{R(F)SI}}). \tag{3.3}$$

This shows that $\mathbf{Alg}^*Ł_\infty$ is an example of a subdirectly representable class of algebras, where the representation theorem is not a consequence of Birkhoff's theorem (recall that $\mathbf{Alg}^*Ł_\infty$ is not a quasi-variety). The same is true for every infinitary axiomatic expansion of $Ł_\infty$.

Moreover, as another consequence of Proposition 3.21, we can obtain a description of the class $\mathbf{Alg}^*Ł_\infty$.

Proposition 3.24. *The equivalent algebraic semantics of* $Ł_\infty$ *is the prevariety generated by the algebra* $[0,1]_Ł$*, that is*

$$\mathbf{Alg}^*Ł_\infty = \mathbf{ISP}([0,1]_Ł).$$

Proof. The inclusion from right to left clearly holds (recall, for instance, that $\mathbf{Alg}^*Ł_\infty = \mathbf{ISP}_{\mathbf{R}_{\aleph_1}}([0,1]_Ł)$). For the other inclusion we can prove

$$\mathbf{Alg}^*Ł_\infty = \mathbf{P_{SD}}(\mathbf{Alg}^*Ł_{\infty\mathrm{RSI}}) \subseteq \mathbf{ISP}([0,1]_Ł),$$

where the equality is (3.3) and the inclusion follows from Proposition 2.22, which says that $\mathbf{Alg}^*Ł_{\infty\mathrm{RSI}} \subseteq \mathbf{IS}([0,1]_Ł)$. □

The proof of Proposition 3.21 suggests a general methodology to obtain an upper bound for the cardinality of a logic in κ-many variables defined by a class of matrices $\mathbb{K}$, call it $\mathrm{L}_{\mathbb{K},\kappa}$. The proof of the theorem is omitted, since it can be easily abstracted from that of Proposition 3.21.

Theorem 3.25. *Suppose λ is a regular cardinal and $\mathbb{K}$ is a class of matrices, such that $|\mathbb{K}| < \lambda$. Then, for every cardinal κ the logic $\mathrm{L}_{\mathbb{K},\kappa}$ has cardinality at most λ whenever the following conditions hold for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$:*

(i) *There is a compact topology τ on A such that all connectives are interpreted by continuous functions w.r.t. τ.*
(ii) *F can be written as an intersection of strictly less than λ open sets in τ.*
(iii) *$A \setminus F$ is open in τ.*

We will use this theorem later in Section 4.3. A special case of the theorem is a well-known result saying that *strongly finite* logics, i.e. those complete w.r.t. finite class of finite matrices, are finitary.

Interestingly enough, we used topological compactness to argue for larger cardinalities than simple finitarity on the logical side. A natural question arises: would a weaker notion of topological compactness suffice? Recall that a topological space is *Lindelöf* if each of its open covers has a countable subcover. In fact, for our proof it would be enough if A^κ was Lindelöf (note that, similarly to compact spaces, any closed subspace of a Lindelöf space is Lindelöf). However, unsurprisingly, these spaces are not as well behaved as the compact ones: in general, they are not even closed under finite products—the standard example of this phenomenon is the Sorgenfrey plane which is the product of two Sorgenfrey lines (see e.g. [41]). Moreover, it can be proved that the following are equivalent for every Hausdorff space A:[2]

(i) A is compact,
(ii) A^κ is Lindelöf for every cardinal κ.

Thus, demanding A^κ to be Lindelöf is the same as our original assumption. We conclude this section by presenting an example with the following properties:

- It is given over one matrix with countable algebra in a language with uncountably many variables. We show it has cardinality strictly larger than $\aleph_1$. Thus, it shows that the notion of strongly finite logics does not directly generalize to larger cardinalities.

[2] A proof by Joel David Hamkins can be found online https://mathoverflow.net/questions/9641/how-far-is-lindel%C3%B6f-from-compactness/9651#9651.

- The defining algebra can be endowed with discrete topology which trivially makes the connectives continuous. Justifying that the property 'compact' cannot be replaced by 'Lindelöf' in the formulation of Theorem 3.25.
- The example shows that ω^κ with a discrete topology on ω is not Lindelöf, because it would clearly be equivalent to the fact that the logic of the next example has cardinality $\aleph_1$.

Example 3.26. Consider a matrix $\mathbf{A} = \langle \boldsymbol{\omega}, \{1\}\rangle$, where $\boldsymbol{\omega}$ is an algebra on the set of natural numbers ω in language containing one constant $\overline{0}$ interpreted as 0, and one binary symbol $\neq$ defined as $a \neq^{\boldsymbol{\omega}} b = 1$ if $a \neq b$ and $a \neq^{\boldsymbol{\omega}} b = 0$ otherwise. Let ω be equipped with the discrete topology; it is obviously Lindelöf. Let κ be an uncountable cardinal. It remains to show that the logic in κ variables given by this matrix has high cardinality. However it is easy to prove, since clearly $\{p_\alpha \neq p_\beta \mid \alpha < \kappa, \beta < \kappa, \alpha \neq \beta\} \vdash \overline{0}$, but no countable subset of the premises does prove $\overline{0}$.

3.3 (C)IPEP and expansions

In this section we investigate the preservation of the (C)IPEP under extension and expansions. In Subsection 3.3.1 we see that these properties are in general not preserved when adding rules (even finitary rules in the same language). Then, in Subsection 3.3.2, we show that the IPEP and the CIPEP are always preserved by axiomatic extensions and we specify a condition under which they are also preserved by axiomatic expansions. Moreover, we show that their transferred variants are preserved by axiomatic expansions of protoalgebraic logics. To obtain the results about preservation under expansions we use the notion of natural expansion developed in Section 2.8.

3.3.1 Finitary extensions

To obtain the required result, we consider a logic L given by an axiomatic system consisting of a single infinitary rule. This easy description allows to prove it has the CIPEP. Afterwards, we define its extension L′ by two finitary rules and we show this logic does not enjoy the IPEP.

We define L in a language $\boldsymbol{\mathcal{L}}$ with three unary connectives l, r, o and countable set of variables. We use metavariables $s, s', \dots$ for finite *non-empty* sequences of $\{l, r\}$. We denote the set of all of them as Seq, which is naturally ordered by: $s < s'$ iff s is a strictly initial sequence of s'. Therefore $\langle \mathrm{Seq}, < \rangle$ can be seen as the full binary tree of height ω without root. So we can see ls as the extension of the node s to the left, and rs as the extension to the right in $\langle \mathrm{Seq}, < \rangle$. Recall that $B \subseteq \mathrm{Seq}$ is a branch in $\langle \mathrm{Seq}, < \rangle$ if it is a maximal chain. The logic L is axiomatized by taking the following infinitary rule for each branch B:

$$\{s(\varphi) \mid s \in B\} \vdash o(\varphi). \tag{B}$$

Let us show that L has the CIPEP. Indeed, let T be a theory and φ a formula and suppose that $\varphi \notin T$; then, if φ is not of the form $o(\psi)$ for some formula ψ, we can take the completely intersection-prime theory $T' = Fm_{\mathcal{L}} \backslash \{\varphi\}$ (the formula φ simply cannot be proved from any premises). If $\varphi = o(\psi)$ then define

$$C = \{s \in \mathrm{Seq} \mid s(\psi) \notin T \text{ and whenever } s' < s \text{ then } s'(\psi) \in T\}$$

and let $T' = Fm_{\mathcal{L}} \setminus (\{s(\psi) \mid s \in C\} \cup \{\varphi\})$. First observe that for every branch B, there is a unique $s \in B$ such that $s(\psi) \notin T'$. Indeed, such s always exists, since otherwise one application of (B) would yield $T \vdash_{\mathrm{L}} \varphi$. If there were two, let us say, $s < s'$, then by the definition of C both $s(\psi) \in T$ and $s(\psi) \notin T$.

Let us prove now that T' is an L-theory. Since φ is the only formula starting with o which is not in T', then, by the definition of L, $\mathrm{Th}_{\mathrm{L}}(T') = T'$ or $\mathrm{Th}_{\mathrm{L}}(T') = T' \cup \{\varphi\}$. If it was the second case, then for some branch B, $\{s(\psi) \mid s \in B\} \subseteq T'$, however, as argued above, this is not possible. Moreover T' clearly extends T. T' is a maximal theory w.r.t. φ: If $s(\psi)$ is not in T' and B is any branch containing s, then, by the uniqueness part of the observation above, for every other $s' \in B$ we have $s'(\psi) \in T'$ and consequently $T', s(\psi) \vdash_{\mathrm{L}} \varphi$, as witnessed by one application of (B); thus L has the CIPEP.

Next, we are going to extend L to a logic L′ in such a way that the second one will not enjoy the IPEP, we claim this can be achieved by adding the following two finitary rules:

$$l(\varphi) \rhd \varphi \;\text{ and }\; r(\varphi) \rhd \varphi. \tag{3.4}$$

Thus, if an L′-theory contains the node $s(\varphi)$, then it contains all of its predecessors. We show that L′ does not have the IPEP. Obviously $l(p), r(p) \nvdash_{\mathrm{L}'} o(p)$. Let T be any theory containing $l(p)$ and $r(p)$ such that $T \nvdash_{\mathrm{L}'} o(p)$. It follows that there must be some sequence s_0 such that $s_0(p) \in T$ and there is no succeeding node s' above s_0 such that $s'(p) \in T$ (otherwise the rules (3.4) and (B) would give that $o(p) \in T$). It is now a simple observation that $\{ls_0(p)\} \cup T$ and $\{rs_0(p)\} \cup T$ are L′-theories (it is obviously closed under (3.4)), and any infinitary rules that was not applicable in T

would have to be of the form $\{s(\chi) \mid s \in B\} \vdash o(\chi)$ for some branch B, but any such a rule lacks infinitely many premises in T). It is obvious that $T = (\{ls_0(p)\} \cup T) \cap (\{rs_0(p)\} \cup T)$, and hence L' does not have the IPEP.

3.3.2 Axiomatic expansions

In this section we study the preservation of the IPEP and the CIPEP under axiomatic extensions and expansions. Let us fix a language $\boldsymbol{\mathcal{L}}$ and its type extension $\boldsymbol{\mathcal{L}'}$. We will see that, while the part about extensions is simple. to investigate axiomatic expansions needs more work. Note that, clearly, to expand a logic axiomatically can be seen as a two step process:

- First, consider its natural expansion (see Section 2.8) to the corresponding type (that is extend only the language).
- Second, add the new axioms.

This realization can be helpful—it allows us to divide accordingly the preservation theorem into two parts which is precisely the direction we shall take.

Theorem 3.27. *IPEP, CIPEP, τ-IPEP, and τ-CIPEP are preserved under axiomatic extensions.*[3]

Proof. Let L' be an axiomatic extension of L. Assume, for instance, that L has the τ-IPEP (the proof for the other cases is analogous). Take $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}'$ and $a \in A \backslash F$. Since, clearly, also $F \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$, there is an intersection-prime filter $F' \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$ such that $a \notin F'$. However, we also obtain that $F' \in \mathcal{F}i_{\mathrm{L}'}\, \boldsymbol{A}$: because F' is closed under all rules of L' and, moreover, since $F \subseteq F'$, it is also closed under the new axioms. □

Next we shall inspect the preservation under axiomatic expansions. To this end, we need the following auxiliary result; recall the cardinal $\mathfrak{e}$ defined on page 34.

Proposition 3.28. *Let L be a logic in $\boldsymbol{\mathcal{L}}$ and take $\kappa = \max\{|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|, \mathfrak{e}\}$. Then, the following are equivalent:*

(i) *L^κ has the (C)IPEP,*
(ii) *$\mathrm{L}_{\boldsymbol{\mathcal{L}'}}$ has the (C)IPEP.*

Proof. Take $\mathit{Var}_{\boldsymbol{\mathcal{S}}}$ as in (2.2). The assumptions ensure that $|\mathit{Var}_{\boldsymbol{\mathcal{S}}}| = \kappa$, thus we can identify $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ with L^κ. By Proposition 2.29 and comments below Definition 2.31, we obtain $\mathrm{L}^\kappa \precsim \mathrm{L}_{\boldsymbol{\mathcal{L}'}}$. Thus, the result follows from Proposition 2.32. □

[3] In the case of the IPEP this result was already proved in [27, Lemma 2.8].

Theorem 3.29. *Let* L' *in* $\boldsymbol{\mathcal{L}'}$ *be an axiomatic expansion of* L *in* $\boldsymbol{\mathcal{L}}$ *and assume that* $\mathfrak{c} \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$. *If* L *has the* (C)IPEP, *then so does* L'.

Proof. The assumption $\mathfrak{c} \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$ says that Proposition 3.28 applies for $\kappa = |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$, thus L^κ can be identified with L and we can conclude that $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ has the (C)IPEP. Further, since L' is clearly an axiomatic extension of $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$, it has the (C)IPEP by Theorem 3.27. □

In particular, the theorem always applies if $\mathcal{L}'$ and $\mathit{Var}_{\boldsymbol{\mathcal{L}}}$ are countable. Also observe that axiomatic expansions obtained by adding countably many constants always preserve both IPEP and CIPEP (this kind of expansions has been deeply studied in the field of fuzzy logics, see e.g. [42, 45]). Moreover, the cardinal restriction in Theorem 3.29 is necessary (even for protoalgebraic logics); indeed, the infinitary product logic Π_∞ does have the CIPEP (Example 3.14), but there exists a cardinal κ such that the logic Π_∞^κ does not enjoys the IPEP (Theorem 4.1), therefore, by Proposition 3.28, the natural expansion of Π_∞ to a language with additional κ-many constants enjoys neither the CIPEP nor the IPEP.

Theorem 3.30. *For protoalgebraic logics both* τ-IPEP *and* τ-CIPEP *are preserved under arbitrary axiomatic expansions.*

Proof. Assume L is a protoalgebraic logic in $\boldsymbol{\mathcal{L}}$ and it has the τ-(C)IPEP. Since protoalgebraic logics are closed under arbitrary axiomatic expansions, by Corollary 3.17 and Theorem 3.27, it is enough to argue that any arbitrarily large natural extension $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^\kappa$ (which is a logic in language $\bar{\boldsymbol{\mathcal{L}}} = \langle \mathcal{L}', \kappa \rangle$) of the natural expansion $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ has the (C)IPEP.

This can be proved by a slight modification of the reasoning seen in Section 2.8. We can again prove a variant of Proposition 2.29 for $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^\kappa$, i.e.

$$\Gamma \vdash_{\mathrm{L}^{\boldsymbol{\mathcal{S}}}} \varphi \quad \text{if and only if} \quad \boldsymbol{\tau}[\Gamma] \vdash_{(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^\kappa} \boldsymbol{\tau}(\varphi),$$

which can be done completely analogously, with the difference that instead of the set $X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'}$ we use:

$$X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'(\boldsymbol{\kappa})} = \{c(\varphi_1, .., \varphi_n) \in \mathit{Fm}_{\mathcal{L}'}(\kappa) \mid \varphi_1, .., \varphi_n \in \mathit{Fm}_{\mathcal{L}'(\kappa)}, \text{and } c \in \mathcal{L}' \setminus \mathcal{L}\}$$

and, of course, as the set $\mathit{Var}_{\boldsymbol{\mathcal{S}}}$ we choose $\mathit{Var}_{\boldsymbol{\mathcal{L}}} \cup \{x_\varphi \mid \varphi \in X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'(\boldsymbol{\kappa})}\}$.

Similarly as in Section 2.8, the translation $\boldsymbol{\tau}$ is an isomorphism from $\boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}}$ onto $\boldsymbol{Fm}_{\bar{\boldsymbol{\mathcal{L}}}} \upharpoonright \mathcal{L}$. Therefore for $\lambda = |\mathit{Var}_{\boldsymbol{\mathcal{S}}}|$ we have $\mathrm{L}^\lambda \precsim (\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^\kappa$. Consequently, since L^λ has the (C)IPEP, so does $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^\kappa$ (Proposition 2.32). □

3.4 Conclusion and remarks

We have finished the basic presentation of the hierarchy of infinitary logics. Let us now briefly contemplate on what we have seen. Firstly, let us focus on the meaning of the classes in the hierarchy. Indeed, determining the position of a given logic in the hierarchy essentially amounts to determining how many R(F)SI-models the logic has.

- R(F)SI-completeness: this weak property only ensures that there are enough R(F)SI-models for the logic to enjoy the completeness result.
- $\mathrm{(C)IPEP}$: This property is often a useful intermediate step when proving R(F)SI-completeness. It should be clear that logics with $\mathrm{(C)IPEP}$ have a lot of Lindenbaum–Tarski R(F)SI-models, i.e. those of the form $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*$: indeed, it suffices to consider these models to obtain R(F)SI-completeness.
  Also we would like to recall that in Subsection 3.1.1 we presented a useful characterization for $\mathrm{(C)IPEP}$ logics to be in the class by means of surjective completeness: in protoalgebraic logics it is enough to know a complete class of matrices $\mathbb{K}$ such that $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{Mod}_{\mathrm{R(F)SI}}\,\mathrm{L}$ (Corollary 3.13).
- τ-$\mathrm{(C)IPEP}$: The strongest property in the hierarchy implies that these logics have the most of R(F)SI-models. The representability theorem (Theorem 3.19) and comments below explain that R(F)SI-models suffice to generate the whole class of reduced models, $\mathbf{Mod}^*\,\mathrm{L}$, by means of subdirect products.

Finally, let us summarize what we (don't) know about preservation of these properties under extensions and expansions in Table 3.1. Since in protoalgebraic logics we have the correspondence theorem, which can be a very helpful property in proving of these preservation results, we included the distinction protoalgebraic/non-protoalgebraic in the table. The table shows that we have left many questions open: for example we have no knowledge about preservation of the R(F)SI-completeness.

	Logics	τ-(C)IPEP	(C)IPEP
Finitary extensions	protoalgebraic	?	?
	non-protoalgebraic	?	No (3.3.1)
Axiomatic extensions	all	Yes	Yes
Axiomatic expansions	protoalgebraic	Yes	No (Π_∞), but Yes, if $\mathfrak{c} \leq \lvert Var_{\mathcal{L}} \rvert$
	non-protoalgebraic	?	

Table 3.1: Preservation under extensions/expansions

4 Separating examples

The goal of this chapter is simple: we are going to present three logics that separate the classes, which shows that we have indeed established a new hierarchy (Figure 4.1). We will also see an additional example of an infinitary logic which does not belong to any class of the hierarchy.

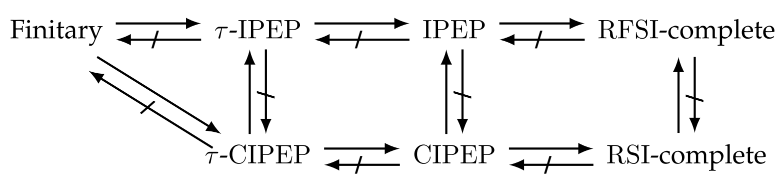


Figure 4.1: The hierarchy

However, there is a small caveat: to introduce some of the examples and to prove the necessary claims we need some basic properties of semilinear and disjunctional logics presented only later in Chapter 5. The ordering has been chosen for the sake of the narrative.

4.1 A non RFSI-complete logic

In this example we use some notions about disjunctions described in Section 5.2. The separating example presented in this section is a modification of [27, Example 3.12], which was proved to have the PCP, but not the sPCP. We present a protoalgebraic version of this logic to ensure the transferred version of PCP, which we consequently use to show that our logic is not RFSI-complete.

Let $\boldsymbol{A}$ be a Heyting algebra, which is not a dual frame. That is there are elements $a_i \in \boldsymbol{A}$ for natural $i > 0$ such that

$$\bigwedge_{i \geq 1}(x_0 \vee x_i) \not\leq x_0 \vee \bigwedge_{i \geq 1} x_i.$$

Note that we can take for example a lattice of open sets on the real line as $\boldsymbol{A}$, where $X \to^{\boldsymbol{A}} Y = \mathrm{int}(\mathbb{R} \setminus X \cup Y)$ for any $X, Y \subseteq \mathbb{R}$ (of course, int is the topological interior). The algebra $\boldsymbol{A}$ is defined in the language of lattice with additional constants $\{c_i\}_{i \geq 0} \cup \{c\}$ which are defined by $c_i^{\boldsymbol{A}} = a_i$ and $c^{\boldsymbol{A}} = \bigwedge_{i \geq 1} a_i$. Finally L is the logic of the following class of matrices:

$$\{\langle \boldsymbol{A} \rangle, F \mid F \text{ is a principal lattice filter on } \boldsymbol{A}\}.$$

It is easy the prove the following description of the entailment:

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \iff \quad \text{for every } \boldsymbol{A}\text{-evaluation } e \text{ we have } \bigwedge_{\gamma \in \Gamma} e(\gamma) \leq e(\varphi).$$

Using this characterization and the fact that $\boldsymbol{A}$ is distributive, it is easy to see that $\vee$ has the PCP. On the other hand, it does not have the sPCP: clearly

$$c_o \vdash_{\mathrm{L}} c_0 \vee c \quad \text{and} \quad \{c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c,$$

thus using sPCP we would obtain $\{c_0 \vee c_i \mid i \geq 1\} \vdash_{\mathrm{L}}$, which is not possible. It remains to show that this logic cannot be RFSI-complete: first observe that, since PCP transfers (Proposition 5.9), we known by Proposition 5.7 that intersection-prime filters coincide with $\vee$-prime. It is easy to see that submodels of models with $\vee$-prime filters are again $\vee$-prime, hence

$$\mathbf{S}(\mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L}) = \mathbf{S}(\mathbf{Mod}^{\vee}\,\mathrm{L}) \subseteq \mathbf{Mod}_{\mathrm{RFSI}}\,\mathrm{L},$$

Consequently, assuming that L is RFSI-complete we obtain that it has the IPEP (Corollary 3.13). However logics with IPEP and the PCP has also the sPCP (Proposition 5.8), which we proved not to be the case. Thus, L cannot be RFSI-complete.

4.2 A logic with CIPEP but without the τ-IPEP

We will now show that in general neither IPEP nor CIPEP transfer. Indeed, we will see that Π_∞ does not have τ-IPEP, but from Example 3.14 we know it has the CIPEP. The crucial difference between Ł$_\infty$ and Π_∞ is in the fact that the connectives of the latter are not continuous on the unit interval topology (in fact, only $\to$ is discontinuous at 0 for Π). In the proof we use some basic properties of semilinear logics—see Section 5.1.

Theorem 4.1. *Π_∞ does not have the τ-IPEP.*

Proof. By virtue of Corollary 3.17, it is enough to prove that some natural extension of Π_∞ does not have the IPEP. Let κ be an arbitrary cardinal strictly larger than the continuum $\mathfrak{c}$ and consider the natural extension Π_∞^κ.

Since Π_∞ is obviously semilinear w.r.t. $\to$, by Theorem 5.2 any filter F is intersection-prime if and only if it is linear. Furthermore, since models of Π_∞ and Π_∞^κ coincide (Observation 2.23), it follows that every theory T of Π_∞^κ is intersection-prime exactly when it is linear. Define

$$\Gamma = \{(x_\alpha \to x_\beta) \to x_0 \mid 0 < \alpha < \beta < \kappa\}.$$

Let us show that $\Gamma \nvdash_{\Pi_\infty^\kappa} x_0$. Indeed, otherwise there would be a countable subset Γ' of Γ such that $\Gamma' \vdash_{\Pi_\infty^\kappa} x_0$ (Π_∞^κ is a natural extension of Π_∞, therefore it has cardinality $\aleph_1$). Then, we could take a substitution σ such that $\sigma[\mathit{Var}[\Gamma']] \subseteq \mathit{Var}$. Then, $\sigma[\Gamma] \vdash_{\Pi_\infty^\kappa} \sigma(x_0)$ and, hence, $\sigma[\Gamma] \vdash_{\Pi_\infty} \sigma(x_0)$. Let us define $v_0 = \sigma(x_0)$ and enumerate the remaining variables of Γ' as $\{v_1, v_2, \ldots\}$ in such a way that $\{(v_n \to v_m) \to v_0 \mid 0 < n < m < \omega\} \vdash_{\Pi_\infty} v_0$. This derivation can be falsified by taking the evaluation $e(v_0) = \frac{1}{2}$ and $e(v_n) = \frac{1}{2^{n+1}}$ for each $n \geq 1$.[1]

If Π_∞^κ had the IPEP, there would be a linear theory $T \supseteq \Gamma$ such that $T \nvdash_{\Pi_\infty^\kappa} x_0$. Consider the Lindenbaum–Tarski model given by T, that is $\langle \boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*, T^* \rangle$. It is easy to see that $\boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*$ is an Archimedean (see the proof of Proposition 2.22) linear product algebra, thus by Proposition 2.22 it embeds into $[0,1]_\Pi$.

Furthermore observe that for any $0 < \alpha < \beta < \kappa$ we have $x_\alpha \to x_\beta \notin T$, which implies that $|Fm_{\mathcal{L}}(\kappa)^*| = \kappa$ (because $\langle \varphi, \psi \rangle \in \Omega(T)$ iff both $\varphi \to \psi \in T$ and $\psi \to \varphi \in T$). However, since $\kappa > \mathfrak{c}$, we have a contradiction. □

First, observe that as a consequence of the fact that the τ-IPEP is preserved under axiomatic extensions (Theorem 3.27), we have the following corollary:

[1] Note that the same argument for Ł$_\infty$ would fail here, i.e. necessarily it is the case that $\{(x_i \to x_j) \to x_0 \mid i \leq j \text{ in } \omega\} \vdash_{\text{Ł}_\infty} x_0$.

Corollary 4.2. BL_∞ *does not enjoy the* τ-IPEP.

By Theorem 3.19, Π_∞ is not finitely subdirectly representable, thus the class $\mathbf{Mod}^* \Pi_\infty$ is not generated as subdirect products of chains. In fact, in the proof of the theorem, we have constructed an algebra, namely $\boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*$, which is not a subdirect product of algebras from $\mathbf{Alg}^*\Pi_{\infty\mathrm{RSI}}$, that is

$$\mathbf{Alg}^*\Pi_\infty \neq \mathbf{P_{SD}}(\mathbf{Alg}^*\Pi_{\infty\mathrm{RFSI}}).$$

Thus, in particular $\mathbf{Alg}^*\Pi_\infty$ is an example of class of algebras which witnesses that the well-known Birkhoff's theorem about subdirect representation does not generalize beyond quasi-varieties: in fact, recall that $\mathbf{Alg}^*\Pi_\infty$ is a generalized quasi-variety which makes it, arguably, the closest generalization of quasi-variety. Moreover, the class is generated by a single chain.

Another consequence of the theorem is that the logic with κ many variables given semantically by the standard product chain need not have cardinality $\aleph_1$ (e.g. whenever $\kappa > \mathfrak{c}$). Moreover, as mentioned in subsection 3.3.2, not every axiomatic expansion of Π_∞ has the IPEP (CIPEP); e.g., by Proposition 3.28 and the previous theorem, adding more than continuum many constants does not preserve any of them.

4.3 A non-RSI-complete logic with the τ-IPEP

In this section, we introduce an infinitary *truth-degrees-preserving Łukasiewicz logic with rational constants* $\text{Ł}^{\leqslant}_{\mathrm{c}}$, which can be seen as a natural combination of the Łukasiewicz logic that preserves degrees of truth studied in [47] and Łukasiewicz logic with rational constants studied in [59, 75]. After an elementary presentation of the logic we shall show that, as desired, it has τ-IPEP but it is not RSI-complete. As in the previous example, in order to argue that it indeed has the desired properties we use some basic facts of semilinear logics (see Section 5.1).

Consider a language of Łukasiewicz logic with rational constants, that is:

$$\mathcal{L} = \{\to, \&, \overline{0}\} \cup \{\overline{q} \mid q \in (0,1] \cap \mathbb{Q}\} \text{ and } \mathit{Var}_{\boldsymbol{\mathcal{L}}} = \omega.$$

Let $[0,1]^{\mathbb{Q}}_{\text{Ł}}$ denote $[0,1]_{\text{Ł}}$ expanded with the natural interpretations of the constants. Let $\uparrow q$ denote the lattice filter generated by a rational number q in $[0,1]$, i.e. $\uparrow q = \{r \in [0,1] \mid r \geq q\}$. Define $\textbf{Ł}_q$ as the matrix $\langle [0,1]^{\mathbb{Q}}_{\text{Ł}}, \uparrow q\rangle$ and define the set $\mathbb{K} = \{\textbf{Ł}_q \mid q \in (0,1] \cap \mathbb{Q}\}$. Thus, we define

$$\text{Ł}^{\leqslant}_{\mathrm{c}} = \models_{\mathbb{K}}.$$

Note that, since rationals form a dense subset of reals, $Ł_c^{\leqslant}$ is indeed the degree-preserving logic over the algebra $[0,1]_Ł^{\mathbb{Q}}$ (i.e. every lattice filter on $[0,1]_Ł^{\mathbb{Q}}$ is a filter of $Ł_c^{\leqslant}$). Indeed, we could define $Ł_c^{\leqslant}$ as

$$\Gamma \vdash_{Ł_c^{\leqslant}} \varphi \quad \Longleftrightarrow \quad \bigwedge v[\Gamma] \leq v(\varphi), \text{ for all } v \in \mathrm{Hom}(\boldsymbol{Fm_{\mathcal{L}}}, [0,1]_Ł^{\mathbb{Q}}).$$

However, as we will see later, it is of significance that the logic can be defined over countably many matrices. Define the following generalized implication connective:

$$x \Rightarrow y = \{(x \to y)^n \mid n \in \omega\},$$

where $(-)^n$ denotes the n times iterated conjunction. It is easy to check that $\Rightarrow$ is a weak implication in L. Indeed, observe that for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and every $a, b \in A$ it holds that $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$ if and only if $a \leq b$, where $\leq$ is the standard order on reals (cf. Observation 2.2).Thus, clearly, $\mathbb{K} \subseteq \mathbf{Mod}^{\ell}\,\mathrm{L}$ and hence $Ł_c^{\leqslant}$ is semilinear.

Recall that on every reduced model $\langle \boldsymbol{A}, F\rangle$, we have the induced order relation $\leq^{\mathbf{A}}$ given by $a{\leq^{\mathbf{A}}}b$ iff $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$. Moreover, it is easy to see that $Ł_c^{\leqslant}$ proves:

$$x \Rightarrow y \vdash_{Ł_c^{\leqslant}} (x \to y) \Leftrightarrow \bar{1},$$

which implies the left to right implication in the following characterization of the order on reduced models (the other implication is obvious):

$$a{\leq^{\mathbf{A}}}b \quad \Longleftrightarrow \quad a \to b = \bar{1}^{\boldsymbol{A}}, \tag{4.1}$$

Now we can easily see that $Ł_c^{\leqslant}$ is infinitary (in fact, $\mathrm{card}\,\mathrm{L} = \aleph_1$); indeed:

$$\bigcup_{q \in (0,1)\cap\mathbb{Q}} \bar{q} \Rightarrow p \vdash_{Ł_c^{\leqslant}} p$$

but no finite subset of premises would entail p.

Note that $Ł_c^{\leqslant}$ is weakly implicational (equivalential), but not algebraizable, which is a consequence of the fact that all the matrices $\mathbf{Ł}_q$ are reduced (on every algebra in algebraizable logics there is at most one filter making the corresponding matrix reduced).

Proposition 4.3. $Ł_c^{\leqslant}$ *has the* τ-IPEP.

Proof. Let $L_{\mathbb{K},\kappa}$ be the logic in κ-many variables semantically given by the class $\mathbb{K}$. We can apply Theorem 3.25 to prove that $L_{\mathbb{K},\kappa}$ has cardinality $\aleph_1$ and thus show that $Ł_c^{\leqslant\kappa} = L_{\mathbb{K},\kappa}$ (the condition necessary to ensure the uniqueness of natural extensions is fulfilled; see Subsection 2.8.1). The theorem applies, because $\mathbb{K}$ is countable, the standard interval topology on $[0,1]$ is compact and all the connectives are continuous w.r.t. it and finally all the filters $\uparrow q$ can obviously be approximated by countably many open subsets of $[0,1]$. So $Ł_c^{\leqslant\kappa}$ is complete w.r.t. $\mathbb{K}$ and, consequently, it is semilinear w.r.t. $\Rightarrow$ (because $\mathbb{K} \subseteq \mathbf{Mod}^{\ell}\,\mathrm{L}$) and has the IPEP (in fact, every semilinear logic has the IPEP; see point (ii) of Theorem 5.2). By Corollary 3.17, since $Ł_c^{\leqslant}$ is protoalgebraic, it has the τ-IPEP. □

Proposition 4.4. $Ł_c^{\leqslant}$ *is not* RSI-*complete.*

Proof. First, observe that $Ł_c^{\leqslant}$ satisfies for every real number $r \in (0,1]$ the following *density* rule:

$$\{\bar{q} \Rightarrow x \mid q < r\} \cup \{\bar{q} \mid q > r\} \vdash_{Ł_c^{\leqslant}} x \tag{4.2}$$

We show that $Ł_c^{\leqslant}$ has no RSI-models which implies that it cannot be RSI-complete because it is not the inconsistent logic. In pursuit of contradiction suppose there is a reduced model $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ in which F is completely intersection-prime. In particular, $F \neq A$ and, since $Ł_c^{\leqslant}$ is semilinear, F is linear (by Theorem 5.2), which implies that $\leq^{\mathbf{A}}$ is a linear order. Consider the following set of rationals

$$C_F = \{q \in (0,1] \mid \bar{q}^{\boldsymbol{A}} \in F\}.$$

Claim 1: For each $a \in A \setminus F$, there is q such that $a <^{\mathbf{A}} \bar{q}^{\boldsymbol{A}}$ and $q \notin C_F$.
Proof: Take $a \in A \setminus F$ and assume that there is no such q. By linearity of $\leq^{\mathbf{A}}$, we have $\bar{q}^{\boldsymbol{A}} \leq^{\mathbf{A}} a$ for every $q \notin C_F$. Thus, applying (4.2) for $r = \inf C_F$, we obtain that $a \in F$; a contradiction.

Let us define $\uparrow\bar{q}^{\boldsymbol{A}} = \{a \in A \mid a \geq^{\boldsymbol{A}} \bar{q}^{\boldsymbol{A}}\}$.
Claim 2: For every $q \notin C_F$, $F \subseteq \uparrow\bar{q}^{\boldsymbol{A}}$.
Proof: Take $a \in F$ and suppose $q \not\leq^{\boldsymbol{A}} a$. By linearity $a <^{\boldsymbol{A}} q$, thus modus ponens of $\Rightarrow$ implies that $q \in F$ –a contradiction.

Claim 3: For every $\boldsymbol{B} \in \mathbf{Alg}^*\mathrm{L}$ and every $q \in (0,1] \cap \mathbb{Q}$, the set $\uparrow\bar{q}^{\boldsymbol{B}}$ is an $Ł_c^{\leqslant}$-filter.
Proof: Since $Ł_c^{\leqslant}$ is equivalential in a countable language with a countable set

of variables we have $\mathbf{Alg}^*\mathrm{L} = \mathbf{ISP}_{\mathbf{R}_{\aleph 1}}([0,1]^{\mathbb{Q}}_{Ł})$, and hence $\boldsymbol{B}$ is embeddable into $\boldsymbol{C} = \prod^{\kappa}[0,1]^{\mathbb{Q}}_{Ł}/\mathcal{F}$ for some $\mathcal{F}$, an $\aleph_1$-complete filter on κ (see e.g. [30, Theorem 1]). Take $G = (\prod^{\kappa}{\uparrow}q)/\mathcal{F}$; it follows that $\langle \boldsymbol{C}, G\rangle$ is a reduced model of $Ł^{\leqslant}_{c}$ (because $\mathbf{Mod}^*\,\mathrm{L}$ is closed under $\mathbf{P}_{\mathbf{R}_{\aleph 1}}$). We show that $G = {\uparrow}\bar{q}^{\boldsymbol{C}}$:

$$
\begin{aligned}
[\bar{a}] \in {\uparrow}\bar{q}^{\boldsymbol{C}} &\iff \{\alpha \in \kappa \mid q \to \bar{a}(\alpha) = 1\} \in \mathcal{F} \\
&\iff \{\alpha \in \kappa \mid \bar{a}(\alpha) \in {\uparrow}q\} \in \mathcal{F} \\
&\iff [\bar{a}] \in G,
\end{aligned}
$$

The claim follows because ${\uparrow}\bar{q}^{\boldsymbol{B}} = {\uparrow}\bar{q}^{\boldsymbol{C}} \cap B$.

It is now easy, by virtue of all the claims above, to conclude that $F = \bigcap_{q \notin C_F} {\uparrow}\bar{q}^{\boldsymbol{A}}$, where for each $q \notin C_F$, ${\uparrow}\bar{q}^{\boldsymbol{A}}$ is a filter and does not coincide with F –a contradiction with the fact that F is completely intersection-prime. □

4.4 An RSI-complete logic without the IPEP

As we have seen in the previous section, when we want to determine whether a given logic is RSI-complete or RFSI-complete, the notions of CIPEP or IPEP are useful sufficient conditions. We only need to verify whether the logic satisfies one of these extension properties (or finitarity). This section is devoted to the problem of separating the classes of logics with the IPEP from RFSI-complete logics, and the classes of logics with the CIPEP from RSI-complete logics. This will be achieved by producing a single example, rather difficult to construct, of an RSI-complete logic which does not enjoy the IPEP. This way we prove that CIPEP and IPEP are not trivial notions, which, as conclusion, allows us to obtain a hierarchy of infinitary logics.

4.4.1 Introducing the example

We are going to describe an RSI-complete *weakly implicative* logic L which does not belong to the IPEP class. Our logic will be given semantically by a suitable matrix $\langle \boldsymbol{A}, F\rangle$. This approach will turn out be very useful in proving RSI-completeness: we only need to check that the matrix is reduced and F is completely intersection-prime filter in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$, i.e. that $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^*\,\mathrm{L}_{\mathrm{RSI}}$.

In order to falsify the IPEP in L, we will implement a full binary tree of height ω into $\boldsymbol{A}$. The motivation is that every node in the tree is $\wedge$-reducible (i.e. can be expressed as a meet of its two immediate successors). To benefit from this idea we make sure that every node s of the tree will correspond to some theory T_s of L. Moreover, we make sure that for every node s and its two immediate successors s_1 and s_2 their corresponding theories T_s, T_{s_1}, T_{s_2} will satisfy $T_s \subsetneq T_{s_1}$, $T_s \subsetneq T_{s_2}$ and $T_s = T_{s_1} \cap T_{s_2}$ (thus ensuring T_s is not intersection-prime for every s). Thus, we define a set of formulas Γ and a formula φ such that $\Gamma \nvdash_{\mathrm{L}} \varphi$, in such a way that any theory extending Γ and not containing φ will correspond to some node in the tree. This way we conclude that Γ cannot be extended to any intersection-prime theory.

To this end, we add a unary connective for each node s, which will allow us to capture (within the logic) which nodes are above s. Interestingly enough, in order to follow through we need to let the logic 'know' something about itself, that is, we include in the algebra $\boldsymbol{A}$ also some substantial subset of $Fm_{\mathcal{L}}(\{p\})$ (algebra of formulas in one variable p). Therefore, an interesting feature of this logic is that it is described semantically but the defining structure is partially based on its own syntax.

Fix $\mathbf{T} = \langle \mathrm{T}, \leq_{\mathrm{T}} \rangle$ to be the full binary tree of height ω. Note that we can view $\mathbf{T}$ as e.g. the collection of all functions which have a natural number n as a domain and a subset of 2 as range, where $\leq_{\mathrm{T}}$ is the inclusion order ($\emptyset$ is the root of this tree). We will use variables s, r, t, u (possibly with indexes, superscripts) for the nodes, moreover we write $\mathbf{r}$ for the root of $\mathbf{T}$.

Let us next focus on the language of L. It has countably many variables with type $\mathcal{L} = \{\overline{0}, \rightarrow, B\} \cup \{B_s \mid s \in \mathrm{T}\}$, where $\overline{0}$ is a nullary connective, $\rightarrow$ is binary and the rest are unary connectives; read B_s as 'bigger than the node s'. Moreover we define a nullary connective $\overline{1}$ as $\overline{0} \rightarrow \overline{0}$.

Next we describe the subset of $Fm_{\mathcal{L}}(\{p\})$ that we will include in $\boldsymbol{A}$. We will call this set $\mathbf{Fm^p}$ and define it recursively as the smallest set satisfying

(i) (a) $\{\boldsymbol{B_s(p)} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
    (b) $\{\boldsymbol{B_s(\overline{0})} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
    (c) $\{\boldsymbol{B_s(\overline{1})} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,

(ii) moreover for every $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{Fm^p}$ also

- (a) $\{\boldsymbol{B_s(\varphi)} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
- (b) $\overline{\mathbf{0}} \Rightarrow \boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi} \Rightarrow \overline{\mathbf{0}} \in \mathbf{Fm^p}$,
- (c) $\overline{\mathbf{1}} \Rightarrow \boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi} \Rightarrow \overline{\mathbf{1}} \in \mathbf{Fm^p}$,
- (d) $\boldsymbol{p} \Rightarrow \boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi} \Rightarrow \boldsymbol{p} \in \mathbf{Fm^p}$,
- (e) $\boldsymbol{\varphi} \Rightarrow \boldsymbol{\psi} \in \mathbf{Fm^p}$

We write the formulas in $\mathbf{Fm^p}$ in boldface (and we use $\Rightarrow$ instead of $\rightarrow$) to have a better distinction between formulas that are part of the syntax and those that we use to build the algebra $\boldsymbol{A}$. Note that $\mathbf{Fm^p}$ is a subalgebra of the $\{B, \overline{0}, \overline{1}\}$-free reduct of $Fm_{\mathcal{L}}(\{p\})$. Let us mention that we have presented $\mathbf{Fm^p}$ in such a way that it has a well-arranged structure for inductive proofs on $\mathbf{Fm^p}$, as we will see later. Having described the two main components of the algebra $\boldsymbol{A}$ we are finally ready to define the logic L.

Definition 4.5. Suppose that all the elements in $\mathbf{Fm^p}$, T and $\{0, 1, \star\}$ are mutually distinct objects and denote $\mathrm{D} = \{\varphi \in \mathbf{Fm^p} \mid \text{there is } \boldsymbol{\psi} \in \mathbf{Fm^p} \text{ such that } \varphi = \psi \Rightarrow \psi\}$. We define:

$$A = \mathbf{Fm^p} \cup \mathrm{T} \cup \{0, 1, \star\} \text{ and } F = \mathrm{D} \cup \mathrm{T} \cup \{1, \star\}.$$

For every $s, t \in \mathrm{T}$ and every $\varphi, \psi \in \mathbf{Fm^p}$ the operations of the algebra $\boldsymbol{A}$ are given by the tables 4.1 and 4.2.

	`Operation` $\rightarrow$
T, T	$s \rightarrow t = s$ if $s = s$ and $s \rightarrow t = 0$ otherwise
T, $\mathbf{Fm^p}$	$s \rightarrow \varphi = \boldsymbol{p} \Rightarrow \varphi$ and $\varphi \rightarrow s = \varphi \Rightarrow \boldsymbol{p}$
T, 0, 1	$s \rightarrow 0 = 0 \rightarrow s = 0$ and $s \rightarrow 1 = 1 \rightarrow s = 0$
$\mathbf{Fm^p}$, $\mathbf{Fm^p}$	$\varphi \rightarrow \psi = \varphi \Rightarrow \psi$
$\mathbf{Fm^p}$, 0	$0 \rightarrow \varphi = \overline{\mathbf{0}} \Rightarrow \varphi$ and $\varphi \rightarrow 0 = \varphi \Rightarrow \overline{\mathbf{0}}$
$\mathbf{Fm^p}$, 1	$1 \rightarrow \varphi = \overline{\mathbf{1}} \Rightarrow \varphi$ and $\varphi \Rightarrow 1 = \varphi \Rightarrow \overline{\mathbf{1}}$
0, 1	$0 \rightarrow 0 = 1 \rightarrow 1 = 1$ and $1 \rightarrow 0 = 0 \rightarrow 1 = 0$
$\star$	As if $\star \in \mathrm{T}$ (e.g. $\star \rightarrow \varphi = \boldsymbol{p} \Rightarrow \varphi$)

Table 4.1: Interpretation of the connective $\rightarrow$

Moreover $B(\star) = \star$ and $B(a) = B_{\mathbf{r}}(a)$, where $\mathbf{r}$ is the root of $\mathbf{T}$ and $a \neq \star$. The constant $\overline{0}$ is interpreted as 0. We define L as the logic of this matrix, i.e. $\mathrm{L} = \models_{\mathbf{A}}$.

To get better acquainted with this definition let us first compute few values of formulas. For this example we will use labels for the nodes as depicted on Figure 4.2.

Example 4.6. Consider an evaluation e and a formula

$$\varphi = B_6(q \rightarrow q) \rightarrow q$$

	`Operations` B_s
T	$B_s(t) = t$ if $t \geq_{\mathrm{T}} s$ and $B_s(t) = \boldsymbol{B_s(p)}$ otherwise
$\mathbf{Fm^P}$	$B_s(\boldsymbol{\varphi}) = \boldsymbol{B_s(\varphi)}$
$0, 1$	$B_s(0) = \boldsymbol{B_s(\overline{0})}$ and $B_s(1) = \boldsymbol{B_s(\overline{1})}$
$\star$	$B_s(\star) = \boldsymbol{B_s(p)}$

Table 4.2: Interpretation of B_i's

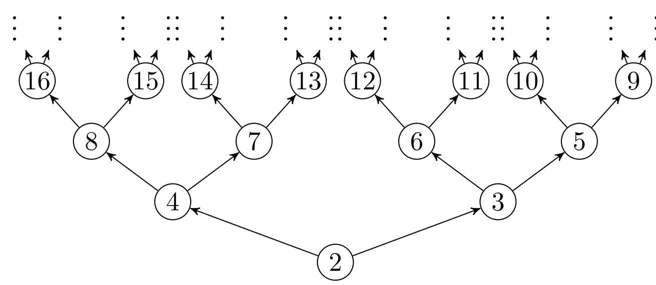


Figure 4.2: The tree T

(a) if $e(q) = 11$, then $e(\varphi) = 11$;
(b) if $e(q) = 3$ (or $5, 10, 14, \dots$ or $\star$), then $e(\varphi) = \boldsymbol{B_6(p) \Rightarrow p}$,
(c) if $e(q) = 0$, then $e(\varphi) = \boldsymbol{B_6(\overline{1}) \Rightarrow \overline{0}}$;
(d) if $e(q) = \boldsymbol{\psi} \in \mathbf{Fm^P}$, then $e(\varphi) = \boldsymbol{B_6(\psi \Rightarrow \psi) \Rightarrow \psi}$.

Observe that for any evaluation e and any formula φ it is true that

$$e(\varphi) \in \mathbf{Fm^P} \iff \text{there is a subformula } B_s(\psi) \text{ of } \varphi \text{ and } e(\psi) \not\geq_{\mathrm{T}} s.$$

Thus, whenever $e(\varphi) \in \mathbf{Fm^P}$ we can say that the formula φ is in some sense falsified by the evaluation e. It is very easy to see that L is a weakly implicative logic using the following observation:

Observation 4.7. For every $a, b \in A$ it holds: $a \to b \in F$ iff $a = b$.

Note that it relies on the fact that we included some part of $\mathbf{Fm^P}$, namely the set D, in the filter.

Corollary 4.8. L *is a weakly implicative logic and* $\langle \boldsymbol{A}, F \rangle$ *is reduced.*

4.4.2 Failure of the IPEP

In this section we prove that L does not satisfy the IPEP (Theorem 4.13). The result follows from the fact that L satisfies two properties which we call *upward persistency* and *infimum property*. The proof of each of these two properties is a rather technical work. Thus, we first disprove the IPEP using these properties and only after that we argue that the logic indeed satisfy them. We believe this presentation will be more convenient to the reader.

Convention 4.9. Every formula mentioned in this section is assumed to contain only the variable p. For every such a formula φ, we denote by φ^s the value of φ under an evaluation e such that $e(p) = s \in \mathrm{T}$. We also write $\varphi =_s \psi$ meaning that $\varphi^s = \psi^s$.

Recall the convention from the beginning of this section that we use variables u, s, r, t for the nodes of $\mathbf{T}$.

Proposition 4.10 (Upward persistency). *For every formula φ and evaluations $s \leq_{\mathrm{T}} t$, it holds: if $\varphi^s \in F$, then $\varphi^t \in F$.*

Proposition 4.11 (Infimum property). *For every formula φ and every s_1, s_2, it holds: if $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$, then also $\varphi^s \in F$, where $s = \inf\{s_1, s_2\}$.*

Now to disprove the IPEP we need to describe a suitable set of formulas Γ_0 and a formula φ such that $\Gamma_0 \nvdash_{\mathrm{L}} \varphi$ and for every theory $T \supseteq \Gamma_0$: if $T \nvdash_{\mathrm{L}} \varphi$, then T is not intersection-prime, i.e. there are two theories T_1 and T_2 strictly containing T such that $T = T_1 \cap T_2$.

Definition 4.12. Let us enumerate all propositional variables as $\{p_i\}_{i\in\omega}$. We then define $\Gamma_0 = \{p_i \to p_j \mid i, j \in \omega\} \cup \{B_{\mathbf{r}}(p_1)\}$.[2]

We extend the notation from Convention 4.9 to sets of formulas: we write T^s for the set $\{\varphi^s \mid \varphi \in T\}$.

Theorem 4.13. *The logic* L *does not satisfy the* IPEP*.*

Proof. We will denote as e_a the evaluation that sends every variable to a fixed element $a \in A$. First, observe that $\Gamma_0 \nvdash_{\mathrm{L}} \overline{0}$, which can be stated as follows: there is an evaluation e which satisfies Γ_0, i.e. $e[\Gamma_0] \subseteq F$. Moreover all the evaluations that satisfy Γ_0 are exactly of the form e_s for some $s \in \mathrm{T}$ (evaluations satisfying Γ_0 can be identified with nodes of T). This allows us to treat formulas as being written in one variable, call it p, and use the notation φ^s instead of $e_s(\varphi)$: because we have for any $T \supseteq \Gamma_0$

[2] Recall that $\mathbf{r}$ is the root of T.

$$e_s[T] \subseteq F \text{ if and only if } \sigma[T]^s \subseteq F,$$

where σ is the substitution sending every variable to p.

Next let us have a theory T containing Γ_0 such that $T \nvdash_{\mathrm{L}} \overline{0}$. Denote the set of all nodes satisfying T as $\mathrm{Sat}(T) = \{s \in \mathrm{T} \mid T^s \subseteq F\}$. Note that $\mathrm{Sat}(T)$ is nonempty, $\mathrm{Sat}(T) \subseteq \mathrm{T}$, and it contains all the evaluations satisfying T.

We first show that there is a $\leq_{\mathrm{T}}$-least element in $\mathrm{Sat}(T)$: pick any $s \in \mathrm{Sat}(T)$ and consider a set $\downarrow s = \{t \leq_{\mathrm{T}} s \mid t \in \mathrm{Sat}(T)\}$ and let s_0 be the $\leq_{\mathrm{T}}$-least element in $\downarrow s$ (such an element always exists, because T is a tree). We show that s_0 is the $\leq_{\mathrm{T}}$-least element in $\mathrm{Sat}(T)$. Suppose it is not, then there is $s_1 \in \mathrm{Sat}(T)$ such that $s_0 \nleq_{\mathrm{T}} s_1$. Let t be the infimum of $\{s_0, s_1\}$ (obviously, $t <_{\mathrm{T}} s_0$). To arrive at contradiction it remains to show that $t \in \mathrm{Sat}(T)$, which is, however, an easy consequence of the infimum property (Proposition 4.11): since for every $\varphi \in T$ we have $\varphi^{s_0} \in F$ and $\varphi^{s_1} \in F$ thus also $\varphi^t \in F$.

Since s_0 is the least element in $\mathrm{Sat}(T)$ we obtain:

$$T \vdash_{\mathrm{L}} \varphi \text{ if and only if } \varphi^{s_0} \in F \tag{4.3}$$

The direction from left to right is obvious. For the other let $\varphi^{s_0} \in F$. Then, since for every $s \in \mathrm{Sat}(T)$ we have $s_0 \leq_{\mathrm{T}} s$, we obtain $\varphi^s \in F$ (by the upward persistency, Proposition 4.10). Therefore every evaluation which satisfies T also satisfies φ.

Now let s_1, s_2 be the two distinct immediate successors of s_0. Obviously, $T \nvdash_{\mathrm{L}} B_{s_1}(p)$ and $T \nvdash_{\mathrm{L}} B_{s_2}(p)$ (this fact is witnessed by the evaluation e_{s_0}). Therefore, both $T_1 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{s_1}(p_1)\})$ and $T_2 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{s_2}(p_1)\})$ strictly contain T. Finally we prove that for every formula φ:

$$\text{if } T_1 \vdash_{\mathrm{L}} \varphi \text{ and } T_2 \vdash_{\mathrm{L}} \varphi, \text{ then } T \vdash_{\mathrm{L}} \varphi \tag{4.4}$$

Suppose $T_1 \vdash_{\mathrm{L}} \varphi$ and $T_2 \vdash_{\mathrm{L}} \varphi$. By (4.3), we need to show $\varphi^{s_0} \in F$. It is easy to see that $T_1^{s_1} \subseteq F$, and consequently $\varphi^{s_1} \in F$, similarly $\varphi^{s_2} \in F$. Thus, the desired result is a consequence of the infimum property.

In particular, the fact (4.4) tells us that the theory T is not intersection-prime ($T = T_1 \cap T_2$), which is exactly what we wanted. □

To conclude the proof we will now demonstrate that L indeed satisfy both of the properties. We start with the upward persistency.

Lemma 4.14. *For every formula φ and every $s \leq_{\mathrm{T}} t$, we have the following: if $\varphi^s \in \mathrm{T} \cup \{\star\}$, then $\varphi^s = s$ and $\varphi^t = t$.*

Proof. The proof proceeds by induction on the complexity of φ. The base step where $\varphi = p$ (or $\varphi = \overline{0}$) is obvious. Induction step:

- If $\varphi = B_u(\psi)$ and $\varphi^s \in \mathrm{T} \cup \{\star\}$, then obviously $\psi^s \in \mathrm{T}$ and, therefore, by induction assumption, $\psi^s = s$ and $\psi^t = t$. It is also clear that $u \leq_{\mathrm{T}} \psi^s \leq_{\mathrm{T}} \psi^t$. We can thus conclude that $\varphi^s = \psi^s = s$ and $\varphi^t = \psi^t = t$.
- The case of $\varphi = B(\psi)$ follows easily because for any $u \in \mathrm{T}$ we have $B(u) = u$.
- Assume that $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s \in \mathrm{T} \cup \{\star\}$. Since ${\varphi_1}^s = {\varphi_2}^s \in \mathrm{T} \cup \{\star\}$, we obtain the result simply from the induction assumption. □

Notice that by this lemma for any formula φ and node s, the value of φ^s can only be $s, 0, 1$, or $\boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$.

Lemma 4.15. *For every formula φ and every $s \leq_{\mathrm{T}} t$ it holds:*

(i) *if $\varphi^s = 0$, then $\varphi^t = 0$,*
(ii) *if $\varphi^s = 1$, then $\varphi^t = 1$.*

Proof. We prove both cases simultaneously using induction over the complexity of the formula φ. The base step is again obvious.

- If $\varphi = B_u(\psi)$, it is trivial (φ^s can be neither 0 nor 1).
- If $\varphi = B(\psi)$, it is trivial for the same reasons.
- If $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s = 0$, then we have the following possibilities:

  (i) ${\varphi_1}^s = 0$ and ${\varphi_2}^s = s$ (or the other way around),
  (ii) ${\varphi_1}^s = 1$ and ${\varphi_2}^s = s$ (or the other way around),
  (iii) ${\varphi_1}^s = 0$ and ${\varphi_2}^s = 1$ (or the other way around),

  All these cases are easy to prove using the induction assumption and Lemma 4.14.
- If $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s = 1$, then both ${\varphi_1}^s$ and ${\varphi_2}^s$ are either 0 or 1. Thus, the result is a simple consequence of induction assumption. □

Lemma 4.16. *For every formulas φ and ψ, for every $s \leq_{\mathrm{T}} t$ and for every $\boldsymbol{\chi} \in \mathbf{Fm^P}$, it holds: if $\varphi =_s \psi = \boldsymbol{\chi}$, then $\varphi =_t \psi$.*

Proof. We prove it by induction over the complexity of $\mathbf{Fm^P}$. In the upcoming proof we will not deal with formulas of the form $B(\varphi')$ because the proof for these instances proceeds exactly in the same way as the proof for $B_s(\varphi')$.

(i) (a) If $\varphi =_s \psi = \boldsymbol{B_u(p)}$, then, obviously, $\varphi = B_u(\varphi_1)$ and $\psi = B_u(\psi_1)$ such that $s = {\varphi_1}^s \ngeq_{\mathrm{T}} u$ and $s = {\psi_1}^s \ngeq_{\mathrm{T}} u$. We use Lemma 4.14 to derive: ${\varphi_1}^t = t = {\psi_1}^t$. The rest is straightforward.
(b) If $\varphi =_s \psi = \boldsymbol{B_u(\overline{0})}$, then $\varphi = B_u(\varphi_1)$, $\psi = B_u(\psi_1)$ and ${\varphi_1}^s = {\psi_1}^s = 0$. The rest is an easy consequence of Lemma 4.15.
(c) If $\varphi =_s \psi = \boldsymbol{B_u(\overline{1})}$, we do it similarly.

(ii) (a) If $\varphi =_s \psi = \boldsymbol{B_u(\chi)}$, then $\varphi = B_u(\varphi_1)$, $\psi = B_u(\psi_1)$ and ${\varphi_1}^s = {\psi_1}^s = \boldsymbol{\chi}$. The rest follows from the induction assumption.
(b) If $\varphi =_s \psi = \overline{\mathbf{0}} \Rightarrow \boldsymbol{\chi}$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and ${\varphi_1}^s = {\psi_1}^s = 0$ and ${\varphi_2}^s = {\psi_2}^s = \boldsymbol{\chi}$. The rest follows from the induction assumption and Lemma 4.15. (The same proof applies to $\boldsymbol{\chi} \Rightarrow \overline{\mathbf{0}}$).
(c) If $\varphi =_s \psi = \overline{\mathbf{1}} \Rightarrow \boldsymbol{\chi}$, we do it similarly.
(d) If $\varphi =_s \psi = \boldsymbol{p} \Rightarrow \boldsymbol{\chi}$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and ${\varphi_1}^s = {\psi_1}^s = s$ and ${\varphi_2}^s = {\psi_2}^s = \boldsymbol{\chi}$. We apply Lemma 4.14 and the induction assumption.
(e) If $\varphi =_s \psi = \boldsymbol{\chi_1} \Rightarrow \boldsymbol{\chi_2}$, we do it similarly. □

We are now ready to prove the upward persistency:

Proof of Proposition 4.10. Let us consider the three possible cases. First, if $\varphi^s = s$, it follows from Lemma 4.14. Second, if $\varphi^s = 1$, it follows by Lemma 4.15. Finally, if $\varphi^s = \boldsymbol{\chi} \Rightarrow \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$, it follows that $\varphi = \varphi_1 \to \varphi_2$ and ${\varphi_1}^s = {\varphi_2}^s = \boldsymbol{\chi}$; then we just apply Lemma 4.16. □

Next we focus on the infimum property. Again we need to prove several technical lemmata first:

Lemma 4.17. *For every formula φ and any $s \leq_T t$, it holds: if $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$, then there is a subformula $B_u(\psi)$ of φ such that $s <_T u \leq_T t$.*

Proof. Suppose we are given s and t satisfying the conditions of this lemma. We prove the conclusion by induction over the complexity of φ. The base step holds trivially. For the induction step we consider the following cases:

- $\varphi = B_u(\psi)$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. Thus, either also ψ satisfies the conditions of this lemma and we are done by the induction assumption or $\psi^s = s$. In the second case, from Lemma 4.14, we get $\psi^t = t$. Finally, since $\varphi^s \neq \varphi^t = \boldsymbol{B_u(p)}$, we conclude that $s <_T u \leq_T t$.
- $\varphi = B(\psi)$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. This case is very similar to the previous one; the only difference is that in this case the second possibility cannot happen.
- $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. There are many cases to discuss (based on the form of $\boldsymbol{\chi}$), however all of them are easy to check (using Lemmas 4.14 and 4.15 and the induction assumption). For example $\boldsymbol{\chi} = \overline{\mathbf{0}} \Rightarrow \boldsymbol{\chi'}$ for some $\boldsymbol{\chi'} \in \mathbf{Fm^P}$, meaning that ${\varphi_1}^s = 0$ and ${\varphi_2}^s = \boldsymbol{\chi'}$. By Lemma 4.15 ${\varphi_1}^s = {\varphi_1}^t$, therefore, since,

$\varphi^s \neq \varphi^s$, we obtain ${\varphi_2}^s \neq {\varphi_2}^s$; the rest follows by the induction applied to φ_2. □

Lemma 4.18. *If $B_s(\psi)$ is a subformula of φ and $t \not\geq_{\mathrm{T}} s$, then $\varphi^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$.*

Proof. Because of Lemma 4.14, we know that ψ^t has one of these values: $t, 0, 1$, or $\boldsymbol{\varphi}$ for some $\boldsymbol{\varphi} \in \mathbf{Fm^P}$. In all these cases we get $B_s(\psi)^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$. The rest is easy (cf. comments right below the definition of the logic L). □

The next auxiliary lemma shows the relation between the presence of $\overline{0}$ in φ and certain values of φ^s.

Lemma 4.19. *For every formula φ and any node $s \in \mathrm{T}$ it holds that $\overline{0}$ is a subformula of φ iff $\varphi^s \in \{0, 1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ such that $\overline{\mathbf{0}}$ is a subformula of $\boldsymbol{\chi}$.*

Proof. We prove it by induction over the complexity of φ. The base step is easy. Now let us write $\mathrm{Prop0}(\varphi)$ if $\varphi^s \in \{0, 1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ such that $\overline{\mathbf{0}}$ is a subformula of $\boldsymbol{\chi}$.

- If $\varphi = B_u(\psi)$, it is easy: $\overline{0}$ is a subformula of φ iff it is a subformula of ψ iff (by the induction assumption) $\mathrm{Prop0}(\psi)$ iff $\mathrm{Prop0}(\varphi)$.
- If $\varphi = B(\psi)$, it works similarly.
- Assume that $\varphi = \varphi_1 \to \varphi_2$. Then: $\overline{0}$ is a subformula of φ iff $\overline{0}$ is a subformula of φ_1 or φ_2 iff $\mathrm{Prop0}(\varphi_1)$ or $\mathrm{Prop0}(\varphi_2)$ iff $\mathrm{Prop0}(\varphi)$. □

Lemma 4.20. *Let φ be a formula and take any $s_1, s_2 \in \mathrm{T}$. Suppose that $s = \inf\{s_1, s_2\}$.*[3] *Then:*

(i) *if $\varphi^{s_1} = s_1$ and $\varphi^{s_2} = s_2$, then $\varphi^s = s$,*
(ii) *if $\varphi^{s_1} = \varphi^{s_2} = 0$ (resp. $\varphi^{s_1} = \varphi^{s_2} = 1$), then $\varphi^s = 0$ (resp. $\varphi^s = 1$),*
(iii) *any other combination of these values is not possible, i.e. the following cannot happen:*
  - *$\varphi^{s_1} = s_1$ and $\varphi^{s_2} \in \{0, 1\}$,*
  - *$\varphi^{s_1} = 0$ and $\varphi^{s_2} = 1$.*

Proof. (i) By the way of contradiction suppose that $\varphi^{s_1} = s_1$, $\varphi^{s_2} = s_2$, and $\varphi^s \neq s$. Therefore, $\varphi^s \in \{0, 1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$, but the first possibility is not true because of Lemma 4.15, thus $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$. Now we use twice Lemma 4.17 to obtain two

[3] $\inf\{s_1, s_2\}$ is the infimum of s_1 and s_2 w.r.t. $\leq_{\mathrm{T}}$. Note that it always exists.

subformulas of φ: $B_{t_1}(\psi_1)$ and $B_{t_2}(\psi_2)$ such that $s <_{\mathrm{T}} t_1 \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} t_2 \leq_{\mathrm{T}} s_2$. Since s is the infimum of s_1, s_2, we obtain that t_1 and t_2 are $\leq_{\mathrm{T}}$-incomparable. Thus, we obtain a contradiction from Lemma 4.18.

(ii) It follows by an analogous argument.

(iii) First point: we argue using Lemma 4.19. If $\varphi^{s_2} \in \{0, 1\}$ we obtain that $\overline{0}$ is a subformula of φ and thus $\varphi^{s_1} \neq s_1$.
Second point: we prove it by induction over the complexity of φ. The base step is immediate.

- If $\varphi = B_u(\psi)$ (or $= B(\psi)$) then we are done (φ^s can be neither 0 nor 1 for any s).
- $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^{s_1} = 0$. Now there are two possibilities. First: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = s_1$. From the induction assumption and from the previous point we get: ${\varphi_1}^{s_2} = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ (or ${\varphi_1}^{s_2} = 0$). The case of $\boldsymbol{\chi}$ is obvious. In the other one in order to obtain $\varphi^{s_2} = 1$ we would need ${\varphi_2}^{s_2} = 0$, but it is not possible, by the previous point (because ${\varphi_2}^{s_1} = s_1$). Second: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = 1$. From the induction assumption (and from the previous point) we obtain: ${\varphi_1}^{s_2} = \boldsymbol{\chi}$ (or ${\varphi_1}^{s_2} = 0$) and ${\varphi_2}^{s_2} = \boldsymbol{\chi'}$ (or ${\varphi_2}^{s_2} = 1$). However, in none of these cases we get $\varphi^{s_2} = 1$. □

Lemma 4.21. *For every nodes s, t and every formulas φ, ψ: if $\varphi^s = 0$ and $\varphi^t = \psi^t = 0$ (or $\varphi^t = \psi^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$), then $\psi^s \neq 1$.*

Proof. We prove it by induction over the complexity of formulas φ and ψ.

- $\varphi = p$: trivial.
- $\varphi = \overline{0}$: if we have $\varphi^t = \psi^t = 0$, then we use Lemma 4.20 to conclude that $\psi^s \neq 1$.
- $\varphi = B_s(\varphi')$ (or $\varphi = B(\varphi')$): trivial.
- $\varphi = \varphi_1 \to \varphi_2$: we discuss two cases. First: if $\varphi^t = 0 = \psi^t$ then, we can again easily use Lemma 4.20 to obtain the conclusion. Second, if $\varphi^t = \boldsymbol{\chi} = \psi^t$, we have that $\psi = \psi_1 \to \psi_2$. We must again deal with two possibilities:

  (i) ${\varphi_1}^s = 0$ and ${\varphi_2}^s = s$. We argue that ${\psi_2}^s = s$ or ${\psi_2}^s = \boldsymbol{\chi'}$ for some $\boldsymbol{\chi'} \in \mathbf{Fm^P}$. Suppose not, i.e. ${\psi_2}^s = 0$ (or $= 1$), by Lemma 4.19 we can conclude that $\overline{0}$ is a subformula also of φ_2 and thus again, by the same lemma, ${\varphi_2}^s \neq s$, a contradiction. From this we infer that it cannot be the case that $\psi^s = 1$.

  (ii) ${\varphi_1}^s = 0$ and ${\varphi_2}^s = 1$. Suppose for contradiction that $\psi^s = 1$, i.e. ${\psi_1}^s = {\psi_2}^s = 0$ (or both are equal to 1). Since the preconditions of this lemma are satisfied for φ_2 and ψ_2 (because, by Lemma 4.20, neither $\psi_2^t = t$ nor $\psi_2^t = 1$), we obtain by the induction assumption that ${\varphi_2}^s \neq 1$, contradiction (the other case is similar). □

Lemma 4.22. *For every formula φ, ψ and every s_1, s_2, it holds: if $\varphi =_{s_1} \psi$ and $\varphi =_{s_2} \psi$, then $\varphi =_s \psi$, where $s = \inf\{s_1, s_2\}$.*

Proof. We prove it by induction over the complexity of φ and ψ.

- $\varphi = p$. It must hold that $\psi^{s_1} = s_1$ and $\psi^{s_2} = s_2$ and, by Lemmas 4.20 and 4.14, we get $\psi^s = s$. Thus, we have verified $\varphi =_s \psi$.
- $\varphi = \bar{0}$. It must hold that $\psi^{s_1} = 0$ and $\psi^{s_2} = 0$. Again, by Lemma 4.20, we get $\psi^s = 0$, and conclude $\varphi =_s \psi$.
- $\varphi = B_t(\varphi')$. Let us inspect what values φ can take. Note that for any s, φ^s can be neither 0 nor 1. Therefore, we have the following possibilities:[4]

cases	value of φ^{s_1}	value of φ^{s_2}
(i)	s_1	s_2
(ii)	$\boldsymbol{\chi}$	s_2
(iii)	s_1	$\boldsymbol{\chi}$
(iv)	$\boldsymbol{\chi_1}$	$\boldsymbol{\chi_2}$

  (i) From Lemma 4.20, we get $\varphi^s = s$ and, since also $\psi^{s_1} = s_1$ and $\psi^{s_2} = s_2$, we can use again Lemma 4.20 and get $\psi^s = s$, i.e. $\varphi =_s \psi$.
  (ii) Since $\psi^{s_1} = \boldsymbol{\chi}$, we can infer that $\psi = B_t(\psi')$ for some ψ'. Now it is not difficult to show that $\varphi' =_{s_1} \psi'$ and $\varphi' =_{s_2} \psi'$. Therefore, by the induction assumption, we get $\varphi' =_s \psi'$, thus clearly also $\varphi =_s \psi$. (Note that in case of $t = \mathbf{r}$ it can also happen $\psi = B(\psi')$, which however behaves in a similar way).
  (iii) Identical.
  (iv) Similar to (ii).

- $\varphi = B(\varphi')$: almost the same.
- $\varphi = \varphi_1 \to \varphi_2$. Here we need to discuss even more cases (realize that we have already rejected a few possibilities in Lemma 4.20):[5]
  (i) Again, easily using Lemma 4.20, we obtain $\varphi^s = s$ and $\psi^s = s$; in other words, $\varphi =_s \psi$.

[4] Note that, thanks to Lemma 4.14, we know that s is the only element in T that can be the value of φ^s.
[5] This time we do not mention symmetric cases.

cases	value of φ^{s_1}	value of φ^{s_2}
(i)	s_1	s_2
(ii)	s_1	$\boldsymbol{\chi}$
(iii)	0	0
(iv)	0	$\boldsymbol{\chi}$
(v)	1	1
(vi)	1	$\boldsymbol{\chi}$
(vii)	$\boldsymbol{\chi_1}$	$\boldsymbol{\chi_2}$

(ii) Since $\varphi^{s_2} = \boldsymbol{\chi}$ and $\varphi = \varphi_1 \to \varphi_2$, we infer that also $\psi = \psi_1 \to \psi_2$. Obviously, $\varphi_1 =_{s_1} \psi_1 = s_1$ and $\varphi_2 =_{s_1} \psi_2 = s_1$. We can also derive $\varphi_1 =_{s_2} \psi_1$ and $\varphi_2 =_{s_2} \psi_2$: we need to distinguish cases based on the formula $\boldsymbol{\chi}$, whether it is: $\boldsymbol{\chi_1} \Rightarrow \boldsymbol{\chi_2}$, $\overline{\mathbf{0}} \Rightarrow \boldsymbol{\chi_1}$, $\overline{\mathbf{1}} \Rightarrow \boldsymbol{\chi_1}$, or $\boldsymbol{p} \Rightarrow \boldsymbol{\chi_1}$ (or some of its symmetric variants). All these cases are easy to check. Now we can apply the induction assumption and obtain $\varphi_1 =_s \psi_1$ and $\varphi_2 =_s \psi_2$ and, hence, conclude $\varphi =_s \psi$.

(iii) As in (i), it follows easily from Lemma 4.20.

(iv) First, since $\varphi^{s_2} = \boldsymbol{\chi}$, we infer $\psi = \psi_1 \to \psi_2$. Now we need to cover two cases based on φ^{s_1}.

1. ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = s_1$: we argue that also ${\psi_1}^{s_1} = 0$ and ${\psi_2}^{s_1} = s_1$. First, if ${\psi_2}^{s_1} = 0$ (or $= 1$), then, by Lemma 4.19, we would get that $\overline{0}$ is a subformula of ψ_2 and, since ${\psi_2}^{s_2} = {\varphi_2}^{s_2}$, we would obtain by the same lemma that $\overline{0}$ is also a subformula of φ_2. Therefore again, by Lemma 4.19, we know ${\psi_2}^{s_1} = s_1$. By Lemma 4.21, we get that ${\psi_1}^{s_1} \neq 1$ thus it must be the case that ${\psi_1}^{s_1} = 0$. Thus, we can conclude $\varphi_1 =_{s_1} \psi_1$ and $\varphi_2 =_{s_1} \psi_2$. It is easy to derive that $\varphi_1 =_{s_2} \psi_1$ and $\varphi_2 =_{s_2} \psi_2$. The rest is an easy consequence of the induction assumption.
2. ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = 1$. If we show that also ${\psi_1}^{s_1} = 0$ and ${\psi_2}^{s_1} = 1$, we are done simply by using the induction assumption. For contradiction suppose it is not the case, i.e. ${\psi_1}^{s_1} = 1$ and ${\psi_2}^{s_1} = 0$ (note that using Lemma 4.19, as in the previous point, we have ${\psi_1}^{s_1} \neq s_1$ and ${\psi_2}^{s_1} \neq s_1$). We get a contradiction from Lemma 4.21 applied on ψ_1 and φ_1.

(v) It is similar to (i) and (iii).

(vi) Again we first argue that also $\psi = \psi_1 \to \psi_2$ (because $\psi^{s_2} = \boldsymbol{\chi}$). Suppose ${\varphi_1}^{s_1} = {\varphi_2}^{s_1} = 0$. Again it is enough to argue that also ${\psi_1}^{s_1} = {\psi_2}^{s_1} = 0$. This is easy to prove, just realize that the only other possibility would be ${\psi_1}^{s_1} = {\psi_2}^{s_1} = 1$, which is by Lemma 4.21 not possible.

(vii) This case is a straightforward application of the induction assumption. □

Finally, we are ready to prove the infimum property:

Proof of Proposition 4.11. For a contradiction suppose that $\varphi^s \notin F$ and both $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$. First, we use Lemma 4.15 to argue that $\varphi^s \neq 0$ which implies $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P} \setminus \mathrm{D}$. Since $\varphi^s \neq \varphi^{s_1}$ and $\varphi^s \neq \varphi^{s_2}$, we can use Lemma 4.17 to infer that there are nodes t_1 and t_2 such that $s <_{\mathrm{T}} t_1 \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} t_2 \leq_{\mathrm{T}} s_2$ and subformulas $B_{t_1}(\psi_1)$ and $B_{t_2}(\psi_2)$ of the formula φ. Then, since $s = \inf\{s_1, s_2\}$, we obtain that $t_1 \not\leq_{\mathrm{T}} s_2$ and $t_2 \not\leq_{\mathrm{T}} s_1$, therefore by Lemma 4.18 it follows $\varphi^{s_1} = \boldsymbol{\chi_1} \in \mathrm{D}$ and $\varphi^{s_2} = \boldsymbol{\chi_2} \in \mathrm{D}$. Consequently, $\boldsymbol{\chi_1} = \boldsymbol{\chi_1'} \Rightarrow \boldsymbol{\chi_1'}$ and $\boldsymbol{\chi_2} = \boldsymbol{\chi_2'} \Rightarrow \boldsymbol{\chi_2'}$ for some $\boldsymbol{\chi_1'}, \boldsymbol{\chi_2'} \in \mathbf{Fm^P}$. We can now easily conclude that $\varphi = \varphi_1 \to \varphi_2$ and $\varphi_1 =_{s_1} \varphi_2 = \boldsymbol{\chi_1'}$ and $\varphi_1 =_{s_2} \varphi_2 = \boldsymbol{\chi_2'}$ thus by Lemma 4.22 also $\varphi_1 =_s \varphi_2$, contradiction (Observation 4.7). □

4.4.3 Proof of RSI-completeness

Finally we prove that L is RSI-complete. From Corollary 4.8 we know that $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is reduced. Moreover, by definition, $\mathbf{A}$ is a complete semantics for L. Therefore to prove RSI-completeness, it is enough to show that $\mathbf{A} \in \mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L}$. Let us now prove that F is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$. To obtain this result we recursively define for every $\boldsymbol{\chi} \in \mathbf{Fm^P}$ a corresponding formula $\varphi_{\boldsymbol{\chi}}$ written in a fixed variable p:

Definition 4.23. We define formulas $\varphi_{\boldsymbol{\chi}}$ recursively as follows:

(i) (a) $\varphi_{\boldsymbol{B_s(p)}} = B_s(B(p))$,
  (b) $\varphi_{\boldsymbol{B_s(\overline{0})}} = B_s(0 \to B(p))$,
  (c) $\varphi_{\boldsymbol{B_s(\overline{1})}} = B_s(\overline{0} \to (\overline{0} \to B(p)))$,

(ii) (a) $\varphi_{\boldsymbol{B_s(\psi)}} = B_s(\varphi_{\boldsymbol{\psi}})$,
  (b) $\varphi_{\boldsymbol{\overline{0} \Rightarrow \psi}} = \overline{0} \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi \Rightarrow \overline{0}}} = \varphi_{\boldsymbol{\psi}} \to \overline{0}$,
  (c) $\varphi_{\boldsymbol{\overline{1} \Rightarrow \psi}} = \overline{1} \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi \Rightarrow \overline{1}}} = \varphi_{\boldsymbol{\psi}} \to \overline{1}$,
  (d) $\varphi_{\boldsymbol{p \Rightarrow \psi}} = p \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi \Rightarrow p}} = \varphi_{\boldsymbol{\psi}} \to p$,
  (e) $\varphi_{\boldsymbol{\psi \Rightarrow \psi'}} = \varphi_{\boldsymbol{\psi}} \to \varphi_{\boldsymbol{\psi'}}$.

Using an easy induction over the complexity of the set $\mathbf{Fm^P}$ we obtain:

Observation 4.24. For every $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and every evaluation e such that $e(p) = \star$, it holds that $e(\varphi_{\boldsymbol{\chi}}) = \boldsymbol{\chi}$.

Note that in the previous observation we benefited from the connective B (namely from the fact that $B(\star) = \star$). In order to obtain the RSI-completeness, we need the following proposition:

Proposition 4.25. *For every* $\boldsymbol{\chi} \in \mathbf{Fm^p} \setminus \mathrm{D}$*:* $\varphi_{\boldsymbol{\chi}} \vdash_{\mathrm{L}} B_{\mathbf{r}}(p)$*.*

We postpone its proof till later and proceed immediately to the main theorem of this section.

Theorem 4.26. *The logic* L *is* RSI*-complete.*

Proof. We show that F is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$. Let us consider a non-trivial $F' \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ which strictly contains F. First, observe that $0 \notin F'$ (because $\overline{0} \vdash_{\mathrm{L}} p$, which would imply that F' is trivial). It follows that there is some $\boldsymbol{\chi} \in \mathbf{Fm^p} \setminus \mathrm{D}$ which is also in F'. However, from the previous proposition, we know that $\varphi_{\boldsymbol{\chi}} \vdash_{\mathrm{L}} B_{\mathbf{r}}(p)$; thus, if we consider an evaluation e such that $e(p) = \star$, we obtain, by Observation 4.24, that $e(\varphi_{\boldsymbol{\chi}}) = \boldsymbol{\chi} \in F'$, which implies that also $e(B_{\mathbf{r}}(p)) = \boldsymbol{B}_{\mathbf{r}}(\boldsymbol{p}) \in F'$. Therefore any non-trivial filter strictly above F contains $\boldsymbol{B}_{\mathbf{r}}(\boldsymbol{p})$. In other words: F is completely intersection-prime, which completes the proof. ☐

Now it remains to prove Proposition 4.25, but first need some technical lemmata. In upcoming proofs we will tacitly be using next observation (easily provable by induction):

Observation 4.27. Let us have an evaluation e such that $e(p) \notin \mathrm{T}$. Then, for every $\boldsymbol{\chi} \in \mathbf{Fm^p}$, it holds that $e(\varphi_{\boldsymbol{\chi}}) \in \mathbf{Fm^p}$.

Further we define a subformula order on $\mathbf{Fm^p}$: we write $\boldsymbol{\chi} \leq_{\mathbf{Fm^p}} \boldsymbol{\chi'}$ iff $\boldsymbol{\chi}$ is a subformula of $\boldsymbol{\chi'}$. It takes again an easy induction argument to prove that for every $\boldsymbol{\chi}, \boldsymbol{\chi'} \in \mathbf{Fm^p}$ and any evaluation e sending p to $\boldsymbol{\chi}$ that

$$\boldsymbol{\chi} <_{\mathbf{Fm^p}} e(\varphi_{\boldsymbol{\chi'}}) \text{ and thus } e(\varphi_{\boldsymbol{\chi'}}) \neq \boldsymbol{\chi} \tag{4.5}$$

Lemma 4.28. *For every* $\boldsymbol{\chi} \in \mathbf{Fm^p}$ *and for every evaluation* e *we have:*

(i) if $e(p) = 0$, then $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$, and $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow (\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}}))$.

(ii) if $e(p) = 1$, then $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}})$, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{1}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}})$, and $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{1}} \Rightarrow (\overline{\mathbf{1}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}}))$

(iii) if $e(p) = \boldsymbol{\chi'}$ for some $\boldsymbol{\chi'} \in \mathbf{Fm^p}$, then $e(\varphi_{\boldsymbol{\chi'}}) \neq \boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi'})$, $e(\varphi_{\boldsymbol{\chi'}}) \neq \overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi'})$ and $e(\varphi_{\boldsymbol{\chi'}}) \neq \overline{\mathbf{0}} \Rightarrow (\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi'}))$.

Proof. We prove only the first point; the others are even simpler. Case of $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}})$: obvious. Second, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}})$: this possible happen only if (1) $\boldsymbol{\chi} = \boldsymbol{p} \Rightarrow \boldsymbol{\chi}'$ (in this case $\varphi_{\boldsymbol{\chi}} = p \to \varphi_{\boldsymbol{\chi}'}$), then $e(\varphi_{\boldsymbol{\chi}}) = \overline{\mathbf{0}} \Rightarrow e(\varphi_{\boldsymbol{\chi}'})$, the rest easily follows from the first inequation; the second possibility $\boldsymbol{\chi} = \overline{\mathbf{0}} \Rightarrow \boldsymbol{\chi}'$ is proved analogously. Finally, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow (\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}}))$: similar but using the second inequation. □

Lemma 4.29. *For every* $\boldsymbol{\chi_1}, \boldsymbol{\chi_2} \in \mathbf{Fm^p}$ *and for every evaluation* e *such that* $e(p) = 0$ *or* $e(p) = 1$ *or* $e(p) = \boldsymbol{\chi}$ *for some* $\boldsymbol{\chi} \in \mathbf{Fm^p}$*, we have:*

$$e(\varphi_{\boldsymbol{\chi_1}}) = e(\varphi_{\boldsymbol{\chi_2}}) \quad \iff \quad \boldsymbol{\chi_1} = \boldsymbol{\chi_2}.$$

Proof. We prove this lemma only for evaluations e such that $e(p) = 0$, the other cases follow almost in the same way (they are only using different points from the previous lemma). This lemma is then proved by induction over the complexity of $\boldsymbol{\chi_1}, \boldsymbol{\chi_2}$ according to the definition of $\mathbf{Fm^p}$:

(i) (a) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\boldsymbol{p})$: we get $\varphi_{\boldsymbol{\chi_1}} = B_s(B(p))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}}))$. Now it is easy to see that the conclusion follows for base step for $\boldsymbol{\chi_2}$ (i.e. points 1.(a),(b),(c)). Moreover, for the induction step there is only one more complicated variant, namely 2.(a) (for the rest obviously $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$): suppose $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$, but, by Lemma 4.28, $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}})$, i.e. $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

(b) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\overline{\mathbf{0}})$: $\varphi_{\boldsymbol{\chi_1}} = B_s(\overline{0} \to B(p))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}}))$; we again deal only with the case 2.(a). So let $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$. However, again by Lemma 4.28, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}})$ and therefore $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

(c) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\overline{\mathbf{1}})$: then $\varphi_{\boldsymbol{\chi_1}} = B_s(\overline{0} \to (\overline{0} \to B(p)))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\overline{\mathbf{0}} \Rightarrow (\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}})))$. Again 2.(a): $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$. However, by Lemma 4.28, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}} \Rightarrow (\overline{\mathbf{0}} \Rightarrow \boldsymbol{B}_{\mathrm{r}}(\overline{\mathbf{0}}))$.

(ii) (a) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\boldsymbol{\chi}_1')$: $\varphi_{\boldsymbol{\chi_1}} = B_s(\varphi_{\boldsymbol{\chi}_1'})$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\boldsymbol{e}(\boldsymbol{\varphi}_{\boldsymbol{\chi}_1'}))$ the base step for $\boldsymbol{\chi_2}$ follows by the first part of this proof. Case 2.(a): $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi}_2')$ and $e(\varphi_{\boldsymbol{\chi_2}}) = \boldsymbol{B_s}(\boldsymbol{e}(\boldsymbol{\varphi}_{\boldsymbol{\chi}_2'}))$. We obtain the result easily from the induction assumption. For the other cases we trivially get $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

(b) $\boldsymbol{\chi_1} = \overline{0} \Rightarrow \boldsymbol{\chi}_1'$: $\varphi_{\boldsymbol{\chi_1}} = \overline{0} \to \varphi_{\boldsymbol{\chi}_1'}$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \overline{\mathbf{0}} \Rightarrow e(\varphi_{\boldsymbol{\chi}_1'})$. The base step for $\boldsymbol{\chi_2}$ is trivial. Moreover the only interesting induction cases for $\boldsymbol{\chi_2}$ are 2.(b) and 2.(d)-which are treated in the same way: we obtain $\boldsymbol{\chi_2} = \overline{\mathbf{0}} \Rightarrow \boldsymbol{\chi}_2'$ and $e(\varphi_{\boldsymbol{\chi_2}}) = \overline{\mathbf{0}} \Rightarrow e(\varphi_{\boldsymbol{\chi}_2'})$. The rest easily follows by induction assumption. In the remaining cases we can simply observe $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

(c),(d),(e) Similar. □

Now we have all the ingredients to prove the key proposition:

Proof of Proposition 4.25. It is enough to show that for any evaluation e such that $e(p) \in \mathbf{Fm}^{\mathbf{P}} \cup \{0, 1, \star\}$ we have $e(\varphi_{\boldsymbol{\chi}}) \notin F$. If $e(p) = \star$ we argue using Observation 4.24. For other evaluations we distinguish two possible scenarios in which we could get $e(\varphi_{\boldsymbol{\chi}}) \in F$ (note that by Observation 4.27 we would get $e(\varphi_{\boldsymbol{\chi}}) \in \mathrm{D}$, therefore the only possible cases are 2.(d) and (e) of Definition 4.23). First, if $\boldsymbol{\chi} = \boldsymbol{p} \Rightarrow \boldsymbol{\chi}'$, we get $e(\varphi_{\boldsymbol{\chi}}) = e(p) \to^{\mathbf{A}} e(\varphi_{\boldsymbol{\chi}'})$. If $e(p) = 0$ or $e(p) = 1$, then obviously $e(\varphi_{\boldsymbol{\chi}}) \notin F$ and, if $e(p) = \boldsymbol{\chi}''$, we conclude $e(\varphi_{\boldsymbol{\chi}}) \notin F$ by (4.5). Second, assume that $\boldsymbol{\chi} = \boldsymbol{\chi_1} \Rightarrow \boldsymbol{\chi_2}$. Since $\boldsymbol{\chi_1} \neq \boldsymbol{\chi_2}$, we can use Lemma 4.29 to obtain $e(\varphi_{\boldsymbol{\chi}}) \notin F$. □

This finishes the proof that L is an RSI-complete logic (Theorem 4.26) without the IPEP (Theorem 4.13).

4.5 Conclusion and remarks

In this chapter we have presented all the examples necessary to separate the classes in the hierarchy (see Figure 4.3). Recall we showed that $Ł_\infty$ has the τ-CIPEP already in Theorem 3.22. Moreover, we have seen that there are examples of logics that lie outside the hierarchy (Section 4.1).

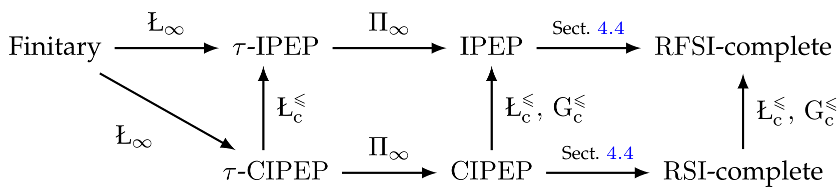


Figure 4.3: The hierarchy with the separating examples

It is of some interest to comment on the 'naturality' of the separating examples. While the first two, $Ł_\infty$ and Π_∞, are arguably quite natural, the rest are rather artificial and *ad hoc* examples built only for the separating purposes (though $Ł_c^{\leqslant}$ can be seen as a well-motivated example). A remaining question is whether we can find more natural examples separating the CIPEP from RFSI-completeness.

Also an interesting feature of $Ł_c^{\leqslant}$, of the logic from Section 4.4, as well as the one from Section 4.1 is that all of them have infinite type. So another question is can we find examples with finite one?

On the other hand, a measure of the good behavior of a logical system, from the point of view of abstract algebraic logic, is given by its position in the Leibniz hierarchy. From this perspective, the presented examples feature quite well. The first two $Ł_\infty$ and Π_∞ are at the very top of the hierarchy (they are Rasiowa-implicative), $Ł_c^\leqslant$ is weakly implicational (equivalential), the example of Section 4.4 is weakly implicative.[6] And finally the example outside the hierarchy (Section 4.1) is again as good as possible, namely Rasiowa-implicative. This also yields the question whether all separating examples can be found in the class of algebraizable logics. To this end, we remark that a version of $Ł_c^\leqslant$ based on Gödel logic G, we denote it $\mathrm{G}_c^\leqslant$, again has no RSI-models and has the IPEP (we do not know about the τ-IPEP). On the other hand $\mathrm{G}_c^\leqslant$, unlike $Ł_c^\leqslant$, is even Rasiowa-implicative (top of the Leibniz hierarchy). We present $\mathrm{G}_c^\leqslant$ in Subsection 5.1.1.

Finally, let us recall that the properties τ-(C)IPEP are in protoalgebraic logics equivalent to subdirect representation theorem (Theorem 3.19). Therefore, the example of section 4.2, Π_∞, is not subdirectly representable. Consequently, also the equivalent algebraic semantics of Π_∞, i.e. the generalized quasi-variety generated by $[0,1]_\Pi$, is not subdirectly representable. From the perspective of universal algebra, we have shown that Birkhoff's theorem does not generalize beyond quasi-varieties.

[6] One can check, that it is not even order-algebraizable in the sense of [17]. This, in particular, means it is not algebraizable.

Part II:

Theories and Connectives

5 | Linear and prime theories

In the first part of the thesis we have described a hierarchy of infinitary logics. In this part we will study the interaction of between the hierarchy and some well-known classes of (non-)classical logics concentrating on the role of some connectives.

First, in this chapter, we will examine two properties of intersection-prime filters, theories, and models:

- linearity (for logics with implication),
- primality (for logics with disjunction).

Then, in Chapter 6, we will investigate another property, this time related to **completely** intersection-prime theories, filters, and models:

- simplicity (for logics with negation).

In the first section of this chapter, we start with logics that are complete w.r.t. linearly ordered models known as *semilinear* logics (which actually contain many examples seen in Part I). We will recall the basic facts about this well-studied class of logics [25, 29, 30] and describe their interaction with the general theory of Part I.

In the remaining part of this chapter, we focus on logics with a disjunction, which again thoroughly studied in the literature [27, 32, 35, 48, 51, 89]. After a general presentation, we will see that having some form of a disjunction can be very useful to prove completeness results for (not only) infinitary logics. Most importantly, we prove an abstract form of the Lindenbaum lemma for logics with countably axiomatizable logics with disjunction and we demonstrate its applicability. We also relate the Lindenbaum lemma to the *pair extension property* [40] and various forms of cut rules.

5.1 Semilinear logics

In this section we will describe a subclass of weakly p-implicational logics which is complete w.r.t. subclass of reduced models, where the natural order induced by the weak p-implication is linear. Such class of logics was introduced in [25] as an abstraction of fuzzy logics.

Let L be a logic with a weak p-implication $\Rightarrow$. We say that a reduced model $\langle \boldsymbol{A}, F\rangle$ of L is *linear w.r.t.* $\Rightarrow$ if the order $\leq^{\mathbf{A}}_{\Rightarrow}$ is linear. We denote the class of all linear models w.r.t. $\Rightarrow$ as $\mathbf{Mod}^{\ell}_{\Rightarrow}\mathrm{L}$. Weak p-implication is called *semilinear* if the corresponding linear models are complete semantics for L, that is $\vdash_{\mathrm{L}} \;=\; \models_{\mathbf{Mod}^{\ell}_{\Rightarrow}\mathrm{L}}$. Finally, L is called *semilinear* if it has a semilinear weak p-implication. For example $\mathrm{CL}, \mathrm{G}, \mathrm{BL}, \mathrm{Ł}_{\infty}, \Pi_{\infty}$ are all semilinear with $\to$ being a semilinear implication—however note that $\leftrightarrow$ is still an implication in these logics, but it is not semilinear. Also, $\mathrm{Ł}^{\leqslant}_{\mathrm{c}}$ is semilinear with implication

$$x \Rightarrow y = \{(x \to y)^n \mid n \in \omega\}.$$

As the next example shows, a logic can have two semilinear implications which are not mutually interderivable (they define different orders).

Example 5.1. Let us consider a logic L in language with two binary connectives $\to_1$ and $\to_2$, given semantically by matrix $\langle \boldsymbol{A}, \{1\}\rangle$ where the algebra $\boldsymbol{A}$ has universe $A = \{1, a, b\}$. Consider two linear orders on A: first $b \leq_1 a \leq_1 1$ and second $a \leq_2 b \leq_2 1$. The operations on $\boldsymbol{A}$ are given by $x \to_i y = 1$ if $x \leq_i y$ and $x \to_i y = a$ otherwise for $i \in \{1, 2\}$. Clearly both $\to_1, \to_2$ are semilinear implications, but $p \to_1 q \nvdash_{\mathrm{L}} p \to_2 q$.

Even though, in principle, a logic can have more semilinear implications, it is always the case that they induce the same linear models (it follows from Theorem 5.2 point 3e). Therefore, for a semilinear logic L, we can simply denote the class of its linear models as $\mathbf{Mod}^{\ell}\,\mathrm{L}$.

An L-filter F on $\boldsymbol{A}$ is called *linear* w.r.t. to a weak p-implication $\Rightarrow$ (or for short $\Rightarrow$-linear) if for every $a, b \in A$ either $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$ or $b \Rightarrow^{\boldsymbol{A}} a \subseteq F$ or equivalently if $\leq^{\mathbf{A}^*}_{\Rightarrow}$ is a linear order, where $\mathbf{A}^* = \langle \boldsymbol{A}, F\rangle^*$. Analogously we define *linear theories*. L is said to have *linear extension property*, LEP for short, if the family of $\Rightarrow$-linear theories forms a basis of $\mathrm{Th}\,\mathrm{L}$ for some weak p-implication $\Rightarrow$. Similarly, a logic has the *transferred linear extension property*, τ-LEP, when linear filters forms a basis of $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$. A logic is said to have have a *semilinearity property* if there is a weak p-implication $\Rightarrow$ validating the following meta-rule for every $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$:

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

A logic L has a transferred version of this property, τ-SLP, if for every $X \cup a, b \subseteq A$ on every algebra $\boldsymbol{A}$ we have:

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X \cup a \Rightarrow^{\boldsymbol{A}} b) \cap \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X \cup b \Rightarrow^{\boldsymbol{A}} a) = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X).$$

The following theorem, proved in [29, Theorem 3], characterizes semilinear logics by means of the aforementioned notions.

Theorem 5.2. *Let* L *be a logic with a weak p-implication* $\Rightarrow$*. Then, the following are equivalent*

(i) $\Rightarrow$ *is semilinear in* L*.*
(ii) L *has the* LEP *for* $\Rightarrow$*.*
(iii) L *has the* IPEP *and any of the following holds:*
   3a. L *has the* SLP *for* $\Rightarrow$*.*
   3b. L *has the* τ-SLP *for* $\Rightarrow$*.*
   3c. $\Rightarrow$*-linear and intersection-prime filters coincide on each* $\mathcal{L}$*-algebra.*
   3d. $\Rightarrow$*-linear and intersection-prime theories coincide.*
   3e. $\mathbf{Mod}^{*}_{\mathrm{RFSI}}\,\mathrm{L} = \mathbf{Mod}^{\ell}_{\Rightarrow}\,\mathrm{L}$*.*

If furthermore L *is finitary, we can add two more equivalent conditions:*

(iv) L *has* τ-LEP *witnessed by* $\Rightarrow$*.*
(v) $\mathbf{Mod}^{*}_{\mathrm{RSI}}\,\mathrm{L} \subseteq \mathbf{Mod}^{\ell}_{\Rightarrow}\,\mathrm{L}$*.*

Let us comment a little on what the theorem says. First of all observe that the linear extension property amounts to intersection-prime extension property plus the meta-rule SLP:

$$(\tau)\text{-LEP} \quad \Longleftrightarrow \quad (\tau)\text{-IPEP} \quad + \quad \text{SLP}.$$

Secondly, semilinear logics are by definition RFSI-complete, and by the theorem they always enjoys the IPEP (resp. LEP). This can be seen in the following way: the IPEP and RFSI-completeness are actually equivalent properties in the presence of the meta-rule SLP. This fact is a consequence of two important properties of semilinear logics:

(i) Semilinear logics are by definition protoalgebraic.
(ii) In semilinear logics the class $\mathbf{Mod}_{\mathrm{RFSI}}\,\mathrm{L}$ is closed under submatrices.

The second point is an obvious consequence of the point 3c. Thus, we could use e.g. Corollary 3.13 to see that all semilinear logics enjoy the IPEP.

While describing the hierarchy we have already come across some new results regarding semilinear logics. We will briefly summarize them now:

- IPEP is the smallest class in the hierarchy that contains all semilinear logics. Indeed, Π_∞ (Theorem 4.1) does not enjoy the τ-IPEP and $Ł_c^{\leqslant}$ is not RSI-complete (Section 4.3). Note that Π_∞ shows that not even the LEP transfers in general.
- All semilinear logics are RFSI-complete, but not all of them are RSI-complete as shown again by $Ł_c^{\leqslant}$.
- Semilinear logics take their name from the fact their finitely subdirectly irreducible models are linear, that is, $\mathbf{Mod}^\ell\,\mathrm{L} = \mathbf{Mod}^*\,\mathrm{L}_{\mathrm{RFSI}}$, and moreover they are complete w.r.t. these models. However, this does not entail that they are always finitely subdirectly representable (i.e. representable by chains), as shown again by the infinitary product logic Π_∞ (see Section 4.2).
- Our preservation result of the τ-CIPEP and the τ-IPEP under axiomatic expansions allows to show that many other interesting infinitary semilinear logics enjoy these properties, such as the usual expansions with the projection connective Δ or other truth hedges, logics with additional truth-constants, logics with additional involutive negation, etc. [44].

Lastly, we show that the additional assumption on finitarity in Theorem 5.2 is indeed necessary: we already saw that the infinitary product logic Π_∞ does not have the τ-IPEP, thus it cannot have the τ-LEP (see Section 4.2). For the second case, point (v) of the theorem, we can use the logic $Ł_c^{\leqslant}$ defined in Section 4.3: indeed $\Leftrightarrow$, the symmetrization of $\Rightarrow$, is not semilinear in $Ł_c^{\leqslant}$ (on reduced models $\Leftrightarrow^{\mathbf{A}}$ is always the identity relation), but clearly $\emptyset = \mathbf{Mod}^*_{\mathrm{RSI}}\,Ł_c^{\leqslant} \subsetneq \mathbf{Mod}^\ell_{\Leftrightarrow}\,Ł_c^{\leqslant}$: in the next subsection we show that there is even Rasiowa-implicative logic with this property (Recall that $Ł_c^{\leqslant}$ is not even algebraizable).

5.1.1 Truth-degrees-preserving Gödel logic with constants

Consider a language with denumerable set of variables and type

$$\mathcal{L} = \{\to\} \cup \{\bar{q} \mid q \in (0,1] \cap \mathbb{Q}\}.$$

For every $0 < q \leq 1$, define an $\mathcal{L}$-algebra $\boldsymbol{A}_q$ with domain $[0, q]$, $a \to b = q$ if $a \leq b$ and $a \to b = b$ otherwise (i.e. it is a Gödel implication); and for constants $\bar{r}^{\boldsymbol{A}_q} = \min\{r, q\}$. Define $\mathbb{K} = \{\mathbf{A}_q = \langle \boldsymbol{A}_q, \{q\}\rangle \mid q \in (0,1] \cap \mathbb{Q}\}$ and let $\mathrm{G}_c^{\leqslant}$ be the logic given by $\mathbb{K}$.

Till the end of this section we will use letters q, r, s as variables for rational numbers in the interval $[0, 1]$. Given any two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ with $q < r$

and given any $\mathbf{A}_r$-evaluation e, we define an $\mathbf{A}_q$-evaluation e^q as follows: $e^q(p) = \min\{e(p), q\}$ for each variable p. Using induction on the complexity of formulas we can easily prove the following observation. Consider two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ such that $q < r$. Then, for every $\mathbf{A}_r$-evaluation e and every formula φ we have:

(i) $e(\varphi) \in [q, r]$ iff $e^q(\varphi) = q$,
(ii) if $e(\varphi) < q$ then $e(\varphi) = e^q(\varphi)$.

Using the previous observation we can easily characterize the consequence in $\mathrm{G}_\mathrm{c}^\leqslant$: for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$

$$\Gamma \vdash_{\mathrm{G}_\mathrm{c}^\leqslant} \varphi \quad \Longleftrightarrow \quad \bigwedge v[\Gamma] \leq v(\varphi), \text{ for all } v \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A}_1),$$

which justifies to call $\mathrm{G}_\mathrm{c}^\leqslant$ the *truth-degrees-preserving Gödel logic with constants*. It is easy to see that $\mathrm{G}_\mathrm{c}^\leqslant$ is semilinear and Rasiowa-implicative w.r.t. $\to$. Moreover, it is infinitary, as witnessed by the rule

$$\{\bar{q} \to \varphi \mid q \in (0,1)\} \vdash_{\mathrm{G}_\mathrm{c}^\leqslant} \varphi.$$

The set of all constants $\{\bar{q} \mid q \in (0,1] \cap \mathbb{Q}\}$ is clearly an infinite antitheorem (cf. Corollary 2.10). On the other it can easily be seen that $\mathrm{G}_\mathrm{c}^\leqslant$ has no finite antitheorem: any finite set of formulas Γ contains only finitely many constants, consequently Γ can be satisfied in $\mathbf{A}_q$ for q smaller then all of the constants in Γ (or q arbitrary if there are no constants in Γ) by an evaluation sending every variable to q. It follows that $\mathrm{G}_\mathrm{c}^\leqslant$ is not compact.

Presenting $\mathrm{G}_\mathrm{c}^\leqslant$ by the class $\mathbb{K}$ allows us to easily compute the class of all reduced models, $\mathbf{Mod}^*\,\mathrm{L}$. We will see that $\mathrm{G}_\mathrm{c}^\leqslant$ in fact has no subdirectly irreducible models: i.e. $\mathbf{Mod}^*\,{\mathrm{G}_\mathrm{c}^\leqslant}_{\mathrm{RSI}} = \emptyset$. Consequently, since $\Leftrightarrow$, the symmetrization of $\to$, is clearly a weak p-implication and $\mathbf{Mod}^*\,{\mathrm{G}_\mathrm{c}^\leqslant}_{\mathrm{RSI}} \subseteq \mathbf{Mod}^\ell_\leftrightarrow(\mathrm{G}_\mathrm{c}^\leqslant)$, we can conclude that $\mathrm{G}_\mathrm{c}^\leqslant$ is the desired counterexample, indeed $\leftrightarrow$ is not semilinear, because on every reduced model it defines the identity relation.

It remains to show that $\mathrm{G}_\mathrm{c}^\leqslant$ really has no subdirectly irreducible models. To this end, let K denote the class of algebraic reducts of matrices in $\mathbb{K}$. Knowing that L is Rasiowa-implicative, we obtain two important consequences:

- $\mathbf{Alg}^*\mathrm{L}$ is the generalized quasi-variety generated by K, i.e.

$$\mathbf{Alg}^*\mathrm{L} = \mathbf{GQ}(\mathsf{K}).$$

- For every $\boldsymbol{A} \in \mathbf{Alg}^*\mathrm{L}$, there is a unique filter F making $\langle \boldsymbol{A}, F\rangle$ reduced, namely $F = \{\bar{1}^{\boldsymbol{A}}\}$.

Therefore we obtain the following simple characterization of all reduced matrix models of L:

$$\mathbf{Mod}^* \mathrm{L} = \{\langle \boldsymbol{A}, \{\bar{1}^{\boldsymbol{A}}\}\rangle \mid \boldsymbol{A} \in \mathbf{UISP}(\mathsf{K})\}.$$

Thus, on every algebra $\boldsymbol{A} \in \mathbf{Alg}^*\mathrm{L}$, we have a canonical ordering given by the unique filter on $\boldsymbol{A}$: $a \leq^{\boldsymbol{A}} b$ iff $a \to^{\boldsymbol{A}} b = \bar{1}^{\boldsymbol{A}}$, for each $a, b \in A$. For every algebra $\boldsymbol{A} \in \mathbf{Alg}^*\mathrm{L}$, we will denote as $\uparrow^{\boldsymbol{A}} q$ the set of all elements in A bigger than $\bar{q}^{\boldsymbol{A}}$ w.r.t. $\leq^{\boldsymbol{A}}$, in symbols: $\uparrow^{\boldsymbol{A}} q = \{a \in A \mid \bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} a\}$.

Lemma 5.3. *For any $\boldsymbol{A}_r$ and $q < r$, $\uparrow^{\boldsymbol{A}_r} q = [q, r]$ is a $\mathrm{G}_{\mathrm{c}}^{\lessgtr}$-filter on $\boldsymbol{A}_r$. In particular, $\mathbf{A}_r \notin \mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L}$.*

Proof. Suppose that $\Gamma \vdash_{\mathrm{G}_{\mathrm{c}}^{\lessgtr}} \varphi$ and $e[\Gamma] \subseteq [q, r]$. We know that $e^q[\Gamma] \subseteq \{q\}$ and, since $\mathbf{A}_q$ is a model, we obtain $e^q(\varphi) = q$ and again, from the observation, $e(\varphi) \in [q, r]$. For the last claim, clearly $\{r\} = \bigcap_{q<r}[q, r]$. □

Lemma 5.4. *The unique reduced matrix based on each algebra from $\mathbf{SP}(\mathsf{K})$ is not relatively subdirectly irreducible in $\mathbf{Mod}^*\,\mathrm{L}$.*

Proof. Let $\boldsymbol{B} \in \mathbf{SP}(\mathsf{K})$. $\boldsymbol{B}$ is a subalgebra of some direct product of algebras $\boldsymbol{C} = \prod_{i\in I} \boldsymbol{C}_i$ for $\boldsymbol{C}_i \in \mathsf{K}$. The only filter that makes $\boldsymbol{B}$ reduced is $\{\bar{1}^{\boldsymbol{C}}\}$; we show it is completely intersection-prime: it is easy to observe that for any system of filters $F_i \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{C}_i$ we have $\prod_{i\in I} F_i \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{C}$. In particular, if we choose $F_i = \uparrow^{\boldsymbol{C}_i} q$, then $\prod_{i\in I} F_i = \uparrow^{\boldsymbol{C}} q$ is a filter on $\boldsymbol{C}$.

Define $Z = \{q \in (0, 1) \mid$ there is some $\boldsymbol{C}_i$ with domain $[0, r]$ and $q < r\}$. Observe that for every $q \in Z$ we have $\{\bar{1}^{\boldsymbol{C}}\} \subsetneq \uparrow^{\boldsymbol{C}} q$ and moreover $\{\bar{1}^{\boldsymbol{C}}\} = \bigcap_{q\in Z} \uparrow^{\boldsymbol{C}} q$.

Further $\uparrow^{\boldsymbol{B}} q = \uparrow^{\boldsymbol{C}} q \cap B$ is an L-filter on $\boldsymbol{B}$ and, since, for every $q \in Z$: $\bar{1}^{\boldsymbol{B}} = \bar{1}^{\boldsymbol{C}} \neq \bar{q}^{\boldsymbol{C}} = \bar{q}^{\boldsymbol{B}} \in \uparrow^{\boldsymbol{B}} q$ we conclude $\{\bar{1}^{\boldsymbol{B}}\} \subsetneq \uparrow^{\boldsymbol{B}} q$ and finally $\{\bar{1}^{\boldsymbol{B}}\} = \bigcap_{q\in Z} \uparrow^{\boldsymbol{B}} q$. Thus, $\langle \boldsymbol{B}, \{\bar{1}^{\boldsymbol{B}}\}\rangle \notin \mathbf{Mod}^*_{\mathrm{RSI}}\,\mathrm{L}$. □

Now we are heading towards the same claim for $\mathbf{UISP}(\mathsf{K})$. We first show some properties of chains in $\mathbf{SP}(\mathsf{K})$:

Lemma 5.5. *For any chain $\boldsymbol{A} \in \mathbf{SP}(\mathsf{K})$ and any $a \in A$ such that $a < \bar{1}^{\boldsymbol{A}}$ there is some q such that $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.*

Proof. Start with a subalgebra $\boldsymbol{A}$ of a direct product of algebras $\boldsymbol{B} = \prod_{i\in I} \boldsymbol{B}_i$. Let us have $a \in A$ such that $a < \bar{1}^{\boldsymbol{A}}$. Clearly there is i such that $a(i) < \bar{1}^{\boldsymbol{A}_i}$ and consequently some $q \in (0, 1)$ such that $a(i) < \bar{q}^{\boldsymbol{A}_i} < \bar{1}^{\boldsymbol{A}_i}$. From linearity we know that either $a < \bar{q}^{\boldsymbol{A}}$ or $\bar{q}^{\boldsymbol{A}} \leq a$ is true. Clearly, the second possibility would lead to contradiction. Thus, since obviously $\bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$, we are done. □

Proposition 5.6. *The unique reduced matrix based on each algebra from* $\mathbf{UISP}(\mathsf{K})$ *is not relatively subdirectly irreducible in* $\mathbf{Mod}^*\,\mathrm{G}^{\leqslant}_{\mathrm{c}}$*. That is,* $\mathbf{Mod}^*\,\mathrm{G}^{\leqslant}_{\mathrm{c}\,\mathrm{RSI}} = \emptyset$*.*

Proof. In pursuit of a contradiction suppose that there is $\boldsymbol{A}$ in $\mathbf{UISP}(\mathsf{K})$ such that the unique L-filter $\{\bar{1}^{\boldsymbol{A}}\}$ is completely intersection-prime. First note that this implies that $\langle \boldsymbol{A}, \leq\rangle$ is linear with maximum element $\bar{1}^{\boldsymbol{A}}$ (see Theorem 5.2).

Claim 1: *For every* $a \in A$ *such that* $a < \bar{1}^{\boldsymbol{A}}$ *there is* $q \in (0,1)$ *such that* $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.

Proof (of Claim 1): Let $\langle a\rangle$ be the subalgebra generated by the element a. Since it is countable generated, we have

$$i : \langle a\rangle \simeq \boldsymbol{B} \hookrightarrow \textstyle\prod_{i\in I} \boldsymbol{B}_i.$$

Further, since $\boldsymbol{B}$ is a chain (due to the isomorphism i) and $\boldsymbol{B} \in \mathbf{SP}(\mathsf{K})$, we can find the desired q by applying i and Lemma 5.5.

Claim 2: $\uparrow^{\boldsymbol{A}} q$ *is a filter on* $\boldsymbol{A}$ *for every* $q \in (0,1)$.

Proof (of Claim 2): Suppose $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq \uparrow^{\boldsymbol{A}} q$. It is clear that $e[\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}]$ is a countably generated subalgebra of $\boldsymbol{A}$ thus we have

$$i : e[\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}] \simeq \boldsymbol{B} \hookrightarrow \textstyle\prod_{i\in I} \boldsymbol{B}_i.$$

For any $\psi \in \Gamma$ we have $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\psi)$. Since i is an isomorphism, also $i(\bar{q}^{\boldsymbol{A}}) = \bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\psi))$. We know that $\uparrow^{\boldsymbol{B}} q$ is a filter on $\boldsymbol{B}$ (see the proof of Lemma 5.4), which implies $\bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\varphi))$. Thus, it follows that $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\varphi)$, as we wanted.

To finish the proof observe that if $\boldsymbol{A}$ is not trivial then it is, by Claim 1, infinite. Define $Z = \{q \in (0,1) \mid \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}\}$. Now, using both claims, we can easily decompose $\{\bar{1}^{\boldsymbol{A}}\}$ by means of the collection of $\uparrow^{\boldsymbol{A}} q$ ranging over Z. □

5.2 Logics with disjunction

In this section we introduce another example of intersection-prime theories, this time defined by means of disjunction.

Let $\nabla(p, q, \bar{r})$ be a set of formulas. Then, for any elements a, b of an algebra $\boldsymbol{A}$, we use the following notational convention

$$a \nabla^{\boldsymbol{A}} b = \{\varphi^{\boldsymbol{A}}(a, b, \bar{c}) \mid \varphi \in \nabla \text{ and } \bar{c} \in A^{Var_{\boldsymbol{\mathcal{L}}}}\}.$$

As usual, we write simply $\varphi \nabla \psi$ when $\boldsymbol{A} = \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}$. We also write

$$X \nabla^{\boldsymbol{A}} Y = \bigcup_{x \in X, y \in Y} x \nabla^{\boldsymbol{A}} y,$$

for subsets X, Y of A. Similarly we write $\Phi \nabla \Psi$ for sets of formulas Φ, Ψ. We say ∇ is a *p-protodisjunction* (or simply a *protodisjunction* if ∇ has no parameters $\bar{r}$) in L if it enjoys the following basic property disjunction should have:

$$\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \quad \text{and} \quad \psi \vdash_{\mathrm{L}} \varphi \nabla \psi \tag{PD}$$

For the rest of the chapter as a convention we will always assume that a set ∇ is a protodisjunction. An L-filter F on $\boldsymbol{A}$ is called ∇-*prime* if for every $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies that either $a \in F$ or $b \in F$ for every $a, b \in A$, analogously we define ∇-*prime* theories. We usually just call the filter (theory) prime, when the protodisjunction ∇ is known from the context. We denote the class of all reduced models of L with ∇-prime filters as $\mathbf{Mod}^{\nabla}$ L. It is easy to show that every ∇-prime filter (theory) is intersection-prime. The converse holds when ∇ enjoys a meta-rule characteristic for disjunctions: we say that ∇ enjoys the *proof by cases property* in L, if for every set of formulas $\Gamma \cup \{\varphi, \psi, \chi\}$ we have:

PCP If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

By *transferred* PCP, τ-PCP, we mean the corresponding version of PCP valid for filter generation on arbitrary algebra, that is for every $\langle \boldsymbol{A}, F \rangle \in \mathbf{Mod}\,\mathrm{L}$ and every $a, b, c \in A$ we have:

$$c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(F \cup \{a \nabla^{\boldsymbol{A}} b\}) \iff c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(F \cup \{a\}) \text{ and } c \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(F \cup \{b\}). \tag{5.1}$$

We then obtain the desired characterization:

Proposition 5.7 ([27, Lemma 4.16]). *If* ∇ *has* (τ)-PCP *then every filter (theory) is intersection-prime if and only if it is* ∇*-prime.*

In paper [27] the authors propose a hierarchy of logics with disjunction based on variants of PCP: the first one is called the *weak proof by cases property* and it say that for every formulas φ, ψ, and χ we have:

wPCP If $\varphi \vdash_{\mathrm{L}} \chi$ and $\psi \vdash_{\mathrm{L}} \chi$, then $\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$.

And finally the strongest version discussed in the literature is the *strong proof by cases property* saying that for every sets of formulas Γ, Φ, Ψ and a formula χ we have:

sPCP If $\Gamma, \Phi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \Phi \nabla \Psi \vdash_{\mathrm{L}} \chi$.

It is easy to see that every ∇ with the wPCP satisfies:

$$\varphi \nabla \psi \vdash \psi \nabla \varphi \tag{C}$$
$$\varphi \nabla \varphi \vdash \varphi \tag{I}$$
$$\varphi \nabla (\psi \nabla \chi) \vdash (\varphi \nabla \psi) \nabla \chi \tag{A}$$

We say that ∇ is a *strong p-disjunction* (resp. *(p)-disjunction*, resp. *weak (p)-disjunction*) if it enjoys the sPCP (resp. PCP, resp. wPCP). We drop the prefix 'p-', if ∇ has no parameters. Further we say L is *strongly (p)-disjunctional* (resp. *p-disjunctional*, resp. *weakly p-disjunctional*) provided it has a strong (p)-disjunction (resp. p-disjunction, resp. weak (p)-disjunction). Finally L is said to be *strongly disjunctive* (resp. *disjunctive*, resp. weakly disjunctive) if it has a strong disjunction (resp. disjunction, resp. weak disjunction) given by one parameter free formula. The hierarchy together with basic relations of the properties just defined is given in Figure 5.1.[1] Note that all of the implications are proper (counterexamples can be found in [27]).

We say that L has *(transferred) prime extension property*, (τ)-PEP, if the prime theories form a basis of $\mathrm{Th\,L}$ (prime filters forms a basis of $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$). It is easy to prove the basic relations between PEP and PCP:

Proposition 5.8 ([27, Theorem 4.17]). *Let* L *be a logic with the* IPEP *and a p-protodisjunction* ∇. *Then, the following are equivalent:*

(i) ∇ *enjoys the* sPCP.
(ii) ∇ *enjoys the* PCP.
(iii) L *has the* PEP.

The next proposition was already proved in [27], but with stronger assumptions; we also present a simpler proof.

Proposition 5.9. *Suppose* L *is a protoalgebraic logic with enough variables. If* ∇ *has the* PCP, *then it has the* τ-PCP.

Proof. Since the logic has enough variables we can easily use the method from Proposition 2.24 to prove that ∇ has the PCP in every natural extension of L. The left-to-right direction follows easily by (PD). Moreover, clearly natural extensions of protoalgebraic logics are protoalgebraic (e.g. they share the same protoimplication). Let κ be an infinite cardinal such that there is a surjective $e\colon \boldsymbol{Fm}_{\mathcal{L}}(\kappa) \twoheadrightarrow \boldsymbol{A}$. It is easy to see (use e.g. the correspondence theorem of protoalgebraic logics) for every strict and surjective homomorphism $h\colon \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \twoheadrightarrow \langle \boldsymbol{A}, F\rangle$ between models of L and every set of formulas $\Gamma \cup \{\varphi\}$:

[1] The figure is taken from [27, Figure 1].

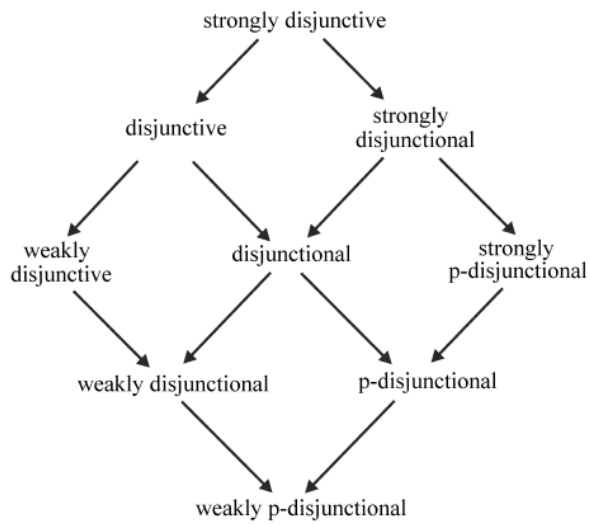


Figure 5.1: The hierarchy of disjunctional logics

$$T, \Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad h(\varphi) \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup h[\Gamma]).$$

Thus, the right-hand side of (5.1) corresponds to $T, \varphi \vdash_{\mathrm{L}^{\kappa}} \chi$ and $T, \psi \vdash_{\mathrm{L}^{\kappa}} \chi$ for $T = e^{-1}[F]$ and some φ, ψ, χ such that $e(\varphi) = a, h(\psi) = b$, and $e(\chi) = c$. The rest of the argument should be clear. □

Next we describe a useful characterization by means of an axiomatic system of a given logic of the fact that ∇ has the sPCP and thus is a strong disjunction. A ∇-*form* of a consecution $\Gamma \rhd \varphi$ is a consecution of the form $\Gamma \nabla \psi \rhd \varphi \nabla \psi$.

Proposition 5.10 ([27, Proposition 4.6]). *Assume that $\mathcal{AS}$ is a presentation of* L. *Then,* ∇ *enjoys the* sPCP *if and only if* ∇ *satisfies* (C), (I), *and all the* ∇*-forms of every consecution in* $\mathcal{AS}$ *is provable in* L.

5.2.1 Lindenbaum and pair extension lemma

In the section we shall meet some conditions that will ensure that a given logic has the IPEP. Since the main result of this subsection does not hold for general disjunction ∇, we restrict here to a simple disjunctions given by a single formula without parameters (although many results would still be true in general).

Before we prove the lemma, we introduce a useful technical tool which is interesting on its own. First observe that if $\vee$ is a disjunction in logic L and $\Delta = \{\varphi_1, \ldots, \varphi_n\}$ is a finite non-empty set of formulas we can define $\bigvee \Delta = \varphi_1 \vee (\varphi_2 \vee \cdots \vee \varphi_n) \ldots)$ and, thanks to (C) and (A), the bracketing does not matter if the derivability is all we are interested in. Therefore for each such logic we can define a relation $\Vdash_{\mathrm{L}}$ between sets of formulas as:[2]

$$\Gamma \Vdash_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite non-empty } \Delta' \subseteq \Delta \text{ and } \Gamma \vdash_{\mathrm{L}} \bigvee \Delta'.$$

It is known that if L is a finitary logic, then $\Vdash_{\mathrm{L}}$ is the so-called *symmetric consequence relation* as defined e.g. in [40]: that is

- $\Gamma \Vdash_{\mathrm{L}} \Gamma$. (Reflexivity)
- If $\Gamma \Vdash_{\mathrm{L}} \Delta$ and $\Gamma \subseteq \Sigma$, then $\Sigma \Vdash_{\mathrm{L}} \Delta$. (Left-monotony)
- If $\Gamma \Vdash_{\mathrm{L}} \Delta$ and $\Delta \subseteq \Sigma$, then $\Gamma \Vdash_{\mathrm{L}} \Sigma$. (Right-monotony)
- If $\Gamma, \Sigma_1 \Vdash_{\mathrm{L}} \Delta, \Sigma \setminus \Sigma_1$ for each $\Sigma_1 \subseteq \Sigma$ then $\Gamma \Vdash_{\mathrm{L}} \Delta$ (Symmetric cut)

In general (for infinitary L) the relation $\Vdash_{\mathrm{L}}$ need not satisfy the symmetric cut (see Example 5.15). On the other hand, we can show it satisfies a particular form of cut, which will be instrumental to prove the Lindenbaum lemma.

Lemma 5.11. *Let* L *be a logic with strong disjunction* $\vee$*. Then, the relation* $\Vdash_{\mathrm{L}}$ *has the so called* (finite) strong cut property*, i.e., for each sets* Γ_1, Γ_2, Φ *of formulas and each (finite) sets* Δ_1, Δ_2 *of formulas we have:*

$$\frac{\{\Gamma_1 \Vdash_{\mathrm{L}} \Delta_1 \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma_2 \cup \Phi \Vdash_{\mathrm{L}} \Delta_2}{\Gamma_1 \cup \Gamma_2 \Vdash_{\mathrm{L}} \Delta_1 \cup \Delta_2}.$$

Proof. Let us by χ denote the formula $\bigvee(\Delta_1 \cup \Delta_2)$. From the assumption and properties of $\vee$ we obtain $\Gamma_1, \Gamma_2, \Phi \vdash_{\mathrm{L}} \chi$ and $\Gamma_1, \Gamma_2 \vdash_{\mathrm{L}} \varphi \vee \chi$ for each $\varphi \in \Phi$. As clearly $\Gamma_1, \Gamma_2, \chi \vdash_{\mathrm{L}} \chi$ we can use sPCP to obtain $\Gamma_1, \Gamma_2, \Phi \vee \chi \vdash_{\mathrm{L}} \chi$ and the regular cut of L to get the claim. □

The final ingredient used in the proof of the Lindenbaum lemma below is the following: we say that $\langle \Gamma, \Delta \rangle$ is a *pair* in L if $\Gamma \nVdash_{\mathrm{L}} \Delta$. Furthermore we say that a pair $\langle \Gamma, \Delta \rangle$ is *full* if $\Gamma \cup \Delta = Fm_{\boldsymbol{\mathcal{L}}}$; note in such case Γ has to be a prime theory.

Theorem 5.12 (Lindenbaum lemma for infinitary logics). *Suppose* L *is a countably axiomatizable logic with a strong disjunction. Then* L *enjoys the* PEP.

[2] By way of convention we say that $\Gamma \Vdash_{\mathrm{L}} \emptyset$ iff $\Gamma \vdash_{\mathrm{L}} \varphi$ for each formula φ.

Proof. Assume that a theory T and $\chi \notin T$ are given. We construct a prime theory $T' \supseteq T$ such that $\chi \notin T'$. We first enumerate all rules $\Lambda_i \rhd \varphi_i$ in the existing countable axiomatic system, and then define two increasing sequences of sets of formulas Γ_i and Δ_i such that each $\langle \Gamma_i, \Delta_i \rangle$ is a pair with Δ_i finite. We start by putting $\Gamma_0 = T$ and $\Delta_0 = \{\chi\}$. In the induction step, we distinguish two cases:

- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is a pair, we define $\Delta_{i+1} = \Delta_i$ and $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i\}$.
- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is not a pair, then there has to be $\chi_i \in \Lambda_i$ such that $\langle \Gamma_i, \Delta_i \cup \{\chi_i\} \rangle$ is a pair; indeed, otherwise we would have:

$$\frac{\dfrac{\{\Gamma_i \Vdash \Delta_i \cup \{\chi_i\} \mid \chi_i \in \Lambda_i\} \qquad \Lambda_i \Vdash \varphi_i}{\Gamma_i \Vdash \Delta_i \cup \{\varphi_i\}} \qquad \Gamma_i \cup \{\varphi_i\} \Vdash \Delta_i}{\Gamma_i \Vdash \Delta_i}.$$

  Thus, we can define $\Gamma_{i+i} = \Gamma_i$ and $\Delta_{i+i} = \Delta_i \cup \{\chi_i\}$.

Finally, define $T' = \bigcup \Gamma_i$ and $\Delta = \bigcup \Delta_i$. We can assume w.l.o.g. that our axiomatic system contains a 'dummy' rule $\varphi \rhd \varphi$ for each formula φ, so that, due to the construction, $T' \cup \Delta = Fm_{\boldsymbol{\mathcal{L}}}$. So when we show that $\langle T', \Delta \rangle$ is a pair, we have that T' is a prime theory and the proof is done. Note that we can add the rules $\varphi \rhd \varphi$ because the assumptions assures there are only countably many formulas: since the logic proves e.g. $p \vdash_{\mathrm{L}} p \vee q$, there must be some rule in $\mathcal{AS}$ employing variables and since $\mathcal{AS}$ is countable the claim follows.

First we show that for each φ we have: if $T' \vdash \varphi$, then $\varphi \in \Gamma_j$ for some j. Let us fix a proof of φ from T'; we prove the claim for each formula which is a label of some of its nodes. If the node is a leaf the claim is trivial. Consider a node obtained using rule $\Lambda_i \rhd \varphi_i$. If we have proceeded by the first case of the induction step we have $\varphi_i \in \Gamma_{i+1}$. Let us see that we couldn't have proceeded by the second case: consider $\chi_i \in \Lambda_i$ selected by the procedure. We know that $T' \vdash \chi_i$ (it is a label of a node preceding φ_i) and so the induction assumption gives us j such that $\Gamma_j \vdash \chi_i$ and so $\langle \Gamma_{\max\{i+1,j\}}, \Delta_{\max\{i+1,j\}} \rangle$ is not a pair, yielding a contradiction.

Now we can conclude the proof that $\langle T', \Delta \rangle$ is a pair. Assume otherwise, then we have $T' \vdash \bigvee \Delta_0$ for some finite Δ_0. Then, $\Delta_0 \subseteq \Delta_i$ for some i and so by the previous claim there is j such that $\langle \Gamma_{\max\{i,j\}}, \Delta_{\max\{i,j\}} \rangle$ is not a pair, a contradiction. □

Observe that it is in the last paragraph of the proof where the argument would fail for a generalized disjunction ∇. Also, it is to be noted that neither of the two assumptions of Theorem 5.12 (of the countability of the axiomatic system and of having a strong disjunction) can be omitted. We present two examples of logics satisfying only one of these conditions and failing the Lindenbaum lemma (and thus also the other condition).

Example 5.13 (A logic with a strong disjunction and no countable presentation). Consider a type $\mathcal{L}$ consisting of a binary connective $\vee$, a unary connective s, and two constants $\mathbf{0}$ and $\boldsymbol{\omega}$. Let us by $\mathbf{n}$ denote the formula defined inductively as $(\mathbf{n}+\mathbf{1}) = s(\mathbf{n})$.

Let L be a logic $\mathcal{L}$ axiomatized by the rules (PD), (C), (I), and (A), $\vee$-forms of these rules, and the following rules for each infinite set $C \subseteq \omega$:

$$\{\mathbf{i} \vee \psi \mid i \in C\} \rhd \psi. \qquad (\mathrm{Inf}_C)$$

First we use the characterization from Proposition 5.10 to show that $\vee$ is a strong disjunction in L. Indeed $\vee$-forms of some rules are directly part of its presentation and for the remaining ones we use (A), e.g. we know that $(\mathbf{i} \vee \psi) \vee \chi \vdash_{\mathrm{L}} \mathbf{i} \vee (\psi \vee \chi)$ and so obviously $\{(\mathbf{i} \vee \psi) \vee \chi \mid i \in C\} \vdash_{\mathrm{L}} \psi \vee \chi$.

We prove that Lindenbaum lemma fails in L. Consider a subset A of 2^{ω}:

$$A = \{\omega\} \cup \{X \subseteq \omega \mid X \text{ finite and for each } i \in \omega\colon 2i \notin X \text{ or } 2i+1 \notin X\}.$$

Note that A is closed under arbitrary intersections and so $\langle A, \vee\rangle$ with

$$X \vee Y = \bigcap_{Z \in A, X \cup Y \subseteq Z} Z$$

is a complete join-semilattice and observe that

$$X \vee Y = \begin{cases} X \cup Y & \text{if } X \cup Y \in A \\ \omega & \text{otherwise (i.e., when } \{2i, 2i+1\} \subseteq X \cup Y). \end{cases}$$

Consider an algebra $\boldsymbol{A} = \langle A, \vee, s, \mathbf{0}, \boldsymbol{\omega}\rangle$ where $\mathbf{0} = \{0\}$, $\boldsymbol{\omega} = \omega$, and $s(\{i\}) = \{i+1\}$ and $s(X) = \emptyset$ otherwise (note that $\mathbf{n} = \{n\}$). We show that $\langle \boldsymbol{A}, \{\omega\}\rangle \in \mathbf{Mod}\,\mathrm{L}$. Obviously it suffices to check the rules:

- Soundness of the rules (C), (I), and (A) and their $\vee$-forms is straightforward (recall that the join of any sets from A is ω if and only if its union contains $\{2i, 2i+1\}$ for some i)
- Next consider a rule (Inf_C) and an evaluation e such that $e(\psi) \neq \omega$. We know that $e(\psi)$ is a finite set and so for $m = \max(e(\psi))$ we have $e(\mathbf{m}+\mathbf{2} \vee \psi) = e(\psi) \cup \{m+2\} \neq \omega$.

To conclude the proof we show that for a theory T generated by the set of formulas $\{\mathbf{2i} \vee \mathbf{2i}+\mathbf{1} \mid i \in \omega\}$ we have $T \nvdash_{\mathrm{L}} \mathbf{0}$ while $T' \vdash_{\mathrm{L}} \mathbf{0}$ for each prime theory $T' \supseteq T$. To show the first claim just observe that for an arbitrary homomorphism $e\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{A}$ we have $e(\mathbf{2i} \vee \mathbf{2i+1}) = \omega$. As regards the second claim, since T' is prime, we obtain for each i that $\mathbf{2i} \in T'$ or $\mathbf{2i+1} \in T'$. Thus, there is an infinite set C such that $\{\mathbf{i} \vee \mathbf{0} \mid i \in C\} \subseteq T'$ and so by (Inf_C) we obtain $T' \vdash_{\mathrm{L}} \mathbf{0}$.

Example 5.14 (A countably axiomatizable logic without a strong disjunction). Consider a language with one unary operation box $\Box$, we write $\Box^n$ as a shortcut for the n-fold application of $\Box$. In this example we consider a logic L axiomatized by the infinitary rules (Inf_n) for each $n \in \omega$:

$$\{\Box^m(\varphi) \mid m > n\} \rhd \varphi. \qquad (\mathrm{Inf}_n)$$

Clearly this logic is countably axiomatizable. We show that Lindenbaum lemma fails in L. First we show that if $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$, then $\chi = \varphi$ or $\Gamma \vdash_{\mathrm{L}} \chi$. We prove it by induction for each δ in the proof of χ from $\Gamma \cup \{\varphi\}$. The only non-trivial case is if δ follows by the application of an infinitary rule $\{\Box^m(\delta) \mid m > n\} \rhd \delta$. Let us set $n' = k$ if $\varphi = \Box^k(\delta)$ for some $k > n$ and $n' = n$ otherwise. Due to the induction assumption we have $\Gamma \vdash_{\mathrm{L}} \Box^m(\delta)$ for each $m > n'$. Thus, $\Gamma \vdash_{\mathrm{L}} \delta$.

Therefore if T is a theory so is $T \cup \{\psi\}$ and so the only finitely meet-irreducible theory is $Fm_{\mathcal{L}}$. Now, since obviously our logic has non-trivial theories (e.g., the $\emptyset$), the finitely meet-irreducible theories do not form a basis of $\mathrm{Th}\,\mathrm{L}$.

The Lindenbaum lemma is closely related to the so called *pair extension lemma*, which says that each pair can be extended into a full pair (a pair $\langle \Gamma', \Delta' \rangle$ is an *extension* of $\langle \Gamma, \Delta \rangle$ when $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$).

The pair extension lemma is known to be equivalent to the symmetric cut (the proof is straightforward). On the other hand, as our proof of Theorem 5.12 suggests, it should be very easy to see that the restriction of pair extension lemma to pairs with *finite* right-hand sides is equivalent with the Lindenbaum lemma.

The next example shows that Ł$_\infty$ does not enjoy the pair extension lemma and so $\Vdash_{\text{Ł}_\infty}$ is not a symmetric consequence relation, while, as we will see in the next subsection, it enjoys the Lindenbaum lemma.

Example 5.15. Let us consider the infinitary Łukasiewicz logic Ł$_\infty$ and a set $\Delta = \{\neg p^n \mid n > 0\} \cup \{p\}$. Obviously $\langle \emptyset, \Delta \rangle$ is a pair: just use evaluation $e(p) = \frac{m}{m+1}$ and observe that $e(p \vee \bigvee_{0<i\leq m} \neg p^i) = \frac{m}{m+1} \neq 1$. But it cannot be extended to a full pair: indeed, suppose $\langle T, \Delta' \rangle$ is a full pair, then we know that T is a prime theory. We show it must contain a formula from Δ: it is easy to observe that in Ł$_\infty$ we have:

$$\{\neg\varphi \to \varphi^n \mid n \in \omega\} \vdash_{\text{Ł}_\infty} \varphi,$$

thus if $p \notin T$ there is some $n \in \omega$ such that $\neg p \to p^n \notin T$, but then from primeness of T using the prelinearity axiom ($\vdash_{\text{Ł}} \varphi \to \psi \vee \psi \to \varphi$) we obtain $p^n \to \neg p \in T$, but this formula is clearly equivalent to $\neg p^{n+1}$.

In fact, already in a weaker setting (assuming that $\vee$ is merely a disjunction) we can prove a stronger claim which also illuminates the role of the finite strong cut rule and strong proof by cases property.

Theorem 5.16. *Let* L *be a logic with a countable axiomatization* $\mathcal{AS}$ *and a disjunction* $\vee$*. Then, the following are equivalent:*

(i) $\Vdash_{\mathrm{L}}$ *enjoys the pair extension lemma for pairs with finite right-hand sides, i.e., each pair* $\langle \Gamma, \Delta\rangle$ *where* Δ *is finite can be extended into a full one.*

(ii) $\Vdash_{\mathrm{L}}$ *enjoys the finite strong cut rule, i.e., for each sets* Γ_1, Γ_2, Φ *of formulas and each* finite *sets* Δ_1, Δ_2 *of formulas:*

$$\frac{\{\Gamma_1 \Vdash_{\mathrm{L}} \Delta_1 \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma_2 \cup \Phi \Vdash_{\mathrm{L}} \Delta_2}{\Gamma_1 \cup \Gamma_2 \Vdash_{\mathrm{L}} \Delta_1 \cup \Delta_2}.$$

(iii) *For each rule* $\Gamma \rhd \varphi$ *in* $\mathcal{AS}$ *and each formula* χ *we have*

$$\{\gamma \vee \chi \mid \gamma \in \Gamma\} \vdash_{\mathrm{L}} \varphi \vee \chi,$$

i.e., $\vee$ *is a strong disjunction.*

(iv) L *enjoys the Lindenbaum lemma, i.e., prime theories form a basis of* $\mathrm{Th}\,\mathrm{L}$*.*

Proof. (i)→(ii): Assume (a) $\Gamma_1 \Vdash_{\mathrm{L}} \Delta_1 \cup \{\varphi\}$ for each $\varphi \in \Phi$ and (b) $\Gamma_2 \cup \Phi \Vdash_{\mathrm{L}} \Delta_2$. To obtain contradiction suppose that $\Gamma_1 \cup \Gamma_2 \nVdash_{\mathrm{L}} \Delta_1 \cup \Delta_2$. Thus, there is a full pair $\langle \Gamma', \Delta'\rangle$ extending $\langle \Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2\rangle$. Due to (a) there can be no $\varphi \in \Phi \cap \Delta'$. Therefore $\Phi \subseteq \Gamma'$ and so by (b) we get $\Gamma' \Vdash_{\mathrm{L}} \Delta'$, a contradiction.

(ii)→(iii): A simple application of the strong cut rule:

$$\frac{\{\{\gamma \vee \chi \mid \gamma \in \Gamma\} \Vdash_{\mathrm{L}} \{\chi, \gamma\} \mid \gamma \in \Gamma\} \qquad \Gamma \Vdash_{\mathrm{L}} \{\varphi\}}{\{\gamma \vee \chi \mid \gamma \in \Gamma\} \Vdash_{\mathrm{L}} \{\varphi, \chi\}}.$$

(iii)→(iv): This follows by Lindenbaum lemma (Theorem 5.12).

(iv)→(i): The proof is obvious. □

Let us turn our attention to the full pair extension lemma for pairs $\langle \Gamma, \Delta\rangle$ with an arbitrary Δ. We can still prove a similar theorem. It moreover illuminates some interesting limitations of the pair extension lemma: namely, the full pair extension lemma, in presence of infinitely many propositional variables in the language, entails the finitarity of the logic in question. Thus, in particular $\Vdash_{\mathrm{L}}$ is a symmetric consequence relation if and only if L is a finitary logic.

Theorem 5.17. *Let* L *be a logic, whose language contains countably many propositional variables, with a countable axiomatization* $\mathcal{AS}$ *and a disjunction* $\vee$. *Then, the following are equivalent:*

(i) $\Vdash_{\mathrm{L}}$ *enjoys the pair extension lemma, i.e., each pair* $\langle \Gamma, \Delta\rangle$ *can be extended into a full one.*
(ii) $\Vdash_{\mathrm{L}}$ *enjoys the strong cut rule, i.e., for each sets* $\Gamma_1, \Gamma_2, \Phi, \Delta_1, \Delta_2$ *of formulas:*

$$\frac{\{\Gamma_1 \Vdash_{\mathrm{L}} \Delta_1 \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma_2 \cup \Phi \Vdash_{\mathrm{L}} \Delta_2}{\Gamma_1 \cup \Gamma_2 \Vdash_{\mathrm{L}} \Delta_1 \cup \Delta_2}.$$

(iii) *For each rule* $\Gamma \rhd \varphi$ *in* $\mathcal{AS}$, *each set of formulas* Δ *and each surjective function* $f\colon \Gamma \to \Delta$ *we have*

$$\{\gamma \vee f(\gamma) \mid \gamma \in \Gamma\} \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\}.$$

(iv) L *is finitary.*

Proof. The implication (iv)→(i) is a well-known fact valid in general, for a proof see e.g. [40, 84].[3] The proof of the implication (i)→(ii) is analogous to the proof of the corresponding claim in the previous theorem; and the implication (ii)→(iii) is a simple application of the strong cut rule:

$$\frac{\{\{\gamma \vee f(\gamma) \mid \gamma \in \Gamma\} \Vdash_{\mathrm{L}} \Delta \cup \{\gamma\} \mid \gamma \in \Gamma\} \qquad \Gamma \Vdash_{\mathrm{L}} \{\varphi\}}{\{\gamma \vee f(\gamma) \mid \gamma \in \Gamma\} \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\}}.$$

To prove the remaining implication (iii)→(iv) assume that L is not finitary. There has to be a proper infinitary rule $\Gamma \rhd \varphi$ in $\mathcal{AS}$ (i.e. for no finite $\Gamma' \subseteq \Gamma$ we have $\Gamma' \vdash_{\mathrm{L}} \varphi$) and, since $\mathcal{AS}$ is countable and closed under substitutions, there are only finitely many variables occurring in Γ. Assume that $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ and let $\Delta = \{p_1, p_2, \dots\}$ be the infinite set of all variables not occurring in $\Gamma \cup \{\varphi\}$.

We define a function $f\colon \Gamma \to \Delta$ as follows: $f(\gamma_i) = p_i$. Clearly, f is surjective and so from our assumption we know that

$$\{\gamma_i \vee p_i \mid i > 0\} \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\}.$$

Therefore there is a non-empty finite set of variables $\Delta' \subseteq \Delta$ such that

$$\{\gamma_i \vee p_i \mid i > 0\} \vdash_{\mathrm{L}} \varphi \vee \bigvee \Delta'.$$

[3] The simplest proof would be to enumerate all formulas, then in each step add the processed formula either to the left or to the right side of the pair (which can always be done thanks to the cut rule). Finitarity guarantees that the union of the pair will in the end still be a pair.

Pick any $p_n \in \Delta'$ and define a substitution σ in the following way: $\sigma(p) = p$ if p occurs in $\Gamma \cup \{\varphi\}$, $\sigma(p_i) = \varphi$ for $p_i \in \Delta'$, and $\sigma(p_i) = \gamma_n$ otherwise. Then, by structurality we obtain

$$\{\gamma_i \vee \varphi \mid p_i \in \Delta'\} \cup \{\gamma_i \vee \gamma_n \mid p_i \notin \Delta'\} \vdash_{\mathrm{L}} \varphi \vee \varphi \vee \cdots \vee \varphi.$$

Thus, by the properties of disjunction we obtain

$$\{\gamma_i \mid p_i \in \Delta'\} \vdash_{\mathrm{L}} \varphi,$$

a contradiction with the fact that $\Gamma \vdash_{\mathrm{L}} \varphi$ is a proper infinitary rule. □

Remark 5.18. In the proof of (iii)→(iv), we have substantially used the fact that there are infinitely many variables in the language. In Example 5.19 below we demonstrate that there is indeed an infinitary logic with finitely many variables which still has full pair extension lemma.

To do so, we need to show that the properties (i)–(iii) are equivalent even without assuming existence of infinitely many variables. Clearly, the proofs of the implications from top to bottom do not use the assumption and the implication (iii)→(i) can be proved by the following simple modification of the proof of Lindenbaum lemma (Theorem 5.12).

Let $\langle \Gamma, \Delta \rangle$ be the pair we want to extend; we start by Γ_0 being the theory generated by Γ and $\Delta_0 = \Delta$. We only need to show that the second case of the induction procedure can still be carried out, the rest of the proof remains the same. We have a pair $\langle \Gamma_i, \Delta_i \rangle$ and need to process the rule $\Lambda_i \rhd \varphi_i$. We assume that $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is not a pair (i.e., $\Gamma_i, \varphi_i \vdash_{\mathrm{L}} \bigvee \Delta'_i$ for some finite $\Delta'_i \subseteq \Delta_i$) and $\langle \Gamma_i, \Delta_i \cup \{\chi\} \rangle$ is not a pair for any $\chi \in \Lambda_i$. Thus, we can define $f(\chi) = \bigvee \Delta_\chi$, where $\Delta_\chi \subseteq \Delta_i$ is a finite set such that $\Gamma_i \vdash_{\mathrm{L}} \chi \vee \bigvee \Delta_\chi$. By the assumption (iii) we obtain

$$\{\chi \vee \bigvee \Delta_\chi \mid \chi \in \Lambda_i\} \Vdash_{\mathrm{L}} \{\bigvee \Delta_\chi \mid \chi \in \Gamma\} \cup \{\varphi_i\}.$$

Thus, $\Gamma_i \vdash \varphi_i \vee \bigvee \Delta''_i$ for some $\Delta''_i \subseteq \Delta_i$. Using PCP we obtain $\Gamma_i \vdash \bigvee(\Delta'_i \cup \Delta''_i)$ (cf. the proof of Lemma 5.11), i.e., $\langle \Gamma_i, \Delta_i \rangle$ is not a pair, a contradiction.

Example 5.19 (An infinitary logic with finitely many variables and full pair extension lemma). We consider a language with only finitely many variables $Var = \{p_1, \ldots, p_n\}$, a disjunction connective $\vee$, and a constant $\mathbf{n}$ for every natural number n. A logic L in this language is presented by an axiomatic system consisting of rules (PD), (I), (A), (C), an infinitary rule

$$\{\mathbf{n} \mid n \in \omega\} \rhd p_1, \tag{Inf}$$

and $\vee$-forms of all these rules.

It is easy to see that L is infinitary—it is enough to consider models of size 2. As it is clearly countably axiomatizable and $\vee$ is a strong disjunction, if we show that it satisfies condition (iii) of Theorem 5.17 we obtain full pair extension lemma due to the previous remark. For the rules (PD), (I), (A), and (C) and their $\vee$-forms it is a consequence of the fact that they have finitely many premises and are closed under its $\vee$-forms. Thus, it remains to be shown that the rule (Inf) satisfies it too (the argument for its $\vee$-form is basically the same).

Consider a set Δ and a surjective function $f\colon \{\mathbf{n} \mid n \in \omega\} \to \Delta$. Note that $f(\mathbf{n})$ is a disjunction of constants and variables (in the limit case with just one disjunct). First assume that there are $m, l \in \omega$ such that $\mathbf{l}$ is one of the disjunct of $f(\mathbf{m})$. Then, $\mathbf{l} \vdash f(\mathbf{m})$ and so $\mathbf{l} \vdash f(\mathbf{m}) \vee f(\mathbf{l}) \vee p_1$, both by (PD). By (PD) we also obtain $f(\mathbf{l}) \vdash f(\mathbf{m}) \vee f(\mathbf{l}) \vee p_1$, and so by PCP

$$\{\mathbf{n} \vee f(\mathbf{n}) \mid n \in \omega\} \vdash f(\mathbf{m}) \vee f(\mathbf{l}) \vee p_1.$$

If there are no such $m, l \in \omega$, then each formula from Δ is a disjunction of variables. Since there are only finitely many variables, there is a finite set $\Delta' \subseteq \Delta$ such that each variable from some formula from Δ occurs in some formula from Δ', and so, by the properties of disjunction, for each $\chi \in \Delta$ we have $\chi \vdash \bigvee \Delta'$. Thus, also for each n we have $\mathbf{n} \vee f(\mathbf{n}) \vdash \mathbf{n} \vee \bigvee \Delta'$, and therefore by $\{\mathbf{n} \vee \bigvee \Delta' \mid n \in \omega\} \vdash p_1 \vee \bigvee \Delta'$ (which is the $\vee$-form of our infinitary rule) we obtain

$$\{\mathbf{n} \vee f(\mathbf{n}) \mid n \in \omega\} \vdash p_1 \vee \bigvee \Delta'.$$

Observe that the point (iii) of the previous theorem is a stronger version of of the closure under $\vee$-forms which characterizes strong disjunctions (Proposition 5.10). Thus, we could possibly study an extension of the disjunctional hierarchy by 'super' strong disjunctions (protodisjunctions validating (iii))—nevertheless the theorem and the remark suggests very restricted applicability of the notion, it provides new inside only to logics with finitely many variables.

5.2.2 Axiomatizing infinitary logics

We start this subsection demonstrating the applicability of the infinitary version of the Lindenbaum lemma presenting a simple proof of completeness for infinitary Łukasiewicz logic $Ł_\infty$. It is a folklore result which is implicit e.g. in [61], or could be obtained from a more complicated axiomatization of this algebra from [65], where the author proves a completeness result for the infinitary basic fuzzy logic BL_∞. We show it can be axiomatized by adding

$$\{\neg\varphi \to \varphi^n \mid n \in \omega\} \rhd \varphi \qquad (\text{Ł}_\infty)$$

to any presentation of Ł; let us denote such a presentation as $\mathcal{AS}$— this presentation is clearly countable. First we show that $\vee$ is a strong disjunction in $\vdash_{\mathcal{AS}}$:

Lemma 5.20. *The connective $\vee$ is a strong disjunction in $\vdash_{\mathcal{AS}}$.*

Proof. It is known that $\vee$ is a (strong) disjunction in Ł (see e.g. [26]). Therefore, thanks to Proposition 5.10, Ł and also Ł_∞ prove (PD), (I), (C), and $\vee$-form of *modus ponens*. Thus, by the same proposition, it suffices to prove the $\vee$-form of (Ł_∞) i.e.,

$$\{(\neg\varphi \to \varphi^n) \vee \chi \mid n \in \omega\} \vdash_{\mathcal{AS}} \varphi \vee \chi.$$

To do so we prove (in Łukasiewicz logic Ł):

$$(\neg\varphi \to \varphi^n) \vee \chi \vdash_{\text{Ł}} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n$$

and observe that a simple use of (Ł_∞) completes the proof: since we know that $\vee$ has sPCP in Ł it suffices to show that $\neg\varphi \to \varphi^n \vdash_{\text{Ł}} \neg(\varphi\vee\chi) \to (\varphi\vee\chi)^n$ and $\chi \vdash_{\text{Ł}} \neg(\varphi\vee\chi) \to (\varphi\vee\chi)^n$. The first one is provable because $\to$ is antitone in the first argument and monotone in the second. The second one holds by $\chi \vdash_{\text{Ł}} (\varphi \vee \chi)^n$ and $\psi \vdash_{\text{Ł}} \delta \to \psi$. □

Proposition 5.21. *$\mathcal{AS}$ is a presentation for the infinitary Łukasiewicz logic Ł_∞, the logic of the standard* MV*-algebra* $[0,1]_{\text{Ł}}$.

Proof. It is easy to check that the semantics is sound. For completeness assume that $\Gamma \nvdash_{\text{Ł}_\infty} \varphi$ and that T is a prime theory of Ł_∞ separating Γ from φ (its existence follows form the Lindenbaum lemma—Theorem 5.12). Clearly T is a consistent theory of Ł, we show that it is a maximal such theory: consider $\psi \notin T$ then, thanks to the infinitary rule (Ł_∞), there has to be some $n \in \omega$ such that $\neg\psi \to \psi^n \notin T$. Due to the prelinearity theorem $\vdash_{\text{Ł}} (\varphi \to \chi) \vee (\chi \to \varphi)$ and primeness of T we obtain $\psi^n \to \neg\psi \in T$. Thus, $T, \psi \vdash_{\text{Ł}} \neg\psi$ (using *modus ponens* and the fact that $\psi \vdash_{\text{Ł}} \psi^m$) and so $T \cup \{\psi\}$ is inconsistent (because we have $\psi, \neg\psi \vdash_{\text{Ł}} \chi$).

By the standard techniques in algebraic logic we can find a counterexample to $\Gamma \nvdash_{\text{Ł}_\infty} \varphi$ over the Lindenbaum–Tarski matrix $\langle \boldsymbol{Fm}^*_{\mathcal{L}}, T^* \rangle$. Clearly, since MV-algebras are the equivalent algebraic semantics of Ł, we known $\boldsymbol{Fm}^*_{\mathcal{L}}$ is an MV-algebra. We show it is simple: firstly, since Ł is algebraizable, we know that $\boldsymbol{\Omega}\colon \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{Fm}^*_{\mathcal{L}} \to \mathrm{Con}\,\boldsymbol{Fm}^*_{\mathcal{L}}$ is a lattice isomorphism and since $\boldsymbol{\Omega}T^*$ is the identity, we obtain it is the smallest filter on $\boldsymbol{Fm}^*_{\mathcal{L}}$. Thus, since T is maximally consistent, by the correspondence theorem there are precisely two filters (and hence also two congruences) on $\boldsymbol{Fm}^*_{\mathcal{L}}$, namely T^* and $Fm^*_{\mathcal{L}}$. However, e.g. from [11] we know that simple MV-algebras are up to isomorphism subalgebras of $[0,1]_{Ł}$ and so the rest of the proof is straightforward. □

Corollary 5.22. *Alternatively,* $Ł_\infty$ *can be axiomatized by extending any presentation of* Ł *by an additional infinitary rule:*

$$\{\varphi \to \psi^n \mid n \in \omega\} \rhd \neg\varphi \vee \psi. \qquad (Ł_\infty 2)$$

Proof. Clearly the new rule is sound for $Ł_\infty$, thus it suffices to show that the rule ($Ł_\infty$) is provable in our system, but it is easy:

$$\{\neg\varphi \to \varphi^n \mid n \in \omega\} \vdash_{Ł_\infty} \neg\neg\varphi \vee \varphi \vdash_{Ł_\infty} \varphi,$$

where the first one is the rule ($Ł_\infty 2$) and the second one is already valid in Ł and follows using PCP and the fact that $\vdash_{Ł} \neg\neg\varphi \to \varphi$. □

We present another known application of the rule ($Ł_\infty$) to Ł and $Ł_\infty$ (it can be extracted e.g. from [59, Theorem 5.4.11]).

Proposition 5.23. *For every set of formulas of Łukasiewicz logic* $\Gamma \cup \{\varphi\}$*, we have*

$$\Gamma \vdash_{Ł_\infty} \varphi \quad\Longleftrightarrow\quad \Gamma \vdash_{Ł} \neg\varphi \to \varphi^n \text{ for every } n \in \omega.$$

Proof. The right-to-left direction is obvious. For the other one, first observe, that $\Gamma, \neg\varphi^n \vdash_{Ł_\infty} \emptyset$. Then, since $Ł_\infty$ is compact and Ł is its finitary companion, we obtain also that $\Gamma, \neg\varphi^n \vdash_{Ł} \emptyset$. Next, since linear Ł-theories form basis of $\mathrm{Th}\,Ł$, it is enough to show that every linear Ł-theory T extending Γ contains all the formulas $\neg\varphi \to \varphi^n$. For contradiction suppose that $\neg\varphi \to \varphi^n \notin T$ for some $n \in \omega$. Then, by linearity of T, $\varphi^n \to \neg\varphi \in T$. By residuation (and definition of $\neg$), we get that $\neg\varphi^{n+1} \in T$. This implies that T is inconsistent—contradiction with $\neg\varphi \to \varphi^n \notin T$. □

In [65] it was shown that BL_∞ can be axiomatized by extending a presentation of BL by

$$\{\varphi \to \psi^n \mid n \in \omega\} \vee \chi \rhd (\varphi \to \varphi \mathbin{\&} \psi) \vee \chi. \qquad (\mathrm{BL}_\infty)$$

Observe that unlike in the case of $Ł_\infty$ the additional infinitary rule is already added in a $\vee$-form (we expect that (BL_∞) cannot simplified by its non $\vee$-form version). It is to be noted that the techniques used in the paper to prove the completeness are analogous to the one used for $Ł_\infty$, although we in our proof we used many general statements: for example the author proves that the logic induced by the presentation has the PCP and proves a particular version of the Lindenbaum lemma. Additionally, it was shown in the paper that both $Ł_\infty$ and Π_∞ can be axiomatized by adding (BL_∞) to a presentation of the corresponding finitary companion, that is to Ł and Π. Finally let us remark about the possible simplifications of the just mentioned axiomatization for Π_∞:

Proposition 5.24. *Alternatively, Π_∞ can be axiomatized by extending any presentation of Π by the infinitary rule ($Ł_\infty 2$).*

Proof. Again the soundness of the rule is easy to prove. We want to show that Π_∞ proves (BL_∞), which is again easy: first observe that we can think that our axiomatization has the same rules as the one of Corollary 5.22 (both logics can be presented as axiomatic extensions of BL). Thus, by Lemma 5.20 and Proposition 5.21, $\vee$ is a strong disjunction even in the logic given by the current axiomatization. Thus, by Proposition 5.10 it is enough to check a non $\vee$-form version of (BL_∞):

$$\{\varphi \to \psi^n \mid n \in \omega\} \vdash \neg\varphi \vee \psi \vdash \varphi \to \varphi \,\&\, \psi.$$

The second one easily follows by PCP and that clearly $\neg\varphi \vdash_\Pi \varphi \to \varphi \,\&\, \psi$ and $\psi \vdash_\Pi \varphi \to \varphi \,\&\, \psi$. □

On the other hand it is easy to see that the rule ($Ł_\infty$) is not sound for Π_∞. In fact, we can show even more: Π_∞ simply cannot be axiomatized by an additional infinitary rule written in onevariable; indeed we show that the one-variable fragment of Π_∞ is finitary. Easily we can describe the functions on the algebra $[0,1]_\Pi$ induced by a formula $\varphi(p)$: either there is $n \in \omega$ such that that $\varphi^{[0,1]_\Pi}(x) = x^n$ or $\varphi^{[0,1]_\Pi}(x) = 0$ on the interval $(0,1]$ and $\varphi^{[0,1]_\Pi}(0) = 1$ or $\varphi^{[0,1]_\Pi}(0) = 0$, which can be proved by an elementary inductive argument on the complexity of φ or we can use a particular case of the full characterization of the formula induced functions via monomial functions (see [26, Chapter IX]). This allows use to divide formulas into six groups:

Proposition 5.25. *One-variable fragment of Π_∞ is finitary. In particular, Π_∞ cannot be axiomatized by adding rules in one variable to Π.*

Proof. If $\Gamma(p) \vdash_{\Pi_\infty} \varphi(p)$ then we can partition Γ into six groups (a)–(f) based on the functions they induce. Then, we can select Γ' to contain one representative for every non-empty group. Since clearly for every evaluation e and every two formulas ψ, χ from one group we have $e(\psi) = 1$ if and only if $e(\chi) = 1$, we conclude that $\Gamma' \vdash_{\Pi_\infty} \varphi$. □

	$x \in (0,1]$	$x = 0$
(a)	1	1
(b)	1	0
(c)	$x^n, n > 0$	1
(d)	$x^n, n > 0$	0
(e)	0	1
(f)	0	0

5.2.3 Conclusion and remarks

We have proved a general form of Lindenbaum lemma for a wide class of infinitary logics, that is, for countably axiomatizable logics with disjunction, and explored its relation with the pair extension lemma. We have seen how it can be used to obtain some known completeness results. Thus, not only we can subsume numerous *ad hoc* proofs of similar results scattered in the literature, but more importantly, we can easily prove them for newly defined logics (here especially the characterization of strong disjunction comes in handy). On the other hand in our comparison to pair extension lemma we saw the limitations of our approach in Theorem 5.17. The reason is that the relation $\Vdash_{\mathrm{L}}$ has finitarity built-in on the right-hand side. However there are other possible natural symmetrizations of logics that we shall investigate in the future: for example for a logic L we could define

$$\Gamma \Vvdash_{\mathrm{L}} \Delta \quad \Longleftrightarrow \quad \text{for every completely intersection-prime theory } T \supseteq \Gamma \text{ there is } \delta \in \Delta \cap T.$$

Of course the reading is that Γ entails infinite meta-disjunction of formulas in Δ. Thus, it allows to speak about infinite disjunctions even in finitary syntax. Moreover, observe that this notion is a natural extension of finite disjunctions: we precisely have that if $\vee$ is a disjunction then

$$\Gamma \vdash_{\mathrm{L}} \varphi \vee \psi \quad \Longleftrightarrow \quad \text{for every intersection-prime theory (or equivalently } \vee\text{-prime) } T \supseteq \Gamma \text{ either } \varphi \in T \text{ or } \psi \in T.$$

$\Vvdash_{\mathrm{L}}$ is a symmetric consequence relation. Indeed it always enjoys the full pair extension lemma: if $\Gamma \not\Vvdash_{\mathrm{L}} \Delta$, then there is completely intersection-prime theory T extending Γ such that $T \cap \Delta = \emptyset$, thus clearly $\langle T, Fm_{\boldsymbol{\mathcal{L}}} \setminus T \rangle$ is the pair extending $\langle \Gamma, \Delta \rangle$. Next proposition is straightforward to prove and it explains the expected importance of the CIPEP for this kind of symmetrization:

Proposition 5.26. *A logic* L *has the* CIPEP *if and only if* $\Vdash_{\mathrm{L}}$ *is its conservative extension, that is for every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash_{\mathrm{L}} \varphi.$$

As an example we can prove in infinitary Łukasiewicz logic that

$$\Vdash_{Ł_\infty} \{\varphi\} \cup \{\neg\varphi^n \mid n \in \omega\},$$

which expresses the fact that every non 1 value on $[0,1]_{Ł}$ reaches zero after finitely many applications of conjunction $\&$.

Indeed, we know from the proof of Proposition 5.21 that prime theories in $Ł_\infty$ are maximally consistent. Thus, in particular, every completely intersection-prime theory is maximally consistent. Then, since in $Ł_\infty$ (Example 6.30) we have

$$\Gamma, \varphi \vdash_{Ł_\infty} Fm_{\mathcal{L}} \quad \Longleftrightarrow \quad \Gamma \vdash_{Ł_\infty} \neg\varphi^n \text{ for some } n \in \omega,$$

the claim easily follows.

Observe that there is no finite $\Delta \subset \{\varphi\} \cup \{\neg\varphi^n \mid n \in \omega\}$ such that $\Vdash_{Ł_\infty} \Delta$ which follows from the fact that $Ł_\infty$ has the CIPEP (Example 3.14) and

$$\nvdash_{Ł_\infty} p \vee \neg p \vee \neg p^2 \vee \cdots \vee \neg p^n.$$

Thus, $\Vdash_{Ł_\infty}$ is a symmetric consequence relation infinitary on both sides.

Finally, in the previous section we have explored axiomatizations of some infinitary extensions of the basic fuzzy logic BL, which we shall now summarize:

- BL_∞ can be axiomatized relative to BL by adding
  (i) $\{\varphi \to \psi^n \mid n \in \omega\} \vee \chi \rhd (\varphi \to \varphi \,\&\, \psi) \vee \chi$.
- Π_∞ can be axiomatized relative to Π by adding
  (i) $\{\varphi \to \psi^n \mid n \in \omega\} \vee \chi \rhd (\varphi \to \varphi \,\&\, \psi) \vee \chi$, or
  (ii) $\{\varphi \to \psi^n \mid n \in \omega\} \rhd \neg\varphi \vee \psi$.
- $Ł_\infty$ can be axiomatized relative to Ł by adding
  (i) $\{\varphi \to \psi^n \mid n \in \omega\} \vee \chi \rhd (\varphi \to \varphi \,\&\, \psi) \vee \chi$, or
  (ii) $\{\varphi \to \psi^n \mid n \in \omega\} \rhd \neg\varphi \vee \psi$, or
  (iii) $\{\neg\varphi \to \varphi^n \mid n \in \omega\} \rhd \varphi$.

6 | Simple theories

In the final chapter, we will investigate a special instance of completely intersection-prime theories, which we call *simple* (usually known as *maximally consistent*). Unlike in the case of linear and prime theories, the simple ones were not properly studied in the literature *per se*. Thus, we shall provide the first step into the systematic study of the role of simple theories in non-classical logics. In particular, the contribution of this chapter is not restricted to infinitary logics.

An the center of this investigation there is a generalization of the following well-known property of classical logic:

$$\Gamma, \varphi \vdash_{\mathrm{CL}} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \neg\varphi.$$

It was first introduced in [82] and called the *inconsistency lemma*. We introduce and study the hierarchy of inconsistency lemmas in a similar fashion to the hierarchy of deduction-detachment theorems. To this end, we define the class of protonegational logics, which extends that of protoalgebraic logics. Intuitively, protonegational logics retain all properties of protoalgebraic logics though restricted to simple theories. Interestingly, we will see that a natural dual property to inconsistency lemmas is a syntactical counterpart of semisimplicity. We again prove an abstract form of the Lindenbaum lemma, this related to negation and simple theories.

In the third section we investigate a notion dual to structural completeness, which we call *antistructural completeness* (it bears the same connection to antitheorems as the structural completeness to theorems). We will see that in most cases the antistructural completion will be semisimple.

The final three sections of this chapter provide initial insight into directions of possible future research:

- We will see that antistructural completions and inconsistency lemmas are very useful in the study of Glivenko-like theorems.
- We will propose a notion of infinitary deduction-detachment theorem.
- We will suggest a possible alternative presentation of protonegational logics. The class of protoalgebraic logics is generalized by splitting its defining properties into pairs of logics.

6.1 Protonegational logics

First we shall introduce and describe the basic properties of protonegational logics. As suggested above, this is achieved by restricting the properties of protoalgebraic logics to simple theories. Let us start at the begging: we say that an L-theory (L-filter) is *simple* provided it is maximally consistent, i.e. maximal non-trivial in $\mathrm{Th}\,\mathrm{L}$ (resp. in $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$). We denote the collection of all them as $\mathrm{MaxTh}\,\mathrm{L}$ (resp. $\mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$). A model $\langle \boldsymbol{A}, F\rangle$ is called *simple* if F is simple. We denote the collection of all (reduced) simple models as $\mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}$ (resp. $\mathbf{Mod}^{*}_{\mathrm{Max}}\,\mathrm{L}$). The motivation to call these theories, filters, and models *simple* comes from universal algebra, where an algebra is called *simple* provided $\mathrm{Con}\,\boldsymbol{A}$ has two elements. Observe that, if L is algebraizable, then for every reduced model $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}\,\mathrm{L}$ we have:

$$\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}^{*}_{\mathrm{Max}}\,\mathrm{L} \iff |\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}| = 2.$$

Definition 6.1. A logic L is called *protonegational* if the Leibniz operator is monotone for simple theories: $\boldsymbol{\Omega} T \subseteq \boldsymbol{\Omega} S$, whenever $T \subseteq S$, $T \in \mathrm{Th}\,\mathrm{L}$, and $S \in \mathrm{MaxTh}\,\mathrm{L}$.

Definition 6.2. A set of formulas is $\Rightarrow(p, q, \bar{r})$ is called a *parametrized protonegation* if it satisfies

- $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$[1] (reflexivity)
- If $T \in \mathrm{MaxTh}\,\mathrm{L}$ and $T \vdash_{\mathrm{L}} \varphi, \varphi \Rightarrow \psi$, then $T \vdash_{\mathrm{L}} \psi$ (simple MP)

If $\Rightarrow$ has no parameters we call it simply a *protonegation*.

Let us motivate why we call $\Rightarrow$ a *protonegation*, though it still resembles an implication: recall, that, in (non)-classical logics with an implication $\to$ and a constant $\bot$ for contradiction, it is common to define a negation as $\neg p = p \to \bot$. This negation is *explosive*, meaning that $\varphi, \neg\varphi \vdash \emptyset$. The idea behind the simple *modus ponens* is that it allows to retain only this property of implication (in a general sense): if $\Rightarrow$ enjoys the simple *modus ponens* and $\mathcal{A}$ is an antitheorem, then we can define a parametrized negation

$$\neg\varphi = \bigcup_{\alpha \in \mathcal{A}} \varphi \Rightarrow \alpha.$$

However, it turns out that to guarantee that a parametrized protonegation behaves property (e.g. that the negation defined above is explosive), we need to ensure that a given logic has enough simple theories, i.e. that the logic satisfies what we call the maximal consistency property:

[1] Recall the notation from Section 2.4.

Definition 6.3. We say that L enjoys the *maximal consistency property*, MCP, if every non-trivial L-theory T (i.e. $T \neq Fm_{\mathcal{L}}$) is contained in a simple L-theory. Similarly, L has the transferred MCP, τ-MCP, if every non-trivial L-filter is contained in a simple L-filter.

On the other hand, both the MCP and the τ-MCP are rather weak conditions, indeed

Proposition 6.4. *Every logic* L *which is compact (on every algebra) has the* MCP *(resp.* τ-MCP*). In particular, finitary logics with an antitheorem enjoy the* τ-MCP*.*

Proof. Let $\langle \boldsymbol{A}, F\rangle$ be a non-trivial model of L. The result then follows by an easy application of the maximality principle (Zorn's lemma): clearly, every maximal element in

$$\{G \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A} \mid F \subseteq G \text{ and } G \neq A\}$$

is a simple filter extending F. Compactness ensures that the chain condition of maximality principle is met. Therefore such a maximal element always exists. □

Clearly, every protoalgebraic logic is protonegational and every protoimplication is a protonegation. Here are some non-trivial examples:

Example 6.5. The intuitionistic logic, IL, has a protonegation without parameters $p \Rightarrow q = \neg(p \wedge \neg q)$: it is easy to see that it validates reflexivity and it has simple MP, because every simple theory of intuitionistic logic is a theory of classical logic (by the deduction theorem clearly either φ or $\neg\varphi$ belongs to every simple theory of intuitionistic logic), and in classical logic $p \Rightarrow q$ is equivalent with $p \to q$, which, of course, satisfies the *modus ponens*. Moreover, $\Rightarrow$ is also a protonegation in the implication-less fragment of IL, which is not protoalgebraic (thus has no protoimplication)—see [51].

Example 6.6. A similar situation arises for the basic fuzzy logic BL, which has protonegation $p \Rightarrow q = \neg(p \,\&\, \neg q)$. *Modus ponens* for simple theories is valid, because simple theories of BL are theories of Łukasiewicz logic Ł,[2] where again $p \Rightarrow q$ is equivalent to $p \to q$. Note that again implication-less fragment of BL has a protonegation, but it is not protoalgebraic (clearly protoalgebraicity is preserved by extensions).

[2] Later we will see that that $Ł_\infty$ is the strongest extension of BL which has the same simple theories (Example 6.82). In particular, since $\mathrm{BL} \leq Ł \leq Ł_\infty$, the claim follows.

Definition 6.7. A logic L is said to have a *parametrized local inconsistency lemma*, PLIL , if for every natural number n, there is a family of sets of formulas $\Psi_n(p_1, \dots, p_n, \bar{r})$ such that for every $\{\bar{\varphi}\} \cup \Gamma \subseteq Fm_{\mathcal{L}}$

$$\Gamma, \varphi_1, \dots, \varphi_n \vdash_{\mathrm{L}} \emptyset \iff \Gamma \vdash_{\mathrm{L}} I(\varphi_1, \dots, \varphi_n, \bar{\delta}) \text{ for some } I \in \Psi_n \text{ and some } \bar{\delta} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}.$$

The collection $\{\Psi_n\}_{n \in \omega}$ is called an *inconsistency sequence*.

Theorem 6.8. *Every compact logic* L *with the* PLIL *has a parametrized protonegation.*

Proof. Let $\Psi_n \subseteq P(Fm_{\mathcal{L}})$ be collections of sets witnessing the PLIL for L, by compactness, we can assume that every $I \in \Psi_1$ is finite. It is easy to observe that for every $I(p, \bar{u}) \in \Psi_1$ the set $p, I(p, \bar{u})$ is a finite antitheorem. Thus, by PLIL we obtain

$$\vdash_{\mathrm{L}} J_I(p, \varphi_1, \dots, \varphi_n, \bar{\chi}),$$

where $I(p, \bar{u}) = \{\varphi_1, \dots, \varphi_n\}$, $J_I(p_1, \dots p_{n+1}, \bar{v}) \in \Psi_{n+1}$, and $\bar{\chi} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}$. Then, we define $\Rightarrow(p, q, \bar{r})$ as a union of all

$$J_I(p, \varphi_1(q, \bar{u}), \dots, \varphi_n(q, \bar{u}), \bar{\chi}), \tag{6.1}$$

for every such $I(p, \bar{u})$, where $\varphi_i(q, \bar{u})$ is a formula obtained by substituting q for p in φ_i.

Clearly $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$. To see it is indeed a parametrized protonegation let T be a simple theory and assume $T \vdash_{\mathrm{L}} \varphi, \varphi \Rightarrow \psi$, we show $T \vdash_{\mathrm{L}} \psi$. If it where not the case, then, because T is simple, $T, \psi \vdash_{\mathrm{L}} \emptyset$. By the PLIL there is $I(p, \bar{u}) = \{\varphi_1, \dots, \varphi_n\} \in \Psi_1$ such that $T \vdash_{\mathrm{L}} I(\psi, \bar{\delta})$ for some $\bar{\delta} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}$. Then, by the definition in (6.1) we obtain that the set

$$J_I(\varphi, \varphi_1(\psi, \bar{\delta}), \dots, \varphi_n(\psi, \bar{\delta}), \overline{\chi'})$$

is contained in $\varphi \Rightarrow \psi$, where each χ' is obtained from χ substituting $\varphi, \psi, \bar{\delta}$ respectively for $p, q, \bar{r}$. Finally, since $I(\psi, \bar{\delta}) = \{\varphi_1(\psi, \bar{\delta}), \dots, \varphi_n(\psi, \bar{\delta})\}$, we conclude by the PLIL that

$$T \supseteq \varphi, I(\psi, \bar{\delta}), J_I(\varphi, \varphi_1(\psi, \bar{\delta}), \dots, \varphi_n(\psi, \bar{\delta}), \overline{\chi'}) \vdash_{\mathrm{L}} \emptyset,$$

a contradiction. □

We are now ready to prove the main characterization theorem for protonegational logics. Recall the definition of the *fundamental set*

$$\Sigma_{\mathrm{L}}(p, q, \bar{r}) = \{\chi(p, q, \bar{r}) \in Fm_{\mathcal{L}} \mid \emptyset \vdash_{\mathrm{L}} \chi(p, p, \bar{r})\}$$

and note that even though $\Sigma_{\mathrm{L}}\langle\varphi, \psi\rangle$ need not be an L-theory, we still have $\langle\varphi, \psi\rangle \in \boldsymbol{\Omega}(\Sigma_{\mathrm{L}}\langle\varphi, \psi\rangle)$ and $\mathrm{Th}_{\mathrm{L}}(\emptyset) \subseteq \Sigma_{\mathrm{L}}\langle\varphi, \psi\rangle$.

Theorem 6.9. *For every compact logic* L*, the following are equivalent:*

(i) L *is protonegational.*

(ii) L *enjoys the* weak form of the correspondence theorem:
if $h\colon \langle Fm_{\mathcal{L}}, T\rangle \to \langle \boldsymbol{A}, F\rangle$ *is a strict and surjective homomorphism between models* $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$ *and* $\langle \boldsymbol{A}, F\rangle$*, then whenever* S *is a simple* L*-theory extending* T*, we have* $S = h^{-1}[h[S]]$ *(i.e.* h *is strict between* S *and* $h[S]$*) and* $h[S]$ *is a simple* L*-filter.*

(iii) L *enjoys the* surjective substitution swapping for antitheorems*:*
for every L*-theory* T*, every set of formulas* Δ*, and every surjective substitution* σ *we have*

$$T, \sigma[\Delta] \vdash_{\mathrm{L}} \emptyset \quad \iff \quad \sigma^{-1}[T], \Delta \vdash_{\mathrm{L}} \emptyset.$$

(iv) *Surjective substitution swapping for antitheorems for finite* Δ*s.*

(v) L *enjoys the* PLIL *.*

(vi) L *has a parametrized protonegation.*

(vii) *The* Leibniz congruence is formula definable on simple L-theories*:*
there is a reflexive $\Delta(p, q, \bar{r}) \subseteq Fm_{\mathcal{L}}$ *(for every* φ *we have* $\vdash_{\mathrm{L}} \Delta\langle\varphi,\varphi\rangle$*) such that* $\Delta\langle\varphi,\psi\rangle \subseteq T$ *if and only if* $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega} T$ *for every* $T \in \mathrm{MaxTh}\,\mathrm{L}$*.*

Proof. Recall that every compact logic has the MCP (Proposition 6.4).

(i)→(ii): Suppose a strict and surjective $h\colon \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \to \langle \boldsymbol{A}, F\rangle$ is given. Then, if S is a simple theory extending T, by protonegationality, $\ker(h) \subseteq \boldsymbol{\Omega}(T) \subseteq \boldsymbol{\Omega}(S)$, thus h is strict between S and $h[S]$, which implies $h[S]$ is an L-filter and it is easy to see that $h[S]$ is indeed simple.

(ii)→(iii): right-to-left direction follows by structurality. For the other one observe that σ is strict surjective between $\sigma^{-1}[T]$ and T. Arguing by contraposition: if $\sigma^{-1}[T], \Delta \nvdash_{\mathrm{L}} \emptyset$, then by the MCP there is simple theory S containing $\sigma^{-1}[T]$ and Δ. By (ii) we know that $\sigma[S]$ is simple theory containing T and $\sigma[\Delta]$, hence $T, \sigma[\Delta] \nvdash_{\mathrm{L}} \emptyset$.

(iii)→(iv): trivial.

(iv)→(v): define $\Psi_n = \{I(p_1, \dots, p_n, \bar{r}) \subseteq Fm_{\mathcal{L}} \mid p_1, \dots, p_n, I \vdash_{\mathrm{L}} \emptyset\}$. The right-to-left direction of the definition follows, because antitheorems are closed under substitution (Corollary 2.9). Next if $\Gamma, \varphi_1, \dots, \varphi_n \vdash_{\mathrm{L}} \emptyset$, then by the substitution swapping $\sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)], p_1, \dots, p_n \vdash \emptyset$, where σ is an appropriate surjective substitution. The set $I(p_1, \dots, p_n, \bar{r}) = \sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)]$ belongs to Ψ_n and by structurality $\Gamma \vdash_{\mathrm{L}} I(\varphi_1, \dots, \varphi_n, \overline{\sigma(r)})$

(v)→(vi): Theorem 6.8.

(vi)→(i): if $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}(T)$ then $\langle\chi(\varphi,\bar{\delta}), \chi(\psi,\bar{\delta})\rangle \in \boldsymbol{\Omega}(T)$. Consequently, since $\Rightarrow$ is reflexive, $\chi(\varphi,\bar{\delta}) \Rightarrow \chi(\psi,\bar{\delta}) \subseteq T \subseteq S$. Thus, *modus ponens* ensures that $\chi(\varphi,\bar{\delta}) \in S$ if and only if $\chi(\psi,\bar{\delta}) \in S$—which is equivalent to $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}(S)$.

(i)→(vii): we show that $\Sigma_{\mathrm{L}}(p, q, \bar{r})$ is the desired set of congruence formulas. Indeed if $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}(T)$ then by reflexivity $\Sigma_L\langle\varphi,\varphi\rangle \subseteq T$ and consequently $\Sigma_L\langle\varphi,\psi\rangle \subseteq T$. Conversely, suppose $\Sigma_L\langle\varphi,\psi\rangle \subseteq T$. We want to show that for every formula $\chi(x,\bar{u})$ and every tuple of formulas $\bar{\delta}$ if $\chi(\varphi,\bar{\delta}) \in T$ then also $\chi(\psi,\bar{\delta}) \in T$ and *vice versa*: but $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}(\Sigma_L\langle\varphi,\psi\rangle)$ implies that $\langle\chi(\varphi,\bar{\delta}),\chi(\psi,\bar{\delta})\rangle \in \boldsymbol{\Omega}(\Sigma_L\langle\varphi,\psi\rangle)$. Therefore, by protonegationality $\langle\chi(\varphi,\bar{\delta}),\chi(\psi,\bar{\delta})\rangle \in \boldsymbol{\Omega}(T)$, thus we can conclude $\chi(\psi,\bar{\delta}) \in T$ if and only if $\chi(\varphi,\bar{\delta}) \in T$. Note that, since $\Sigma_L\langle\varphi,\psi\rangle$ need not be an L-theory, we are, strictly speaking, using more than the plain definition of protonegationality: in fact, for the monotonicity to hold it is enough that $\mathrm{Th}_{\mathrm{L}}(\emptyset) \subseteq \Sigma_L\langle\varphi,\psi\rangle$ (see the proof of (vi)→(i)), which is the case.

(vii)→(vi): the set Δ is clearly a parametrized protonegation. □

Let us mention that the assumption of compactness was only important in the proof of (vi)→(i) (Theorem 6.8)—thus every protonegational logic with the MCP has all the properties of the previous theorem. Compactness could be in fact replaced by the MCP, provided that we strengthen the notion of the PLIL . We will discuss this issue in Section 6.5.

A natural question is whether every protonegational logic also has a protonegation without parameters (as in the case of protoalgebraic logics): so far we do not know the answer to this question, but we strongly suspect it will not be the case. However, in sections to come we shall meet some natural conditions, which ensures that such protonegation exists (see e.g. Corollary 6.85).

We saw that in protoalgebraic logics compactness transfers to all algebras (Theorem 2.18). In fact it is enough to assume protonegationality to obtain the result. Firstly, we can prove the following variant of the filter generation property of protoalgebraic logics (Proposition 2.16):

Proposition 6.10. *Let* L *be a protonegational logic with the* MCP *and either a* ***small type or a*** $|Var_{\mathcal{L}}|$***-small protonegation and enough variables****. For every algebra* $\boldsymbol{A}$ *and every* $X \cup \{a\} \subseteq A$*,* $a \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ ***only if*** *there is a* $|Var_{\mathcal{L}}|$*-small* Γ *and a consecution* $\Gamma \rhd \varphi$ *such that* ***every simple theory containing*** Γ ***also contains*** φ *and there is an* $\boldsymbol{A}$*-evaluation* h *such that* $h[\Gamma] \subseteq X \cup \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\emptyset)$ *and* $h(\varphi) = a$*.*

Proof. We proceed by induction on the complexity of the proof tree witnessing $a \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$—see Proposition 2.19. In the base case there are two option (i) $a \in X$ and (ii) $a = h(\varphi)$ for some theorem φ. For (i) consider $p \rhd p$ and for (ii) $\emptyset \rhd \varphi$ and the homomorphism h.

For the inductive step there is a $|Var_{\mathcal{L}}|$-small Γ, a rule $\Gamma \vdash \varphi$, and an evaluation h such that $h(\varphi) = a$. Moreover, by the induction assumption we know that for every $\gamma \in \Gamma$ there is an appropriate consecution $\Gamma_\gamma \rhd \varphi_\gamma$ and an evaluation h_γ such that $h_\gamma(\varphi_\gamma) = h(\gamma)$. By the cardinality assumptions we may consider each $\Gamma_\gamma \rhd \varphi_\gamma$ is written using unique set of variables which are different from the variables in $\Gamma \rhd \varphi$ (modulo a suitable renaming of variables in $\Gamma \rhd \varphi$ we may assume that there are $|Var_{\mathcal{L}}|$-many variables which are not used in this consecution).

Next choose an evaluation f which agrees with h_γ on every variables from $\Gamma_\gamma \rhd \varphi_\gamma$ for every $\gamma \in \Gamma$ and with h on variables from $\Gamma \rhd \varphi$. Let $\Rightarrow(p, q, \bar{r})$ be a parametrized protonegation (resp. let $\Rightarrow(p, q)$ be a $|Var_{\mathcal{L}}|$-small protonegation) and put

$$\Delta = \bigcup_{\gamma \in \Gamma} \Gamma_\gamma \cup \bigcup_{\gamma \in \Gamma} \varphi_\gamma \Rightarrow \gamma$$

Because $f(\varphi_\gamma) = h_\gamma(\varphi_\alpha) = h(\gamma) = f(\gamma)$, we have that $f[\Delta] \subseteq X \cup \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\emptyset)$ and $f(\varphi) = a$. Moreover, by protonegationality every simple theory which contains Δ contains by induction assumption φ_γ and consequently, by protonegationality, also γ for every $\gamma \in \Gamma$ thus it also contains φ. Clearly, by the construction, Δ is $|Var_{\mathcal{L}}|$-small as desired. □

The previous proposition seems as a rather weak result, but it is enough to prove the trivial filter generation property of protoalgebraic logics (Proposition 2.17) in the precisely same formulation also for protonegational logics with the MCP, which is the key component to prove that the compactness transfers.

Proposition 6.11. *Let* L *be a protonegational logic with the* MCP*, an antitheorem, and either a* ***small type or a*** $|Var_{\mathcal{L}}|$***-small protonegation and enough variables****. Then, for every algebra* $\boldsymbol{A}$ *and every* $X \subseteq A$, $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = A$ *if and only if there is* $|Var_{\mathcal{L}}|$*-small antitheorem* $\mathcal{A}$ *and an* $\boldsymbol{A}$*-evaluation* h *such that* $h[\mathcal{A}] \subseteq X \cup \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\emptyset)$.

Proof. Recall the proof of Proposition 2.17. We can again construct the set

$$\mathcal{A}' = \bigcup_{\alpha \in \mathcal{A}} \Gamma_\alpha \cup \bigcup_{\alpha \in \mathcal{A}} \varphi_\alpha \Rightarrow \alpha.$$

This time of course $\Rightarrow(p, q, \bar{r})$ is a parametrized protonegation. $\mathcal{A}'$ was clearly an antitheorem in Proposition 2.17. Here suppose $\mathcal{A}'$ is not an antitheorem then by the MCP there is a simple T extending $\mathcal{A}'$, but then *modus ponens* for simple theories of protonegation ensures that $T \vdash \mathcal{A}$ thus T is not consistent—contradiction. The rest of the proof is the same. □

Theorem 6.12. *Let* L *be a protonegational logic with an antitheorem, the* MCP*, and either a* ***small type or a*** $|Var_{\mathcal{L}}|$***-small protonegation and enough variables****. Then, if* L *is at most* κ*-compact, then so is* $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *for every algebra* $\boldsymbol{A}$*.*

Proof. Analogous to the one of Theorem 2.18. □

Corollary 6.13. *Let* L *be a protonegational logic with either a small type or* $|Var_{\mathcal{L}}|$*-small protonegation and enough variables. Then, compactness transfers to all algebras.*

Proof. Compact logics always have both an antitheorem (see Corollary 2.10) and the MCP (Proposition 6.4) so the previous theorem applies. □

Corollary 6.14. *Every compact protonegational logic with either a small type or* $|Var_{\mathcal{L}}|$*-small protonegation and enough variables enjoys the* τ*-*MCP*.*

Proof. Combine the previous corollary and the fact that the transferred compactness imply the τ-MCP (Proposition 6.4). □

So far we know that protonegational logics with the MCP have many interesting properties (Theorem 6.9), but they are all restricted to the algebra of formulas. In fact, we do not know whether they transfer in general to all algebras, which motivates the following definition:

Definition 6.15. A logic L is called *fully protonegational* if the Leibniz operator is monotone for simple filters: $\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$, whenever $F \subseteq G$, $F \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$, and $G \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$.

Going along the lines of the proof of Theorem 6.9 we can show:

Proposition 6.16. *For every fully protonegational logic* L *the following is true:*

(i) L *enjoys the* full weak form of the correspondence theorem:
if $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ *is a strict and surjective homomorphism between models* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, G\rangle$*, then whenever* $F \subseteq H \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$*, we obtain that* h *is strict between* H *and* $h[H]$*, and* $h[G] \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$*.*

(ii) *The* Leibniz congruence is formula definable on simple L-filters*:*
there is a reflexive $\Delta(p, q, \bar{r}) \subseteq Fm_{\mathcal{L}}$ *such that* $\Delta^{\boldsymbol{A}}\langle a, b\rangle \subseteq F$ *if and only if* $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$ *for every* $F \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$*.*

As said above, we do not know whether every protonegational logic is fully protonegational. However, we will see that under two different sets of (very weak) assumptions it is always the case. The assumptions are:

(i) L is compact and has enough variables.
(ii) L enjoys the MCP and has a small type.

We will start with the first case, which we prove via natural extensions.

Lemma 6.17. *Let* L *is (fully) protonegational with the (*τ*)-*MCP *and* $\boldsymbol{A}$*,* $\boldsymbol{B}$ *algebras. Then, for every* $F \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ *and every surjective* $\boldsymbol{A}$*-evaluation* h *(resp. homomorphism* $h\colon \boldsymbol{B} \to \boldsymbol{A}$*), we have* $h^{-1}[F] \in \mathrm{Max}\mathrm{Th}\,\mathrm{L}$ *(resp.* $h^{-1}[F] \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$*).*

Proof. We prove only the protonegational case, the other one is analogous. Clearly $h^{-1}[F]$ is a theory of L. If it was not simple, then by the MCP there would be a simple theory T extending it. However, then by the weak correspondence theorem (see Theorem 6.9) we obtain a simple L-filter $h[T]$ which strictly extends F—contradiction. □

Lemma 6.18. *If* L *is a compact logic with the* PLIL *and enough variables, then so is every natural extension* L^κ *to* κ*-many variables. In particular,* L^κ *is protonegational.*

Proof. Recall that in Proposition 2.24 we proved that compactness is preserved under natural extensions. On the other, the general method to prove that result can be easily applied even to the case of PLIL (both logics will have the same inconsistency sequence). Finally L^κ is protonegational by the characterization theorem (Theorem 6.9). □

Theorem 6.19. *Every compact protonegational logic with enough variables is fully protonegational.*

Proof. There is a surjective homomorphism h from $\boldsymbol{Fm}_{\mathcal{L}}(\kappa)$ to $\boldsymbol{A}$ for some cardinal κ. Suppose $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ and pick φ_a, φ_b such that $h(\varphi_a) = a$ and $h(\varphi_b) = b$. Clearly $\langle \varphi_a, \varphi_b\rangle \in \boldsymbol{\Omega} h^{-1}[F]$, by Lemma 6.17 we know that $h^{-1}[G]$ is a simple theory of L^κ and, since L^κ is protonegational by Lemma 6.18, we obtain $\langle \varphi_a, \varphi_b\rangle \in \boldsymbol{\Omega} h^{-1}[G]$. Consequently, we conclude $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} G$. □

For the second case we use the interesting fact that small subsets of RSI (resp. simple) models can be extended to small RSI (resp. simple) submatrices.

Lemma 6.20. *Let* κ *be an infinite cardinal,* L *a logic with type of size* $\leq \kappa$ *and* $\mathrm{card}\,\mathrm{L} \leq \kappa^+$*. Suppose* $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}$*. Then, each* κ*-small subset of* A *extends to a* κ*-small submodel* $\langle \boldsymbol{B}, G\rangle$ *of* $\langle \boldsymbol{A}, F\rangle$ *such that* $\langle \boldsymbol{B}, G\rangle \in \mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}$*.*

Proof. Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L}$, this implies that F is saturated w.r.t. some $a \in A$. We will define a countable chain $\mathbf{A}_i = \langle \boldsymbol{A}_i, F_i\rangle$ of κ-small submatrices of $\mathbf{A}$. Let $H \subseteq A$ be the κ-small set we want to extend. We start with $\mathbf{A}_0 = \langle \mathbf{Sg}^{\boldsymbol{A}}(H \cup \{a\}), F \cap \mathbf{Sg}^{\boldsymbol{A}}(H \cup \{a\})\rangle$.

Suppose $\mathbf{A}_i$ is defined. By the assumption $|A_i| \leq \kappa$, thus, in particular $|A_i \setminus F_i| \leq \kappa$. By the saturation, for every $b \in A_i \setminus F_i$, we obtain $a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F, b)$ ($\mathbf{A}_i$ is a submatrix of $\mathbf{A}$, thus $a \notin F$). Therefore, by Proposition 2.19, there is a well-founded proof tree such that the leaves of the tree are from $F \cup \{b\}$ and a is the root. Define P_b as the union of all subsets of A of the form $e[\mathit{Var}_{\boldsymbol{\mathcal{L}}}]$, where e is an $\boldsymbol{A}$-evaluation such that $\langle e[\Gamma], e(\varphi)\rangle \in V_{\mathcal{AS}}$ was used in the proof tree. The cardinality restriction of L implies that P_b is κ-small. Define $H = A_i \cup \bigcup_{b \in A_i \setminus F_i} P_b$, this set is still κ-small. We can set

$$\mathbf{A}_{i+1} = \langle \mathbf{Sg}^{\boldsymbol{A}}(H), \mathbf{Sg}^{\boldsymbol{A}}(H) \cap F\rangle.$$

Obviously the cardinality of A_{i+1} is bounded by κ. Finally define

$$\mathbf{B} = \langle \mathbf{Sg}^{\boldsymbol{A}}(\bigcup_{i \in \omega} A_i), \mathbf{Sg}^{\boldsymbol{A}}(\bigcup_{i \in \omega} A_i) \cap F\rangle.$$

We claim that $\mathbf{B} = \langle \boldsymbol{B}, G\rangle$ is our desired subdirectly irreducible submatrix of $\mathbf{A}$. It clearly satisfies the cardinality requirements. To end the proof we show that a saturates G. Take an element $b \in B \setminus G$. We want to show that $a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{B}}(G, b)$. By the construction there is $i \in \omega$ such that $b \in A_i$, but then the whole proof P_b is contained in $A_{i+1} \subseteq B$, thus again Proposition 2.19 implies $a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{B}}(G, b)$. □

Proposition 6.21. *Under the assumptions of the previous lemma, if moreover* L *has an antitheorem, then κ-small subsets of simple models extend to κ-small simple submodels.*

Proof. If $\langle \boldsymbol{A}, F\rangle$ is simple then it is in fact saturated w.r.t. every $a \in A \setminus F$. Let $\mathcal{A}$ be an antitheorem (by the cardinality restrictions we may assume $|\mathcal{A}| \leq \kappa$) in one variable, then $h[\mathcal{A}]$ is an inconsistent set in $\boldsymbol{A}$ (that is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(h[\mathcal{A}]) = A$) for an arbitrary homomorphism $h\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{A}$. We can easily modify the proof of the previous lemma so that the matrix $\langle \boldsymbol{B}, G\rangle$ not only contains the set H but also the inconsistent set $h[\mathcal{A}]$, but moreover we can ensure that G is saturated w.r.t. every element in $h[\mathcal{A}] \setminus F$, which implies that G is simple. □

Theorem 6.22. *Every protonegational logic with the* MCP *and a small type is fully protonegational.*

Proof. Note that if L has a small type, then $|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$ satisfies the cardinality restrictions of Lemma 6.20 and Proposition 6.21. Suppose $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Given a formula $\chi(p, \bar{r})$ and a tuple of elements $\bar{c}$ of A, it is enough to show that $\chi^{\boldsymbol{A}}(a, \bar{c}) \in G$ implies $\chi^{\boldsymbol{A}}(b, \bar{c}) \in G$. Apply Proposition 6.21 to obtain $|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$-small simple submatrix $\langle \boldsymbol{B}, H\rangle$ of $\langle \boldsymbol{A}, G\rangle$ in such a way that

$a, b, \chi^{\boldsymbol{A}}(a, \bar{c}), \chi^{\boldsymbol{A}}(b, \bar{c}) \in B$. Denote $F' = F \cap B$. Let $h\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{B}$ be a surjection (it exists because B is $|Var_{\mathcal{L}}|$-small). Take formulas φ_a, φ_b such that $h(\varphi_a) = a$ and $h(\varphi_b) = b$.

From $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ infer that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} F'$. Consequently, we obtain that $\langle \varphi_a, \varphi_b\rangle \in \boldsymbol{\Omega} h^{-1}[F']$. Then, since by Lemma 6.17 we know $h^{-1}[H]$ is simple, we get

$$\langle \varphi_a, \varphi_b\rangle \in \boldsymbol{\Omega} h^{-1}[H]. \tag{6.2}$$

Thus, if $\chi^{\boldsymbol{A}}(a, \bar{c}) \in G$ then also $\chi^{\boldsymbol{A}}(a, \bar{c}) \in H$, but (6.2) implies that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} H$ which allows us to conclude that $\chi^{\boldsymbol{A}}(b, \bar{c}) \in H \subseteq G$. □

We will now show some additional application of the RSI-submodel extension lemma (Lemma 6.20):

Theorem 6.23. *Suppose* L *is a logic with an antitheorem and has a small type. Further, suppose* L *is protoalgebraic or is protonegational with the* CIPEP*. Then, if every completely intersection-prime* L*-theory is simple, then also every completely intersection-prime* L*-filter is simple.*

Proof. Suppose $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ and F is completely intersection-prime. By Lemma 6.20, there is $|Var_{\mathcal{L}}|$-small submatrix $\langle \boldsymbol{B}, G\rangle$ with G completely intersection-prime. Let $h\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{B}$ be a surjection, we can assume that B contains some element $a \in A$, which makes F saturated. First will show that $h^{-1}[G]$ is completely intersection prime in $\mathrm{Th}\,\mathrm{L}$. If L is protoalgebraic the result easily follows by the correspondence theorem. In the second case by the CIPEP we have $h^{-1}[G] = \bigcap T_i$, where all T_i are completely intersection-prime and thus by the assumption of the theorem also simple. As an easy consequence of the CIPEP and the assumption that completely intersection-prime theories are simple we obtain that L has the MCP. Thus, by the second point of Theorem 6.9,[3] it can easily be shown that $G = \bigcap h[T_i]$. Thus, by the primeness of G we obtain that $G = h[T_i]$ for some i, consequently $h^{-1}[G] = T_i$, thus by the same theorem both $h^{-1}[G]$, and G are simple.

Now, if $b \notin F$, then by the saturation $a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup \{b\})$. Since G is simple and $a \in B \setminus G$ we can see that $B \subseteq \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup \{b\})$. Finally, using the fact that L has an antitheorem, we easily prove that $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup \{a\}) = A$. □

In the remaining part of this section we shall focus on the notion of semisimplicity. Let K be a class of algebras. As $\mathsf{K}_{\mathrm{Simple}}$ we denote its simple members, i.e. those $A \in \mathsf{K}$ such that $|\mathrm{Con}_{\mathsf{K}}\, \boldsymbol{A}| = 2$. A quasi-variety K is called *semisimple* if any of the following equivalent conditions is met:

[3] Note that in this case we can forget about the compactness assumption of the theorem—cf. the comments below the theorem.

(i) Relatively subdirectly irreducible members of K are simple.
(ii) For every $\boldsymbol{A} \in \mathsf{K}$, the identity congruence $\Delta_{\boldsymbol{A}}$ is an intersection of maximal members from $\mathrm{Con}_{\mathsf{K}}\,\boldsymbol{A} \setminus \{\nabla_{\boldsymbol{A}}\}$.
(iii) Every algebra in K is a subdirect product of algebras from $\mathsf{K}_{\mathrm{Simple}}$.

We are now going to propose a logical counterpart of semisimplicity. First of all for simple theories (filters) we have a corresponding extension property: a logic L enjoys *(transferred-)simple extension property*, SEP (resp. τ-SEP), if $\mathrm{MaxTh}\,\mathrm{L}$ (resp. $\mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$) forms a basis of $\mathrm{Th}\,\mathrm{L}$ (resp. $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ on every algebra $\boldsymbol{A}$). In logical setting the conditions (i)–(iii) may be formulated as follows:

(i) $\mathbf{Mod}_{\mathrm{RSI}}\,\mathrm{L} \subseteq \mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}$.
(ii) L enjoys the τ-SEP.
(iii) $\mathbf{Mod}^*\,\mathrm{L} \subseteq \mathbf{P_{SD}}(\mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{L})$.

It can easily be shown that these conditions are equivalent for finitary protoalgebraic logics. One can use the following facts about these logics:

- finitarity implies τ-CIPEP (Proposition 3.3), and that
- for protoalgebraic logics we can prove the subdirect representation theorem (Theorem 3.19.)

However these conditions need not be equivalent for arbitrary logic. Since we believe (iii) to be the intended meaning of semisimplicity, we define L to be *semisimple* if every reduced model of L is a subdirect product of reduced simple models of L—that is, the condition (iii) holds.

Proposition 6.24. *For every* L*, the simple filters are completely intersection-prime. Moreover, if* L *has the (*τ*)-*SEP*, then every completely intersection-prime theory (filter) is simple.*

Proof. The first claim is easy to prove: just note that simple theories are saturated w.r.t. every element. Conversely if T completely intersection-prime, it is saturated w.r.t. some formula φ. Then, by SEP there is a simple $T' \supseteq T$ such that $T' \nvdash \varphi$. Then, since φ saturates T, we conclude $T = T'$. □

Proposition 6.25. *Every logic* L *with* τ*-*SEP *is semisimple. Moreover, if* L *is fully protonegational and has* τ*-*MCP*, then the converse is also true.*

Proof. The first claim is an easy consequence of the previous proposition and Theorem 3.19. The other direction can be proved the same ways as in Theorem 3.19: we use the fact that by Lemma 6.17 the preimage of $\pi_i \circ h$ over a simple filter is again simple. □

6.2 Inconsistency lemmas

In this section we study the notion of inconsistency lemmas. We start by extending Definition 6.7 to local and global version without parameters in a similar fashion to deduction-detachment theorems.

Definition 6.26. L is said to have a *local inconsistency lemma*, LIL, if it has an inconsistency sequence without parameters. Moreover, if for each natural number n the set Ψ_n is a singleton, we say that L has a *global inconsistency lemma*, GIL,and we identify each Ψ_n with the unique $I(p_1, \dots, p_n) \in \Psi_n$.

Inconsistency lemmas were first introduced and studied by James Raftery in [82] but only in its global form. Let us start with some examples

Example 6.27. If L is an superintuitionistic logic, i.e. an axiomatic extension of intuitionistic, then it has the GIL in form

$$\Gamma, \varphi_1, \dots, \varphi_n \vdash_{\mathrm{L}} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg(\varphi_1 \wedge \dots \wedge \varphi_n),$$

which can be seen as an easy consequence of the well-known standard deduction-detachment theorem of these logics.

Example 6.28. Every axiomatic extension of the full Lambek calculus with exchange and weakening, $\mathrm{FL_{ew}}$, has the LIL. It is well known [23, 52] that all of these logics enjoy the local DDT in the form

$$\Gamma, \varphi \vdash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \varphi^n \to \psi \text{ for some } n \in \omega.$$

The LIL can be seen as special case where $\psi = \bot$.

$$\begin{aligned} \Gamma, \varphi_1, \dots, \varphi_n \vdash \emptyset \quad &\Longleftrightarrow \quad \Gamma, \varphi_1 \,\&\, \dots \,\&\, \varphi_n \vdash \emptyset \\ &\Longleftrightarrow \quad \Gamma \vdash (\varphi_1 \,\&\, \dots \,\&\, \varphi_n)^n \to \bot = \neg(\varphi_1 \,\&\, \dots \,\&\, \varphi_n)^n \\ &\qquad\qquad \text{for some } n \in \omega. \end{aligned}$$

Thus, the set Ψ_n has form $\{\neg(p_1 \,\&\, \dots \,\&\, p_n)^n \mid n \in \omega\}$. Note that also the implication-less fragment of these logics has the LIL in this from.

Example 6.29. A special case of the previous example is the product logic Π. Observing that $\neg p^n \vdash_{\Pi} \neg p$ for every $n \in \omega$ it easily follows that

$$\Gamma, \varphi_1 \,\&\, \dots \,\&\, \varphi_n \vdash \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash \neg(\varphi_1 \,\&\, \dots \,\&\, \varphi_n).$$

Thus, Π has GIL with $I_n = \{\neg(p_1 \,\&\, \dots \,\&\, p_n)\}$. So Π is an example of logic with global inconsistency lemma but only local DDT (see e.g. [1]).

The same situation arises in the logic of strict continuous t-norms, SBL, which is axiomatized by adding e.g. $\neg(p \,\&\, p) \to \neg p$ to BL (see [21]).

Example 6.30. The infinitary Łukasiewicz logic $Ł_\infty$ also has the LIL in the same form as $\mathrm{FL_{ew}}$ (Example 6.28), which can easily be proved using compactness of $Ł_\infty$ and the fact that it has LDDT of $\mathrm{FL_{ew}}$.

We will also present a semantical argument, which, as we believe, provides some inside into the behavior of the logic. For the proof of the non-obvious direction assume that for every n we have $\Gamma \nvdash_{Ł_\infty} \neg\varphi^n$, that is, we can assume that we have for every n an evaluation e_n on the standard algebra such that $e[\Gamma] \subseteq \{1\}$ and $e_n(\varphi) > 1 - 1/n$. We want to show that there is an evaluation e such that $e[\Gamma \cup \{\varphi\}] \subseteq \{1\}$, i.e. $\Gamma, \varphi \nvdash_{Ł_\infty} \emptyset$. We will use that fact, that the connectives of the logic are continuous w.r.t. the standard interval topology. The evaluation we are looking for will be limit of the evaluations e_n.

It is easy to prove that if $\{v_n\}_{n\in\omega}$ are evaluations such that for every variable p occurring in φ the sequence $\{v_n(p)\}_{n\in\omega}$ has a limit, then for the limit evaluation v, given by $v(p) = \lim\{v_n(p)\}_{n\in\omega}$, the value $v(\varphi)$ is the limit of $\{v_n(\varphi)\}_{n\in\omega}$.

Thus, to finish the proof, we find an infinite $I \subseteq \omega$ such that the sequence $\{e_n(p)\}_{n\in I}$ has a limit for every variable p. We will extend the well-known Bolzano–Weierstrass theorem to infinitely many sequences. Enumerate the variables as $\{p_i\}_{i\in\omega}$. First, for every $n \in \omega$ using induction and the theorem, we can obtain $I_n \subseteq \omega$ such that $\{e_m(p_n)\}_{m\in I_n}$ converges and the intersection of finitely many of these sets is infinite. Let I be an infinite pseudointersection of I_n, that is for every n the set I is contained in I_n up to finitely many elements. Then, clearly for every n the sequence $\{e_m(p_n)\}_{m\in I}$ converges and the desired evaluation assigns the limit of $\{e_m(p_i)\}_{m\in I}$ to the variable p_i.

For the benefit of the reader we show that indeed such pseudointersection of I_n's exists. It can be defined inductively as follows: let $X_0 = \{k\}$ for some $k \in I_0$ and $X_{n+1} = X_n \cup \{l\}$ for some $l \in (\bigcap_{i\leq n} I_i) \setminus X_n$. It is easy to see that $I = \bigcup_{n\in\omega} X_n$ is a pseudointersection of the sets I_n (all the elements added in steps $> n$ belong to I_n).

Example 6.31. Finally, we are going to see that the Full lambek calculus with weakening, $\mathrm{FL_w}$, enjoys the PLIL while it does not enjoy the LIL (for the second claim see Example 6.41). Recall that a left conjugate is defined as a formula $\lambda_\alpha(\varphi) = \alpha \to \varphi \mathbin{\&} \alpha$ and the right conjugate as $\rho_\alpha(\varphi) = \alpha \rightsquigarrow \alpha \mathbin{\&} \varphi$. An iterated conjugate of φ is a composition $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\dots\gamma_{\alpha_n}(\varphi)))$, where $n \in \omega$ and $\gamma_{\alpha_i} \in \{\lambda_{\alpha_i}, \rho_{\alpha_i}\}$. We can obtain the PLIL as a restriction of the PLDDT of $\mathrm{FL_w}$ (see [53, Theorem 2.14]):

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash \emptyset \iff \Gamma \vdash \neg(\prod_i^m \gamma_i), \text{ for some } m, \text{ where each } \gamma_i \text{ is an iterated conjugate of a formula in } \{\varphi_1, \ldots, \varphi_n\}.$$

Of course, by $\prod$ we mean $\&$ over the arguments. In other words $\mathrm{FL_w}$ has the following inconsistency sequence

$$\Psi_n(p_1, \ldots, p_n, \bar{r}) = \{\{\neg(\prod_i^m \gamma_i)\} \mid m \in \omega, \ \gamma_i \text{ iterated conjugate of some } p_i\}.$$

A formula in two variables $c(p, q)$ is called a *conjunction* in L if for every formulas φ, ψ

$$\varphi, \psi \vdash_{\mathrm{L}} c(\varphi, \psi) \quad \text{and} \quad c(\varphi, \psi) \vdash_{\mathrm{L}} \varphi, \psi.$$

Note that such connectives are associative and commutative thus we can write $c(\varphi_1, \ldots, \varphi_n)$ instead of $c(\varphi_1, c(\ldots c(\varphi_{n-1}, \varphi_n)))$. It can easily be seen that in logics with conjunction the definition of inconsistency lemma simplifies, it is enough to have just Ψ_1, indeed:

$$\begin{aligned} \Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \emptyset &\iff \Gamma, c(\varphi_1, \ldots, \varphi_n) \vdash_{\mathrm{L}} \emptyset \\ &\iff \Gamma \vdash_{\mathrm{L}} I(c(\varphi_1, \ldots, \varphi_n), \bar{\delta}) \text{ for some } I \in \Psi_1 \\ &\qquad \text{and some } \bar{\delta} \in Fm_{\mathcal{L}}{}^{Var_{\mathcal{L}}}. \end{aligned}$$

The next result, the transfer of inconsistency lemmas, can be found already in [82, Theorem 3.6], though in less general form and only for GIL, Moreover we present a different proof.

Theorem 6.32. *Let* L *be a protonegational logic with the* MCP*, enough variables, and the* LIL *(or GIL) witnessed by* $\{\Psi_n\}_{n\in\omega}$*. Then, for every* L*-filter* F *on an algebra* $\boldsymbol{A}$ *and every tuple of elements* $a_1, \ldots, a_n$*:*

$$A = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\{F, a_1, \ldots, a_n\}) \iff I^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F \textit{ for some } I \in \Psi_n.$$

Proof. The same proof holds for both LIL and GIL, The right-to-left direction follows easily from Proposition 6.11 because

$$p_1, \ldots, p_n, I(p_1, \ldots, p_n) \vdash_{\mathrm{L}} \emptyset.$$

For the other direction, let $\mathcal{A}(q)$ be an antitheorem in one variable q, we can assume that $|\mathcal{A}| \leq |Var_{\mathcal{L}}|$ (Proposition 2.15). Then, choose $a \in A$, clearly $\mathcal{A}^{\boldsymbol{A}}(a) \subseteq \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\{F, a_1, \ldots, a_n\})$. Using Proposition 2.19 (similarly as in the proof of Lemma 6.20) for every $b \in \mathcal{A}^{\boldsymbol{A}}(a)$ we obtain a set P_b

as the union of all subsets of A of the form $e[Var_{\mathcal{L}}]$, where e is an $\boldsymbol{A}$-evaluation such that $\langle e[\Gamma], e(\varphi)\rangle \in V_{\mathcal{AS}}$ was used in the proof tree witnessing $b \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\{F, a_1, \ldots, a_n\})$. Then, let $X = \bigcup_{b \in A^{\boldsymbol{A}}(a)} P_b$ and observe it is $|Var_{\mathcal{L}}|$-small. Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ generated by $X \cup \{a_1, \ldots, a_n\}$ and let $G = F \cap A$. Since the algebra $\boldsymbol{B}$ contains the proofs P_b we can easily conclude that $\mathrm{Fi}^{\boldsymbol{B}}_{\mathrm{L}}(\{G, a_1, \ldots, a_n\}) = B$. Let $h : \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{B}$ be a surjective homomorphism such that $h(p_i) = a_i$ for $i \leq n$. By protonegationality and the MCP we obtain that $h^{-1}[G], p_1, \ldots, p_n \vdash_{\mathrm{L}} \emptyset$: indeed otherwise the premises $h^{-1}[G], p_1, \ldots, p_n$ could be extended to a simple theory T and by Theorem 6.9 $h[T]$ would be a non-trivial L-filter extending $G, a_1, \ldots, a_n$—contradiction. Now, by the LIL (resp. GIL), we obtain that $h^{-1}[G] \vdash I(p_1, \ldots, p_n)$ for some $I \in \Psi_n$ and, consequently, $h[I] = I^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq G \subseteq F$. □

The previous theorem allows to prove characterizations for both local and global inconsistency lemmas. We shall now focus on the first case:

Definition 6.33. A logic L has the *simple filter extension property* (SFEP) if for any two L-models $\langle \boldsymbol{B}, G\rangle \leq \langle \boldsymbol{A}, F\rangle$ and every $G \subseteq H \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$ there is $F \subseteq H' \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ such that $H = H' \cap B$.

Note that in the definition we do not require H' to be simple.

Lemma 6.34. *If* L *has an antitheorem then its non-trivial models are closed under submatrices.*

Theorem 6.35. *Let* L *be a compact protonegational logic with a small type.*[4] *Then, the following are equivalent:*

(i) L *enjoys the local inconsistency lemma.*
(ii) L *enjoys the simple filter extension property.*
(iii) *The* SFEP *holds on the algebra of formulas* $\boldsymbol{Fm}_{\mathcal{L}}$.

Proof. (i)→(ii): Let $\langle \boldsymbol{B}, G\rangle \leq \langle \boldsymbol{A}, F\rangle$ and let $G \subseteq H \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$. Define $H' = \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(F \cup H)$. We prove that $H = H' \cap B$. To this end, take $b \in B \cap H'$. In order to obtain contradiction suppose $b \notin H$, then $\mathrm{Fi}^{\boldsymbol{B}}_{\mathrm{L}}(H, b) = B$ and, by transfer of LIL (Theorem 6.32—recall that by Proposition 6.4 compactness implies the MCP), $I^{\boldsymbol{B}}(b) \subseteq H$ for some $I \in \Psi_1$. Consequently $I^{\boldsymbol{A}}(b) = I^{\boldsymbol{B}}(b) \subseteq H'$, and thus, again by LIL, $H' = A$ (because $b \in H'$). Therefore it is enough to show that $H' \neq A$: Suppose it is not the case, i.e. $A = H'$. Then, the transferred compactness (Corollary 6.13) implies that

[4] Instead of a small type we can assume that the logic has enough variables and a $|Var_{\mathcal{L}}|$-small protonegation.

$A = \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(f_1, \dots, f_m, h_1, \dots, h_n)$, where $f_i \in F$ and $h_i \in H$. Then, by LIL there is $I \in \Psi_n$ such that $I^{\boldsymbol{A}}(h_1, \dots, h_n) \subseteq \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(f_1, \dots, f_m) \subseteq F$. Then, clearly $I^{\boldsymbol{B}}(h_1, \dots, h_n) = I^{\boldsymbol{A}}(h_1, \dots, h_n) \subseteq G$. Again by LIL we conclude that $B = \mathrm{Fi}^{\boldsymbol{B}}_{\mathrm{L}}(h_1, \dots, h_n, G) \subseteq H$. In other words, H is trivial, thus not simple— contradiction.

(ii)→(iii): Trivial.

(iii)→(i): Define an inconsistency sequence as

$$\Psi_n = \{T \cap Fm_{\mathcal{L}}(\{p_1, \dots, p_n\}) \mid p_1, \dots, p_n, T \vdash \emptyset,\ T \in \mathrm{Th}\,\mathrm{L}\}.$$

First we show that for any $I \in \Psi_n$ we still have $p_1, \dots, p_n, I \vdash \emptyset$. If it was not the case, then, by Lemma 6.34, we would obtain that

$$F = \mathrm{Th}_{\mathrm{L}}(p_1, \dots, p_n, I) \cap Fm_{\mathcal{L}}(\{p_1, \dots, p_n\}) \neq Fm_{\mathcal{L}}(\{p_1, \dots, p_n\}).$$

However, then by the τ-MCP (which the logic enjoys by Corollary 6.14), there would be a simple filter G such that $F \subseteq G \subseteq Fm_{\mathcal{L}}(\{p_1, \dots, p_n\})$ and by the SFEP we would obtain $p_1, \dots, p_n, T \nvdash \emptyset$—contradiction. In particular, we have one direction of LIL:

$$\Gamma \vdash_{\mathrm{L}} I(\varphi_1, \dots, \varphi_n) \text{ implies } \Gamma, \varphi_1, \dots, \varphi_n \vdash_{\mathrm{L}} \emptyset \text{ for every } I \in \Psi_n.$$

For the other direction, suppose $\varphi_1, \dots, \varphi_n, \Gamma \vdash \emptyset$ and take a surjective substitution σ such that $\sigma(p_i) = \varphi_i$. Protonegationality and Theorem 6.9 implies that $\sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)], p_1, \dots, p_n \vdash \emptyset$. Let $I = \sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)] \cap Fm_{\mathcal{L}}(\{p_1, \dots, p_n\})$. Clearly, $I \in \Psi_n$ and thus, by structurality, we conclude that

$$\Gamma \vdash \sigma(I(p_1, \dots, p_n)) = I(\varphi_1, \dots, \varphi_n). \qquad \square$$

There is an another property closely related to the SFEP which as we will see can be characterized in a similar fashion.

Definition 6.36. A logic L is *simple submatrix closed*, SSC for short, if every submodel of a simple L-model is also simple.

It is easy to see that SFEP is a sufficient condition for a given logic to be simple submatrix closed. Interestingly enough, we are going to show that SSC (at least in case of protonegational logics) corresponds to some form of definability of simple theories (filters).

Definition 6.37. A logic L has *definable simple filters*, if there is a family of sets of formulas in one variable $\Psi(p)$ such that for every algebra $\boldsymbol{A}$:

$$\mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A} = \{F \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A} \mid \forall a \in A (a \notin F \Leftrightarrow \exists I \in \Psi \text{ such that } I^{\boldsymbol{A}}(a) \subseteq F)\}.$$

Moreover, L has *definable simple theories* if the property holds for the algebra of formulas.

Lemma 6.38. *Every logic* L *with the* MCP *has definable simple theories if and only if for every simple theory* T *and every formula* φ *the following holds*

$$\varphi \notin T \quad \Longleftrightarrow \quad I(\varphi) \subseteq T \text{ for some } I \in \Psi. \tag{6.3}$$

In particular, every logic with the LIL *or the* dLIL *has definable simple theories. An analogous characterization holds for definability of simple filters.*

Proof. The direction from left to right holds trivially. Conversely, clearly we have every simple theory satisfies the defining condition. Suppose T satisfies it, we show it is simple. From the assumption $\varphi \notin T$ we obtain $T \in \Psi$ such that $I(\varphi) \subseteq T$. If $T \cup \{\varphi\}$ would we a consistent set then (by the MCP) there would be a simple theory extending it, but the defining condition prohibits simple theories to contain both φ and $I(\varphi)$.

Finally, we can see (6.3) as a restriction of the LIL to simple theories and tuples of size one. Therefore, clearly, if $\{\Psi_n\}_{n\in\omega}$ is an inconsistency sequence without parameters, then the family Ψ_1 defines simple theories. Similarly, it is also a restriction of the dLIL to simple theories. □

Lemma 6.39. *Let*L *be a protonegational logic protonegational logic with the MCP, an antitheorem, and small type.Then, if* L *has definable simple theories, then it also has definable simple filters.*

Proof. We will argue using the characterization from Lemma 6.38. Let $\langle \boldsymbol{A}, F\rangle$ be a simple model of L. Take an element $a \in A$ and use Proposition 6.21 to obtain a $|Var_{\mathcal{L}}|$-small simple submatrix $\langle \boldsymbol{B}, G\rangle$ of $\langle \boldsymbol{A}, F\rangle$ such that $a \in B$. There is a surjective $e\colon \boldsymbol{Fm_{\mathcal{L}}} \to \boldsymbol{B}$ which is strict between $T = e^{-1}[G]$ and G. By protonegationality T is simple (Lemma 6.17). Choose a formula φ such that $e(\varphi) = a$, then

$$\begin{aligned} a \notin F \quad &\Longleftrightarrow \quad a \notin G \quad \Longleftrightarrow \quad \varphi \notin T \\ &\Longleftrightarrow \quad I(\varphi) \subseteq T \text{ for some } I \in \Psi \\ &\Longleftrightarrow \quad I^{\boldsymbol{A}}(a) \subseteq G \text{ for some } I \in \Psi \\ &\Longleftrightarrow \quad I^{\boldsymbol{A}}(a) \subseteq F \text{ for some } I \in \Psi, \end{aligned}$$

where the third equivalence is due to (6.3). □

Theorem 6.40. *For every protonegational logic* L *with the MCP and an antitheorem, the following are equivalent:*

(i) L *has definable simple theories.*
(ii) $h^{-1}[T] \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$ *for every* $T \in \mathrm{MaxTh}\,\mathrm{L}$ *and* $h\colon \boldsymbol{A} \to \boldsymbol{Fm_{\mathcal{L}}}$.
(iii) *If* $\langle Fm_{\mathcal{L}}, T\rangle$ *is a simple model of* L *then so is its every submatrix.*

Moreover, if L *has a small type, then we can add:*

(iv) L *has definable simple filters.*
(v) $h^{-1}[F] \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ *for every* $F \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$ *and* $h\colon \boldsymbol{A} \to \boldsymbol{B}$.
(vi) L *is simple submatrix closed.*

Proof. (i)→(ii): Let $a \in A$ be given then

$$\begin{aligned} a \notin h^{-1}[T] \quad &\Longleftrightarrow \quad h(a) \notin T \\ &\Longleftrightarrow \quad I(h(a)) \subseteq T \text{ for some } I \in \Psi \\ &\Longleftrightarrow \quad I^{\boldsymbol{A}}(a) \subseteq h^{-1}[T] \text{ for some } I \in \Psi. \end{aligned}$$

It easily follows that $h^{-1}[T]$ is simple: maximality is due to the fact that $\{\varphi\} \cup I(\varphi)$ is an antitheorem (consequence of the MCP), thus also $\{a\} \cup I^{\boldsymbol{A}}(a)$ generates the trivial filter.

(ii)→(iii): Suppose $\langle \boldsymbol{A}, F\rangle \leq \langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle$, where $T \in \mathrm{MaxTh}\,\mathrm{L}$. Taking the identity injection $i\colon \boldsymbol{A} \hookrightarrow \boldsymbol{Fm_{\mathcal{L}}}$ gives the result.

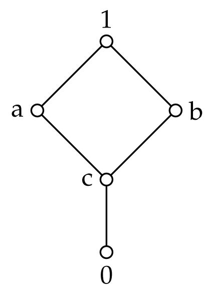
(iii)→(i): We will use the characterization from Lemma 6.38. Define

$$\Psi = \{T \cap Fm_{\mathcal{L}}(\{p\}) \mid T \text{ is simple and } T, p \vdash \emptyset\}.$$

For the right-to-left direction of (6.3) observe that for every $I \in \Psi$ still $I, p \vdash \emptyset$: this follows, since $\langle Fm_{\mathcal{L}}(\{p\}), I\rangle$ is simple by the assumption, and that non-trivial models are closed under submatrices (Lemma 6.34). For the other direction let us have a simple T and $\varphi \notin T$. Protonegationality implies that for any surjective σ such that $\sigma(p) = \varphi$ we have that $S = \sigma^{-1}[T]$ is a simple theory (Lemma 6.17) and $p \notin S$. Thus, $I = S \cap Fm_{\mathcal{L}}(\{p\}) \in \Psi$ and $\sigma[I] = I(\varphi) \subseteq T$, which is what we wanted.

Finally, the directions (iv)→(v)→(vi) can be proved in the same way as (i)→(ii)→(iii). The implication (vi)→(iii) is trivial, and the remaining one, (i)→(iv), was proved in the previous lemma. □

Example 6.41. It is known that the Full Lambek calculus with weakening $\mathrm{FL_w}$ does not have the filter extension property (or equivalently the LDDT). We show that even more is true: it does not enjoy the SCC (in particular, it does not have the LIL and definable simple theories). We present a counterexample taken from [53, pp. 202–203]. The $\mathrm{FL_w}$-algebra presented therein (see Figure 6.1), call it $\boldsymbol{A}$, is simple, i.e. $\langle \boldsymbol{A}, \{1\}\rangle \in \mathbf{Mod}^{*}_{\mathrm{Max}}\,\mathrm{FL_w}$, but it has a subalgebra $\boldsymbol{B}$ with universe $\{1, a, 0\}$, which is equivalent to the 3-element Heyting chain. Thus, clearly $\langle \boldsymbol{B}, \{1\}\rangle$ is not simple.



&	1	a	b	c	0
1	1	a	b	c	0
a	a	a	0	0	0
b	b	c	b	c	0
c	c	c	0	0	0
0	0	0	0	0	0

Figure 6.1: Description of the $\mathrm{FL_w}$-algebra $\boldsymbol{A}$

Next, we are going to present a characterization result for the global inconsistency lemma. It is a stronger version of [82, Theorem 3.7]—the result is proved for compact and protonegational logics (instead of finitary and protoalgebraic), although the proof itself is very similar.

A join semilattice $\langle S, \vee, 0\rangle$ with a least element 0 is called *dually pseudo-complemented* if it has a greatest element 1 and for every $a \in S$ there is a smallest b such that $a \vee b = 1$. We denote such a b as a^*.

Theorem 6.42. *For every compact protonegational logic* L *with a small type,*[5] *the following are equivalent:*

(i) L *enjoys the global inconsistency lemma.*

(ii) *For every algebra* $\boldsymbol{A}$*, the join semilattice of finitely generated* L*-filters is dually pseudo-complemented.*

(iii) *The join semilattice of finitely generated theories of* L *is dually pseudo-complemented.*

Proof. (i)→(ii): Note that by compactness we may assume that the GIL is witnessed by an inconsistency sequence consisting only of finite sets. By the assumptions we know that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}$ is compact (see Theorem 6.12). In particular, A is finitely generated filter. Theorem 6.32 clearly implies that $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\Psi^{\boldsymbol{A}}_n(a_1,\dots,a_n))$ can be chosen as $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\{a_1,\dots,a_n\})^*$.

(ii)→(iii): trivial.

(iii)→(i): Note that by compactness $Fm_{\boldsymbol{\mathcal{L}}}$ is the greatest finitely generated theory. We are going to build an inconsistency sequence for L: let Δ be a finite set of generators for $\mathrm{Th}_{\mathrm{L}}(\{p_1,\dots,p_n\})^*$ and define $\Psi_n(p_1,\dots,p_n)$ as $\sigma[\Delta]$ where σ is the substitution fixing $p_1,\dots,p_n$ and sending the remaining variables to p_1. Observe that for every theory T (non necessarily finitely generated), we have

[5] Instead of a small type we can assume that the logic has enough variables and a $\mathit{Var}_{\boldsymbol{\mathcal{L}}}$-small protonegation.

$$Fm_{\mathcal{L}} = T \vee \mathrm{Th_L}(\{p_1, \ldots, p_n\}) \iff \Delta \subseteq T, \tag{6.4}$$

where the left-to-right direction is due to compactness: indeed by compactness there is finite $\{\varphi_1, \ldots, \varphi_m\} \subseteq T$ such that

$$Fm_{\mathcal{L}} = \mathrm{Th_L}(\{\varphi_1, \ldots, \varphi_m, p_1, \ldots, p_n\}).$$

Therefore $\Delta \subseteq \mathrm{Th_L}(\{p_1, \ldots, p_n\})^* \subseteq \mathrm{Th_L}(\{\varphi_1, \ldots, \varphi_m\}) \subseteq T$.

Let us take $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm_{\mathcal{L}}$ and let σ be a surjective substitution sending p_i to φ_i and all other variables from Γ to α_1. Note that $\sigma[\Gamma] = \Psi_n(\varphi_1, \ldots, \varphi_n)$. We can obtain the result as follows:

$$\begin{aligned}
\Gamma, \varphi_1, \ldots, \varphi_n \vdash \emptyset \quad &\iff \quad \mathrm{Th_L}(\Gamma), \varphi_1, \ldots, \varphi_n \vdash \emptyset \\
&\iff \quad \sigma^{-1}[\mathrm{Th_L}(\Gamma)], p_1, \ldots, p_n \vdash \emptyset \\
&\iff \quad \Delta \subseteq \sigma^{-1}[\mathrm{Th_L}(\Gamma)] \\
&\iff \quad \Psi_n(\varphi_1, \ldots, \varphi_n) = \sigma[\Delta] \subseteq \mathrm{Th_L}(\Gamma) \\
&\iff \quad \Gamma \vdash \Psi_n(\varphi_1, \ldots, \varphi_n),
\end{aligned}$$

where the third equivalence is due to (6.4) and the fourth by the surjective substitution swapping for antitheorems (see the characterization theorem for protonegational logics—Theorem 6.9). □

Note that an interesting feature of this characterization is that in principle the theorem applies also to infinitary logics while we still speak only about finitely generated theories (filters). Indeed, in the next example we show that there is an infinitary logic falling under the scope of the theorem:

Example 6.43. We define L as the logic of all finite Heyting algebras, that is $\mathrm{L} = \models_{\mathbb{HA}_{fin}}$, where

$$\mathbb{HA}_{fin} = \{\langle \boldsymbol{A}, \{1\} \rangle \mid \boldsymbol{A} \text{ is a finite Heyting algebra}\}.$$

We first show that L is compact: clearly, $\mathrm{IL} \leq \mathrm{L} \leq \mathrm{CL}$. And since IL and CL has the same simple theories (see Example 6.5), the same is true also for L. In particular, it easily follows that all these logics has the same antitheorems. Consequently, L is compact by the compactness of CL.

It also easy to verify that it enjoys the same GDDT as IL, i.e.

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \quad \iff \quad \Gamma \vdash_{\mathrm{L}} \varphi \to \psi.$$

Consequently, it also enjoys the GIL in the expected form.

Thus, it remains to show that L is indeed infinitary. To demonstrate this fact we use the Jankov's characteristic formulas (see [10]): For every finite subdirectly irreducible Heyting algebra $\boldsymbol{A}$ there is a formula $\chi(\boldsymbol{A})$ such that for every $\mathbf{B} = \langle \boldsymbol{B}, \{1\}\rangle \in \mathbb{HA}_{fin}$ we have

$$\not\models_{\mathbf{B}} \chi(\boldsymbol{A}) \quad\Longleftrightarrow\quad \boldsymbol{A} \in \mathbf{SH}(\boldsymbol{B}). \tag{6.5}$$

Since up to isomorphism there are only countably many finite subdirectly irreducible algebras, we can put

$$\Gamma = \{\chi(\boldsymbol{A}) \to p \mid \boldsymbol{A} \text{ a finite subdirectly irreducible Heyting algebra}\}$$

and assume that each $\chi(\boldsymbol{A})$ is written using unique set of variables distinct from p. We show that $\Gamma \rhd p$ is a proper infinitary rule of L:

Take a finite Heyting algebra $\boldsymbol{B}$ and assume that for a $\boldsymbol{B}$-evaluation v we have $v[\Gamma] \subseteq \{1\}$. Clearly, for any finite $\boldsymbol{A} \in \mathsf{HA}$ which has more elements than $\boldsymbol{B}$ we obtain by (6.5) that $\models_{\mathbf{B}} \chi(\boldsymbol{A})$, i.e. $v(\chi(\boldsymbol{A})) = 1$ and consequently also $v(p) = 1$.

On the other hand let Γ' be a finite subset of Γ. That is, Γ' is a collection of formulas induced by $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$. Let $\boldsymbol{A}$ be a direct product of these algebras extended by a new top element. That is $\boldsymbol{A}$ is finite, $\boldsymbol{A} \in \mathsf{HA}_{\mathrm{SI}}$, and every $\boldsymbol{A}_i \in \mathbf{SH}(\boldsymbol{A})$. Therefore, by (6.5), we know that $\not\models_{\mathbf{A}} \chi(\boldsymbol{A}_i)$ for every $\boldsymbol{A}_i$. Therefore, we obtain for every i an $\boldsymbol{A}$-evaluation v_i witnessing $\not\models_{\mathbf{A}} \chi(\boldsymbol{A}_i)$. Then, capitalizing on the fact that each $\chi(\boldsymbol{A}_i)$ is written using a unique set of variables, we can easily build an $\boldsymbol{A}$-evaluation v, which agrees with v_i on variables from $\chi(\boldsymbol{A}_i)$ and such that $v(p)$ is the coatom of $\boldsymbol{A}$. Clearly, this evaluation witnesses that $\Gamma' \not\models_{\mathbf{A}} p$ (it must be the case that $v(\chi(\boldsymbol{A}_i)) \leq v(p)$ thus $v[\Gamma'] \subseteq \{1\}$)—which is what we wanted.

Interestingly, inconsistency lemmas also have a natural dual form, which we shall investigate in the remaining part of this section.

Definition 6.44. A logic L has the *dual parametrized local inconsistency lemma*, dPLIL, if there is a family of sets of formulas $\Psi(p, \bar{r})$ such that for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$

$$\text{for every } \bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}} \text{ and every } I(p, \bar{r}) \in \Psi \qquad \Gamma, I(\varphi, \bar{\delta}) \vdash_{\mathrm{L}} \emptyset \quad\Longleftrightarrow\quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Moreover, if Ψ has no parameters, then L is said to have the *dual local inconsistency lemma*, dLIL, and finally the *dual global inconsistency lemma*, dGIL, if additionally Ψ is a singleton (in this case we again identify Ψ with the unique $I \in \Psi$).

The dual version of global inconsistency was already introduced in [82] though it was presented in a different way: it was defined as one property GIL+dGIL. Moreover, the dGIL was postulated for tuples of formulas (both GIL and dGIL witnessed with the same inconsistency sequence). First, we explain that our apparently weaker definition of dual inconsistency lemmas has the same strength. Then, we show that it can be equivalently formulated by means of a generalized *law of excluded middle*. Finally, we explain some interactions between inconsistency lemmas and its duals. Moreover, we will argue that the dual versions of inconsistency lemmas are interesting *per se*, which justifies our decision to separate the original notion into two.

Proposition 6.45. *Suppose* L *has the* dPLIL *witnessed by a family* Ψ *and let* $\{\Psi_n(p_1,\dots,p_n,\bar{r})\}_{n\in\omega}$ *be an inconsistency sequence. Then, for every* $n \in \omega$ *and every* $\Gamma \cup \{\varphi_1,\dots,\varphi_n\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$*:*

$$\text{for every } \bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}} \text{ and every } I(p_1,\dots,p_n,\bar{r}) \in \Psi_n \quad \Gamma, I(\varphi_1,\dots,\varphi_n,\bar{\delta}) \vdash_{\mathrm{L}} \emptyset \iff \Gamma \vdash_{\mathrm{L}} \varphi_1,\dots,\varphi_n.$$

Proof. The direction from right to left follows directly from the PLIL (resp. LIL, GIL). For the other one suppose that $\Gamma \nvdash_{\mathrm{L}} \varphi_i$ for some $i \leq n$. By dPLIL there is $J \in \Psi$ and $\bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}}$ such that $\Gamma, J(\varphi_i,\bar{\delta}) \nvdash_{\mathrm{L}} \emptyset$ and therefore also $\Gamma, J(\varphi_i,\bar{\delta}), \varphi_i \vdash_{\mathrm{L}} \emptyset$ and $\Gamma, J(\varphi_i,\bar{\delta}), \varphi_1,\dots,\varphi_n \vdash_{\mathrm{L}} \emptyset$. Consequently, there is $I \in \Psi_n$ and $\bar{\xi} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}}$ such that $\Gamma, J(\varphi_i,\bar{\delta}) \vdash_{\mathrm{L}} I(\varphi_1,\dots,\varphi_n,\bar{\xi})$. Consequently, $\Gamma, I(\varphi_1,\dots,\varphi_n,\bar{\xi}) \nvdash_{\mathrm{L}} \emptyset$, which is what we wanted. □

In other words if a logic has some form of inconsistency lemma and the dPLIL then it always has the same form of dual inconsistency lemma w.r.t. Ψ_1. Moreover, the duals can be extended to tuple of formulas.

It should not be surprising that the dual inconsistency lemma of classical logic is closely related to the *law of excluded middle*, which can be formulated as the following meta-rule

$$\frac{\Gamma, \varphi \vdash_{\mathrm{CL}} \psi \qquad \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \psi}{\Gamma \vdash_{\mathrm{CL}} \psi}.$$

and to the fact that classical logic is *involutive*, i.e. $\varphi \dashv\vdash_{\mathrm{CL}} \neg\neg\varphi$.

In fact, it is easy (apply the previous proposition) to show that an superintuitionistic logic satisfies the law of excluded middle of classical logic if and only if it has dPLIL if and only if it is a classical logic if and only if it proves $\neg\neg\varphi \to \varphi$—cf. Example 6.27. We will describe a general version of these properties and see that they are equivalent to dPLIL.

Definition 6.46. We say that L satisfies the *law of excluded middle*, LEM, provided there is is family of sets of formulas $\Psi(p,\bar{r})$ (possibly with parameters) such that for every $I \in \Psi$ we have $p, I(p,\bar{r}) \vdash_{\mathrm{L}} \emptyset$ and for every $\Gamma \cup \{\varphi,\psi\} \subseteq Fm_{\mathcal{L}}$:

$$\frac{\Gamma,\varphi \vdash_{\mathrm{L}} \psi \qquad \Gamma, I(\varphi,\bar{\delta}) \vdash_{\mathrm{L}} \psi \text{ for every } \bar{\delta} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}} \text{ and every } I \in \Psi}{\Gamma \vdash_{\mathrm{L}} \psi}.$$

Note that we can see Ψ as a collection of explosive negations which together satisfy the meta-rule law of excluded middle.

Proposition 6.47. *A logic has the* dPLIL *w.r.t. to* Ψ *if and only if it has the* LEM *w.r.t.* Ψ.

Proof. The left-to-right direction: it is clear that $p, I(p,\bar{r})$ is an antitheorem. Suppose that the premises of the meta-rule hold, by the dPLIL it is enough to show that for every $J \in \Psi$ and every $\bar{\xi} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}$ we have $\Gamma, J(\psi,\bar{\xi}) \vdash \emptyset$, but clearly (the right-hand side premises and dPLIL) for every $I \in \Psi$ and every $\bar{\delta} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}$ we have that

$$\Gamma, J(\psi,\bar{\xi}), I(\varphi,\bar{\delta}) \vdash \emptyset.$$

Thus, again by the dPLIL $\Gamma, J(\psi,\bar{\xi}) \vdash \varphi$, therefore the left-hand side premise and the dPLIL give $\Gamma, J(\psi,\bar{\xi}) \vdash \psi$ and consequently $\Gamma, J(\psi,\bar{\xi})$ is an antitheorem. The other direction follows immediately from the meta-rule by taking $\psi = \varphi$. □

Here are some non-trivial examples:

Example 6.48. From the previous proposition we can easily infer that classical logic CL is the only superintuitionistic logic with dPLIL and it has the expected form

$$\Gamma, \neg\varphi \vdash_{\mathrm{CL}} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \varphi.$$

Example 6.49 ([82, §6]). In the k-valued Łukasiewicz logic $Ł_k$ we have

$$\Gamma, \neg\varphi^k \vdash_{Ł_k} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{Ł_k} \varphi.$$

Example 6.50. Another interesting example of both GIL and dGIL arises in extensions of the logic of left-continuous t-norms, MTL [43, 62], expanded with the Monteiro–Baaz connective Δ. These extensions are denoted as L_Δ (see [21, Section 2.2.1]). It is easy to prove that they satisfy

$$\Gamma, \varphi \vdash \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash \neg\Delta\varphi, \text{ and}$$

$$\Gamma, \neg\Delta\varphi \vdash \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash \varphi.$$

Example 6.51. Using a simple semantical argument we can show that the infinitary Łukasiewicz logic $Ł_\infty$ has the dLIL in the following form

$$\varGamma, \neg\varphi^n \vdash_{Ł_\infty} \emptyset \text{ for all } n \in \omega \quad \iff \quad \varGamma \vdash_{Ł_\infty} \varphi.$$

Thus, it has dLIL for $\varPsi(p) = \{\{\neg p^n\} \mid n \in \omega\}$.

Of course, by the last proposition, all the dual inconsistency lemmas can be formulated as the law of excluded middle; for example:

Example 6.52. The infinitary Łukasiewicz has the LEM in the form

$$\frac{\varGamma, \varphi \vdash_{Ł_\infty} \psi \qquad \varGamma, \neg\varphi^n \vdash_{Ł_\infty} \psi \text{ for every } n \in \omega}{\varGamma \vdash_{Ł_\infty} \psi}.$$

Note that the same is true for its finitary companion Ł when we restrict ourselves to finite $\varGamma s$.

Next we will see that having inconsistency lemma and its dual is a stronger condition than having deduction theorem.

Proposition 6.53. *Suppose* L *has the* dPLIL *and the* GIL *(resp. LIL) w.r.t.* $\{\varPsi_i\}_{i\in\omega}$ *such that all sets in* $\varPsi_1$ *are finite. Then, it has the* GDDT *(resp.* LDDT*).*

Proof. We prove only the more general case of LIL. Recall that by Proposition 6.45 L has dLIL w.r.t. $\varPsi_n$.

$$\mathcal{F} = \{f : \varPsi_1 \to \bigcup_{n\in\omega} \varPsi_n \mid f(I) \in \varPsi_{|I|+1} \text{ for every } I \in \varPsi_1\}$$

We define the family $\varPsi \subseteq P(Fm_{\mathcal{L}}(p,q))$ to contain a set I_f for every $f \in \mathcal{F}$ defined as the union of sets

$$f(I)(p, \chi(q)_1, \dots, \chi(q)_n),$$

where $I = \{\chi(q)_1, \dots, \chi(q)_n\}$ and $f(I) \in \varPsi_{n+1}$.

We show that $\varPsi$ is a deduction set. Suppose $\varGamma, \varphi \vdash \psi$, then by the LIL for every $I \in \varPsi_1$ we have $\varGamma, \varphi, I(\psi) \vdash \emptyset$. Thus, again by the LIL, for every $I = \{\psi(q)_1, \dots, \psi(q)_n\}$, there is $f(I) \in \varPsi_{n+1}$ such that $\varGamma \vdash f(I)(\varphi, \chi_1(\psi), \dots, \chi_n(\psi))$. Therefore $\varGamma \vdash I_f(\varphi, \psi)$.

For the other direction, suppose $\varGamma \vdash I_f(\varphi, \psi)$ for some $f \in \mathcal{F}$. We obtain for every $I \in \varPsi_1$ that $\varGamma, \varphi, I(\psi) \vdash \emptyset$. Thus, dLIL implies that $\varGamma, \varphi \vdash \psi$. □

Note that the previous proof was constructive whereas for the PLIL we obtain only an existential proof:

Proposition 6.54. *Let* L *be a logic with the* dPLIL *and with an inconsistency sequence* $\{\Psi_n\}_{n\in\omega}$*, where all sets in* Ψ_1 *are finite. Then,* L *is protoalgebraic (or equivalently has the PLDDT).*

Proof. We prove that L has a protoimplication set. Since every logic satisfies $p \vdash p$ for every $\{\chi(q,\bar{r})_1,\dots,\chi(q,\bar{r})_n\} = I(q,\bar{r}) \in \Psi_1$ and every tuple of formulas $\bar{\delta}$ PLIL implies $p, I(p,\bar{\delta}) \vdash \emptyset$, thus for every such I, again by PLIL , we have $\vdash J(p,\chi(q,\bar{\delta})_1,\dots,\chi(q,\bar{\delta})_n,\bar{\xi})$, for some tuple of formulas $\bar{\xi}$. Define Γ as an union of all sets $J(p,\chi(q,\bar{\delta})_1,\dots,\chi(q,\bar{\delta})_n,\bar{\xi})$ obtained this way. Let σ be a substitution sending p to p and every other variable to q and define $\Rightarrow(p,q) = \sigma[\Gamma]$. By structurality $\vdash \Rightarrow(p,p)$. From the definition of Γ and the PLIL , it follows that for every $I \in \Psi_1$ and every $\bar{\delta}$ it is the case that $p,\Gamma,I(q,\bar{\delta}) \vdash \emptyset$, thus by dPLIL, we have $p,\Gamma \vdash q$. One application of structurality thus concludes the proof that $\Rightarrow$ satisfies modus ponens. □

In fact the constructive aspect of Proposition 6.53 can be useful to discover new deduction theorems, e.g. it allowed us to describe the LDDT for Ł$_\infty$, which as far as we known is the only deduction-detachment theorem to be found in the literature where the validity of $\Gamma,\varphi \vdash_{Ł_\infty} \psi$ is *necessarily* witnessed by infinitely-many inferences from Γ.

Theorem 6.55. *The infinitary Łukasiewicz logic* Ł$_\infty$ *enjoys the* LDDT *in the form*

$$\Gamma,\varphi \vdash_{Ł_\infty} \psi \iff \Gamma \vdash_{Ł_\infty} f(n)(\varphi \to \psi^n) \text{ for some } f:\omega\to\omega \text{ and every } n\in\omega.$$

That is, it enjoys the LDDT *with a family of DD sets* $\Psi(p,q)$ *containing a set of formulas* $I_f = \{f(n)(p \to q^n) \mid n \in \omega\}$ *for every* $f\colon \omega \to \omega$.

Proof. By the example 6.30 we known that it has the LIL w.r.t.

$$\Psi_n = \{\neg(p_1 \,\&\, \dots \,\&\, p_n)^k \mid k \in \omega\}$$

and by the example 6.51 it has dLIL. Thus, Proposition 6.53 implies that it has the LDDT with Ψ containing

$$I_f = \{\neg(p \,\&\, \neg q^n)^{f(n)} \mid n \in \omega\} \text{ for every } f:\omega\to\omega.$$

But in Ł$_\infty$ the formula $\neg(p \,\&\, \neg q^n)^{f(n)}$ is equivalent to $f(n)(p \to q^n)$, which we deem more suitable for the formulation of the theorem because of its implicational form. □

Theorem 6.56. *The infinitary Łukasiewicz logic* $Ł_\infty$ *does not enjoy the* LDDT *with respect to any family of* finite *sets.*

Proof. Suppose that $Ł_\infty$ enjoys the LDDT with respect to some family of finite sets $\Psi(p,q)$. We first show that this implies that $Ł_\infty$ also enjoys the same LDDT as the finitary Łukasiewicz logic Ł, i.e. the LDDT with respect to $\{\{p^n \to q\} \mid n \in \omega\}$.

Clearly, $\Gamma \vdash_{Ł_\infty} \varphi^n \to \psi$ implies $\Gamma, \varphi \vdash_{Ł_\infty} \psi$ (because $\varphi, \vdash_{Ł_\infty} \varphi^n$). Now suppose that $\Gamma, \varphi \vdash_{Ł_\infty} \psi$. Then, $\Gamma \vdash_{Ł_\infty} I(\varphi, \psi)$ for some $I \in \Psi(p,q)$ by the LDDT. But $\varphi, I(\varphi,\psi) \vdash_{Ł_\infty} \psi$ again by the LDDT, therefore $\varphi, I(\varphi,\psi) \vdash_{Ł} \psi$ by the finiteness of I. The LDDT for finitary Łukasiewicz logic then implies that $I(\varphi,\psi) \vdash_{Ł} \varphi^n \to \psi$ for some $n \in \omega$, hence $I(\varphi,\psi) \vdash_{Ł_\infty} \varphi^n \to \psi$ and thus also $\Gamma \vdash_{Ł_\infty} \varphi^n \to \psi$.

It remains to find a counterexample witnessing that $Ł_\infty$ does not enjoy the LDDT with respect to the family of singletons $\{\{p^n \to q\} \mid n \in \omega\}$. Let

$$\Gamma = \{p^{i+1} \to (q_i \leftrightarrow q_{i+1}), \neg q_0 \to q_i^i \mid i \in \omega\}.$$

Observe that $\Gamma, p \vdash_{Ł_\infty} q_0$, since $p, \Gamma \vdash_{Ł_\infty} q_i \leftrightarrow q_{i+1}$ for each $i \in \omega$ and it is easy to check that $\{\neg\varphi \to \varphi^n \mid n \in \omega\} \vdash_{Ł_\infty} \varphi$.

Now take an arbitrary $n \in \omega$. We prove that $\Gamma \nvdash_{Ł_\infty} p^n \to q_0$. Given some $\varepsilon \in (0,1)$, let us define a valuation v such that

$$v(p) = 1 - \frac{\varepsilon}{n+1},$$
$$v(q_i) = \min\left(1, 1 - \varepsilon + \frac{1 + 2 + \cdots + i}{n+1}\,\varepsilon\right).$$

In particular, $v(p^n) = 1 - \frac{n\varepsilon}{n+1} \nleq 1 - \varepsilon = v(q_0)$, hence $v(p^n \to q) < 1$. Moreover, $v(q_{i+1}) - v(q_i) \leq \frac{i+1}{n+1}\varepsilon = 1 - v(p^{i+1})$. It follows that $v(p^{i+1}) \leq v(q_i \leftrightarrow q_{i+1})$ and $v(p^{i+1} \to (q_i \leftrightarrow q_{i+1})) = 1$. Finally, notice that $v(q_i) < 1$ only when $1 + 2 + \cdots + i = \frac{i(i+1)}{2} \leq n$, i.e. only when $i \leq \sqrt{2n}$. But then $v(q_i^i) \geq 1 - i\varepsilon \geq 1 - \sqrt{2n}\varepsilon$. If we now choose ε so that $\varepsilon \leq 1 - \sqrt{2n}\varepsilon$, e.g. $\varepsilon = \frac{1}{2\sqrt{2n}}$, we have $v(\neg q_0) = \varepsilon \leq 1 - \sqrt{2n}\varepsilon \leq v(q_i^i)$ for each $i \in \omega$, i.e. $v(\neg q_0 \to q_i^i) = 1$. The valuation v therefore validates each formula in Γ but not $p^n \to q_0$. □

We know that inconsistency lemma together with its dual imply the deduction-detachment theorem. The converse is not true in general (every superintuitionistic logic besides classical is an example—see Example 6.48), although every logic with deduction-detachment theorem enjoys inconsistency lemma (provided it has an antitheorem):

Proposition 6.57. *Every logic* L *with an antitheorem and the* PLDDT *(resp.* LDDT, *GDDT) has the* PLIL *(resp.* LIL, *GIL). In the case of* PLDDT *assume* L *has a family of DD sets* $\varPsi$ *with* $|\varPsi| \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$.

Proof. We show how to build an inconsistency sequence out of DDT and an antitheorem. Let $\mathcal{A}(p)$ be an antitheorem in variable p. Let $\varPsi(p, q, \bar{r})$ be a family of DD sets for L. Note that the additional assumption allows us to assume that every $I \neq J \in \varPsi$ are written in disjoint sets of parameters. For $f\colon \mathcal{A} \to \varPsi$ define $I_f(p, \bar{r}) = \bigcup_{\alpha \in \mathcal{A}} f(\alpha)(p, \alpha, \bar{r})$. And put

$$\varPsi_1 = \{I_f \subseteq Fm_{\boldsymbol{\mathcal{L}}} \mid f\colon \mathcal{A} \to \varPsi\}.$$

Now we show that $\varPsi_1$ behaves as desired

$$\begin{aligned} \varGamma, \varphi \vdash \emptyset \quad &\Longleftrightarrow \quad \varGamma, \varphi \vdash \mathcal{A}(\varphi) \\ &\Longleftrightarrow \quad \text{for every } \alpha \in \mathcal{A}(p) \text{ there is some } I_\alpha \in \varPsi \text{ and } \bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{\mathit{Var}_{\boldsymbol{\mathcal{L}}}} \\ &\qquad \text{such that } \varGamma \vdash I_\alpha(\varphi, \alpha(\varphi), \bar{\delta}) \\ &\Longleftrightarrow \quad \varGamma \vdash I(\varphi, \bar{\xi}) \text{ for some } I \in \varPsi_1 \text{ and } \bar{\xi} \in Fm_{\boldsymbol{\mathcal{L}}}^{\mathit{Var}_{\boldsymbol{\mathcal{L}}}}. \end{aligned}$$

Analogously we can define all $\varPsi_n$ capitalizing on the fact that we can inductively extend the deduction-detachment theorem to operate on tuples of formulas instead of just on a single one. □

The next theorem and the following results explain the connection of the dual inconsistency lemma (resp. the law of excluded middle) to simple theories:

Theorem 6.58. *For every protonegational logic* L *with an antitheorem, the following are equivalent:*

(i) L *has the* LEM *and the* MCP.
(ii) L *has the* dPLIL *and the* MCP.
(iii) L *has the* SEP.

Proof. (i)↔(ii): Proposition 6.47.

(ii)→(iii): Suppose $T \nvdash_{\mathrm{L}} \varphi$ by the dual PLIL there is $I(p, \bar{r}) \in \varPsi$ and $\bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{\mathit{Var}_{\boldsymbol{\mathcal{L}}}}$ such that $T, I(\varphi, \bar{\delta}) \nvdash_{\mathrm{L}} \emptyset$. From the MCP it follows that $T, I(\varphi, \bar{\delta})$ extends to a simple theory S. Moreover S cannot contain the formula φ—otherwise S would be inconsistent, because $I(\varphi, \bar{\delta}) \subseteq S$.

(iii)→(ii): Define $\varPsi = \{T \in \mathrm{Th}\,\mathrm{L} \mid p, T \vdash_{\mathrm{L}} \emptyset\}$. The right-to-left direction of the definition of dPLIL holds by structurality. For the other one assume $\varGamma \nvdash_{\mathrm{L}} \varphi$ then by the SEP there is a simple theory S extending $\varGamma$ not containing φ. Let σ be a surjective substitution sending variable p to φ. From $S, \varphi \vdash_{\mathrm{L}}$

$\emptyset$ it follows that $\sigma^{-1}[S], p \vdash \emptyset$ (Theorem 6.9 point (iv)), thus $\sigma^{-1}[S] \in \Psi$ and because $\sigma[\sigma^{-1}[S]] = S$ we obtain $\Gamma, \sigma[\sigma^{-1}[S]] \nvdash_{\mathrm{L}} \emptyset$, which is what we wanted. Moreover, it should be clear that the SEP implies the MCP. $\square$

Note that in Theorem 5.12 we proved the Lindenbaum lemma for countably axiomatizable logics with disjunction (i.e. these properties entail the PEP and the IPEP). The previous theorem, on the other hand, postulates some conditions for a logic to have a stronger extension property, the SEP (and consequently the CIPEP—cf. Proposition 6.24). Thus, we can view this result as another form of the Lindenbaum lemma that goes beyond finitary logics. Interestingly, considering again the assumption of countable axiomatization and finite antitheorem we can avoid the assumption of the MCP (the argument is analogous to the one of Theorem 5.12) and obtain an even stronger formulation of the Lindenbaum lemma:

Theorem 6.59. *Let* L *be countably axiomatizable logic with a finite antitheorem, and the* dPLIL. *Then,* L *enjoys the* SEP.

Proof. We first prove that L enjoys the MCP. Let Ψ be a family witnessing the dPLIL and suppose $\Gamma \nvdash \varphi$. We again enumerate all the rules in the axiomatic systems as $\Lambda_i \rhd \varphi_i$ and define a growing chain of sets of formulas Γ_i such that $\Gamma_i \nvdash \emptyset$. We put $\Gamma_0 = \mathrm{Th}_{\mathrm{L}}(\Gamma)$ and for the induction step we define:

- $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_i\}$ provided $\Gamma_i, \varphi_i \nvdash \emptyset$.
- On the other hand, if $\Gamma_i, \varphi_i \vdash \emptyset$, then for some $\lambda \in \Lambda_i$ we obtain that $\Gamma_i \nvdash \lambda$ (because by the induction assumption $\Gamma_i \nvdash \emptyset$). Then, by the dPLIL there is some $I \in \Psi$ and some $\bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}}$ such that $\Gamma_i \cup I(\lambda, \bar{\delta})$ is not an antitheorem. We set $\Gamma_{i+1} = \Gamma_i \cup I(\lambda, \bar{\delta})$.

To prove that $\bigcup \Gamma_i$ is indeed a simple theory is almost the same as in Theorem 5.12. We believe the reader can carry out the details.

We conclude the proof observing that the assumption of protonegationality is not necessary in (ii)→(iii) in the previous theorem. $\square$

Lemma 6.60. *Let* L *be a compact protonegational with a small type. Then,* dLIL *transfers to all algebras. That is, if the family* Ψ *witnesses* dLIL *and* $\boldsymbol{A}$ *is an algebra, then for every* $X \cup \{a\} \subseteq A$*:*

$$\text{for every } I(p) \in \Psi \text{ we have } \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X \cup I^{\boldsymbol{A}}(a)) = A \quad \Longleftrightarrow \quad a \in \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X).$$

Proof. By the standard method (cf. Proposition 2.24), we obtain that the dLIL transfers to all natural extensions: we leave the details as an exercise for the reader. We only note that we essentially used the fact that L is compact and has a small type, which implies:

- L and has enough variables.
- We can assume that Ψ is a family of finite sets.
- $|\Psi| \leq |Var_{\mathcal{L}}|$.

Let κ be a suitable cardinal such that we have a surjective $h\colon \boldsymbol{Fm}_{\mathcal{L}}(\kappa) \twoheadrightarrow \boldsymbol{A}$. The right-to-left direction follows e.g. from the trivial filter generation result (Proposition 6.11), because $I(p) \cup \{p\}$ is an antitheorem.

For the other direction assume $a \notin \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)$ and pick an arbitrary $\varphi \in Fm_{\mathcal{L}}(\kappa)$ such that $h(\varphi) = a$. Then, $T := h^{-1}[\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X)]$ is an L^{κ}-theory such that $\varphi \notin T$. Thus, for some $I \in \Psi$, we have $T, I(\varphi) \nvdash_{\mathrm{L}^{\kappa}} \emptyset$. Then, since L is compact, also L^{κ} is compact (Proposition 2.24), and consequently L^{κ} has the MCP (Proposition 6.4). Therefore, there is a simple L^{κ}-theory S containing $T \cup I(\varphi)$. Finally, since by Lemma 6.18 we know that L^{κ} is protonegational, we can use the second characterization point of Theorem 6.9 to obtain that $h[S]$ is a simple filter (thus non-trivial) containing X and $I^{\boldsymbol{A}}(a)$. In particular, $\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X \cup I^{\boldsymbol{A}}(a, \bar{b})) \neq A$, as we wanted. □

Theorem 6.61. *Every compact protonegational logic* L *with a small type and the* dLIL *has* τ-SEP. *In particular, it is semisimple.*

Proof. Since L has τ-MCP (Corollary 6.14) and the dLIL transfers by the previous lemma, the result can be obtained by a straightforward modification of (ii)→(iii) of Theorem 6.58. The last claim is due to Proposition 6.25. □

Observe that the theorem presents another way to show that $Ł_\infty$ enjoys the τ-CIPEP (clearly τ-SEP is a stronger condition—cf. Proposition 6.24). However, this time we obtain an even stronger representation result: $Ł_\infty$ is semisimple, i.e.

$$\mathbf{Mod}^{*}\, Ł_\infty = \mathbf{P_{SD}}(\mathbf{Mod}^{*}_{\mathrm{Max}}\, Ł_\infty).$$

Proposition 6.62. *Completely intersection-prime* L*-theories are simple, whenever* L *enjoys the* dPLIL.

Proof. If T is completely intersection-prime then some formula φ saturates it. By dPLIL the set $T \cup I(\varphi, \bar{\delta})$ is not an antitheorem for some $I \in \Psi$ and some tuple of formulas $\bar{\delta}$. Thus, by saturation, it must be that $I(\varphi, \bar{\delta}) \subseteq T$ (otherwise the union would prove φ and thus could not be consistent). Consequently, for every ψ not in T we conclude that ψ, T proves both φ and also $I(\varphi, \bar{\delta})$ and consequently is not consistent by the dPLIL. □

Observe that from the previous proposition, combined with Proposition 6.24, characterization for LEM (Proposition 6.47), and Theorem 6.58 we know that, in protonegational logics with an antitheorem, the following equivalence holds:

$$\text{SEP} \iff \text{CIPEP} + \text{LEM}.$$

Moreover, if L has a small type, we obtain an even stronger result:

Proposition 6.63. *Suppose that* L *is a protonegational logic with an antitheorem and a small type. Then, it enjoys*

$$\tau\text{-SEP} \iff \tau\text{-CIPEP} + \text{LEM}.$$

Proof. The left-to-right direction should be obvious: indeed, L enjoys the τ-CIPEP because every simple filter is completely intersection-prime (Proposition 6.24) and the LEM is due to Theorem 6.58. For the converse direction, use the previous proposition and the fact that the property proved therein transfers to all algebras (Theorem 6.23). □

In case of finitary logics (which always enjoy the τ-CIPEP—see Proposition 3.3), we obtain the main connection between the LEM (resp. dPLIL) and semisimplicity.

Corollary 6.64. *For every finitary protonegational logic* L *with an antitheorem and a small type the following are equivalent:*

(i) L *enjoys the* LEM*(resp. the* dPLIL*).*
(ii) L *is semisimple.*

Proof. Recall that the τ-SEP is in our setting equivalent to semisimplicity by Proposition 6.25. Note that the prerequisites of the proposition are satisfied: indeed, finitary logics with an antitheorem are compact (Proposition 2.14) and compact logics enjoy the τ-MCP (Proposition 6.4). Moreover, L is fully protonegational by Theorem 6.19. □

Finally, we will prove the main characterization results for logics which enjoy both the global (resp. local) and a dual inconsistency lemma. Again, the global version can be found in a different form in [82, Theorem 4.3].

Theorem 6.65. *For every compact protonegational logic* L *with a small type*[6] *the following are equivalent:*

(i) L *has the* LIL *and the* dPLIL.
(ii) L *has the* LIL *and the* dLIL.
(iii) L *has the* SFEP *and the* SEP.
(iv) L *has the* SFEP *and the* τ-SEP.
(v) L *has the* FEP *and the* τ-SEP.
(vi) L *has the* LDDT *and the* SEP.

Moreover, we can replace everywhere 'has the τ-SEP*' by 'is semisimple'.*

[6] Instead of a small type we can assume that the logic has enough variables and a $|Var_{\mathcal{L}}|$-small protonegation.

Proof. The equivalence between points (i), (ii), and (iii) can be established using Proposition 6.45 and Theorems 6.35 and 6.58 (recall that the MCP is a consequence of compactness—Proposition 6.4).

(iii)→(iv): Suppose $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}\,\mathrm{L}$ is given and $a \notin F$. Let $\langle \boldsymbol{B}, G\rangle$ be the submatrix of $\langle \boldsymbol{A}, F\rangle$ generated by a. Since there is a surjective homomorphism from $\boldsymbol{Fm_{\mathcal{L}}}$ to $\boldsymbol{B}$, we can use the weak correspondence theorem of protonegational logics and the dLIL to obtain some $I(p) \in \Psi_1$ such that $I^{\boldsymbol{B}}(a) \cup G$ is contained in $H \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$. Then, the SFEP implies that there is $F \subseteq H' \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ such that $H = H' \cap B$. By the τ-MCP (see Corollary 6.14) we can extend H' to $H'' \in \mathrm{Max}\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$. Finally, observe that necessarily $a \notin H''$ since $a, I^{\boldsymbol{A}}(a)$ is an inconsistent set.

(iv)→(v): Let $\langle \boldsymbol{B}, G\rangle \leq \langle \boldsymbol{A}, F\rangle$. We want to prove the FEP, but observe that we already know that the FEP is true restricted to some basis of $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$. In such a situation, the FEP always follows: suppose we are given $H \supseteq G$, then we can express H using the basis of $\mathcal{F}i_{\mathrm{L}}\,\boldsymbol{B}$ as $H = \bigcap H_i$. Consequently, the restricted FEP gives $F \subseteq H'_i \in \mathcal{F}i_{\mathrm{L}}\,\boldsymbol{A}$ such that $H_i = H'_i \cap B$ and, thus, clearly $H = \bigcap H_i \cap B$.

(v)→(vi): It is well known that the FEP is equivalent to LDDT—see [34]. Although the result is actually proved only for finitary logics, still inspecting the proof we can observe that the assumption of finitarity is not needed in the direction we are interested in.

(vi)→(i): Proposition 6.57 and Theorem 6.58.

The final claim is due to Proposition 6.25 (L is fully protonegational due to Theorem 6.19). □

Example 6.66. The infinitary Łukasiewicz logic Ł_∞ satisfies the assumptions of the theorem. Moreover, we already know that it has the LIL (Example 6.30) and dLIL (Example 6.51), thus by the theorem it has the filter extension property. Note that, since the logic is infinitary, we could not simply use the results in [34] (the restriction on finitarity for the equivalence between LDDT and FEP seems unavoidable in the proof presented therein) to obtain the result even though we know that Ł_∞ has the LDDT (Theorem 6.55).

We say that a dually pseudo-complemented join semilattice $\boldsymbol{S} = \langle \boldsymbol{S}, \vee, 0\rangle$ with a least element 0 is a *Boolean lattice* if $a^{**} = a$ for every $a \in S$. In particular, $\boldsymbol{S}$ is a Boolean algebra with $a \wedge b = (a^* \vee b^*)^*$.

In the proof of the theorem we will use the next lemma, which basically asserts that dGIL can be reformulated as a generalized form of *involution* (in classical logic we have $\varphi \dashv\vdash_{\mathrm{CL}} \neg\neg\varphi$). Since the lemma has a straightforward simple proof, we have decided to omit it:

Lemma 6.67. *Suppose* L *enjoys the global inconsistency lemma with a sequence* $\{\Psi_n\}_{n\in\omega}$. *Then, the following are equivalent:*

(i) L *enjoy the* dGIL *w.r.t.* Ψ_1.
(ii) *For every tuple of formulas* $\varphi_1,\dots,\varphi_n$ *we have*

$$\varphi_1,\dots,\varphi_n \dashv\vdash_{\mathrm{L}} \Psi_{|\Psi_n|}\Psi_n(\varphi_1,\dots,\varphi_n,$$

where we write $\Psi_{|\Psi_n|}\Psi_n(\varphi_1,\dots,\varphi_n)$ *as a shortcut for* $\Psi_{|\Psi_n|}(\psi_1,\dots,\psi_{|\Psi_n|})$ *with* $\{\psi_1,\dots,\psi_{|\Psi_n|}\} = \Psi_n(\varphi_1,\dots,\varphi_n)$.

We remark that to obtain the equivalence it is enough in point (ii) to consider just the tuples of length 1.

Theorem 6.68. *For every compact protonegational logic* L *with a small type,*[7] *the following are equivalent:*

(i) L *has the* GIL *and the* dPLIL.
(ii) L *has the* GIL *and the* dGIL.
(iii) L *has the* SEP *and the join semilattice of finitely generated* L*-theories is dually pseudo-complemented.*
(iv) L *has the* τ-SEP *and for every algebra* $\boldsymbol{A}$*, the join semilattice of finitely generated* L*-filters is dually pseudo-complemented.*
(v) *For every algebra* $\boldsymbol{A}$*, the join semilattice of finitely generated* L*-filters is a Boolean lattice.*
(vi) L *has the* GDDT *and the* τ-SEP.
(vii) L *has the* GDDT *and the* SEP.

Again, we can replace everywhere 'has the τ-SEP*' by 'is semisimple'.*

Proof. To establish the equivalence of all points beside (v) use the more general characterization result for LIL+dLIL and its proof (Theorem 6.35) and the characterization of GIL (Theorem 6.42).

(ii)↔(v) Using the characterization for the GIL (Theorem 6.42), we know that both (ii) and (v) imply that there is an inconsistency sequence $\{\Psi_n\}_{n\in\omega}$ for L and that the join semilattice of finitely genereated L-filters on every algebra $\boldsymbol{A}$ is dually pseudo-complemented. Moreover, by the proof of the theorem, we know that

$$\mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\{a_1,\dots,a_n\})^* = \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\Psi_n(a_1,\dots,a_n)).$$

[7] Instead of a small type we can assume that the logic has enough variables and a $|Var_{\mathcal{L}}|$-small protonegation.

In particular,

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\{a_1,\dots,a_n\})^{**} = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\Psi_{|\Psi_n|}\Psi_n(a_1,\dots,a_n)). \tag{6.6}$$

Thus, we can obtain the result by the following chain of reasoning:

On every $\boldsymbol{A}$, the join semilattice of finitely genereated L-filters is Boolean

$\iff$ $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\{a_1,\dots,a_n\})^{**} = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(\{a_1,\dots,a_n\})$ on every algebra $\boldsymbol{A}$

$\iff$ $\varphi_1,\dots,\varphi_n \dashv\vdash_{\mathrm{L}} \Psi_{|\Psi_n|}\Psi_n(\varphi_1,\dots,\varphi_n)$

$\iff$ L enjoys the dual global inconsistency lemma.

The first equivalence is the definition of Boolean lattice. The second one is due to (6.6) (for the top to bottom direction take $\boldsymbol{A} = \boldsymbol{Fm_{\mathcal{L}}}$). The last one follows by the previous lemma. □

6.3 Antistructural completeness

In this section we shall introduce and investigative the notion of *antistructurally complete* logics, which as we will see is strongly connected to the theory developed above. As we will explain, it a is a natural dual notion to the well-known and extensively studied notion of *structurally complete* logics, which was introduced by Pogorzelski in [76] and from many points of view studied e.g. in [2, 77, 85].

A consecution $\Gamma \rhd \varphi$ is called an *admissible rule* in logic L if for every substitution σ it is the case that whenever σ uniforms Γ, i.e. $\vdash_{\mathrm{L}} \sigma[\Gamma]$, it also uniforms φ, i.e. $\vdash_{\mathrm{L}} \sigma(\varphi)$. A logic L is called *structurally complete* if every admissible rule is derivable in L.[8] Equivalently, L is structurally complete if each of its proper extensions admits a new theorem. A logic L′ is the *structural completion* of L if it is the weakest extension of L, which is structurally complete, or, equivalently, if it is its strongest extension with the same theorems. The structural completion of L can either be defined syntactically as the collection of all admissible rules or semantically as the logic of the matrix $\langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle$, where T is the smallest L-theory (i.e. $T = \mathrm{Th}_{\mathrm{L}}(\emptyset)$).

We claim that the notion of antistructural completeness is a natural dual counterpart to structural completeness, because antistructurally complete logics have the following properties:

[8] Note that in the definition of structural completeness one is usually only interested in admissible consecutions with Γ finite, which ensures that the structural completion of finitary logic remains finitary. In our presentation, we are going to be more general.

- Every of its proper extensions admits a new antitheorem.
- They can be characterized by means of *antiadmissible rules*.

Moreover, the antistructural completion of a logic is:

- the strongest extension with the same antitheorems.
- the logic axiomatized by antiadmissible rules.
- defined by means of matrices $\langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle$, where $T \in \mathrm{MaxTh}\,\mathrm{L}$.

Definition 6.69. The *antistructural completion* of L, we denote it $\alpha\mathrm{L}$, is the logic semantically given by the class of all matrices $\langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle$, where T is simple L-theory. Moreover, L is *antistructurally complete* if $\alpha\mathrm{L} = \mathrm{L}$ or equivalently if it is complete w.r.t. the class of all models $\langle \boldsymbol{Fm_{\mathcal{L}}}, T\rangle$ with T simple.

Observe that unfolding the definition we obtain that $\Gamma \vdash_{\alpha L} \varphi$ if and only if for every substitution σ and every simple L-theory T whenever $\sigma[\Gamma] \subseteq T$ then $\sigma(\varphi) \in T$. For the same reason as in the case of protonegational logics, we need to assume at least the MCP to secure the desired behavior of antistructural completions.

Example 6.70. If L is an superintuitionistic logic, then $\alpha\mathrm{L} = \mathrm{CL}$: Indeed, it follows from the fact that L and CL have the same simple theories and that CL is complete w.r.t. them.

Proposition 6.71. *If* L *has the* MCP*, then* $\alpha\mathrm{L}$ *is the strongest extension of* L *with the same simple theories as* L*. Consequently,* $\alpha\mathrm{L}$ *is antistructurally complete and has the* MCP*.*

Proof. Clearly $\mathrm{L} \leq \alpha\mathrm{L}$, we show that they share simple theories: if T is simple in L, then by definition it is an $\alpha\mathrm{L}$-theory and, since $\mathrm{L} \leq \alpha\mathrm{L}$, it is simple. On the other hand, any simple theory of $\alpha\mathrm{L}$ is also an L-theory and by the MCP it must be simple. It is the strongest: if $\Gamma \nvdash_{\alpha\mathrm{L}} \varphi$ then, by the definition of $\alpha\mathrm{L}$, there is a simple theory T and a substitution σ such that $\sigma[\Gamma] \subseteq T$ and $\sigma(\varphi) \notin T$. Thus, if we extend $\alpha\mathrm{L}$ by $\Gamma \rhd \varphi$, then T will no longer be an $\alpha\mathrm{L}$-theory. □

Corollary 6.72. *If* L *has the* MCP*, then* $\alpha\mathrm{L}$ *is its strongest extension with the same antitheorems.*

Proof. Using previous propositions it is easy to see that they have the same antitheorems. Moreover, after extending $\alpha\mathrm{L}$ by some rule, we know that some simple $\alpha\mathrm{L}$-theory T is no longer a theory of the new logic, i.e. T is a new antitheorem. □

Corollary 6.73. *If* L *has the* MCP*, then* L *has an inconsistency sequence* $\{\Psi_n\}_{n\in\omega}$ *if and only if* $\alpha\mathrm{L}$ *does.*

Corollary 6.74. *If* L *has the* MCP *then it is* κ*-compact iff* $\alpha\mathrm{L}$ *is.*

On the other hand, note that $\alpha\mathrm{L}$ of finitary logic can be infinitary—see Example 6.82. In the next example we will see that indeed without MCP the previous statements need not hold.

Example 6.75. The truth-degrees-preserving Gödel logic with constants $\mathrm{G}_{\mathrm{c}}^{\leqslant}$ defined in Subsection 5.1.1 does not have the MCP: indeed, by Proposition 5.6, $\mathrm{G}_{\mathrm{c}}^{\leqslant}$ has no RSI models. In particular, it has no simple theories. Consequently, $\alpha\mathrm{G}_{\mathrm{c}}^{\leqslant}$ is the trivial logic. Thus, $\alpha\mathrm{G}_{\mathrm{c}}^{\leqslant}$ is compact and every set of formulas is an antitheorem, both these properties fail in $\mathrm{G}_{\mathrm{c}}^{\leqslant}$.

Definition 6.76. We say $\Gamma \rhd \varphi$ is an *antiadmissible rule* in L, if for every $\Delta \subseteq Fm_{\mathcal{L}}$ and every substitution σ we have

$$\sigma(\varphi), \Delta \vdash_{\mathrm{L}} \emptyset \quad \Longrightarrow \quad \sigma[\Gamma], \Delta \vdash_{\mathrm{L}} \emptyset.$$

The next proposition characterizes, in case of logics with MCP, antistructural completions via antiadmissible rules.

Proposition 6.77. *If* L *has the* MCP*, then:* $\Gamma \vdash_{\alpha\mathrm{L}} \varphi$ *if and only if* $\Gamma \rhd \varphi$ *is antiadmissible in* L*.*

Proof. Straightforward. □

Interestingly enough, in many cases we can drop the quantification over substitutions in the definition of antiadmissible rules:

Definition 6.78. We say $\Gamma \rhd \varphi$ is a *simply antiadmissible rule* in L, if for every $\Delta \subseteq Fm_{\mathcal{L}}$ we have

$$\varphi, \Delta \vdash_{\mathrm{L}} \emptyset \quad \Longrightarrow \quad \Gamma, \Delta \vdash_{\mathrm{L}} \emptyset.$$

Trivially, every antiadmissible rule is simply antiadmissible. Note that both concepts coincide if simply antiadmissible rules are closed under substitution. We show that this the case at least for logics with definable simple theories:

Proposition 6.79. *In every logic* L *with definable simple theories and the* MCP, *simply antiadmissible rules and antiadmissible rules coincide.*

Proof. Let $\Gamma \rhd \varphi$ be a simply antiadmissible rule and σ a substitution. It is enough to show that also $\sigma[\Gamma] \rhd \sigma(\varphi)$ is simply antiadmissible. Let $\Delta \subseteq Fm_{\mathcal{L}}$. Then, for every simply theory T extending Δ we have

$$\begin{aligned} \sigma(\varphi), \Delta \vdash \emptyset \quad &\iff \quad \sigma(\varphi), T \vdash \emptyset \\ &\iff \quad \varphi, \sigma^{-1}[T] \vdash \emptyset \\ &\implies \quad \Gamma, \sigma^{-1}[T] \vdash \emptyset \\ &\implies \quad \sigma[\Gamma], T \vdash \emptyset, \end{aligned}$$

where the second equivalence is due to the fact the preimages of simple theories are simple (Theorem 6.40). Consequently, if $\sigma(\varphi), \Delta \vdash \emptyset$, then also $\sigma[\Gamma], \Delta \vdash \emptyset$, because otherwise, by the MCP, there would be a simple theory T extending Δ such that $\sigma[\Gamma], T \nvdash \emptyset$, which by the previous reasoning is not possible. □

Recall that we have the following chain of conditions:

$$\begin{aligned} \text{LIL} \quad \text{or} \quad \text{dLIL} \quad &\implies \quad \text{definability of simple theories} \\ &\implies \quad \text{closure of simply antiadmissible rules} \\ &\qquad\quad\ \text{under substitutions.} \end{aligned}$$

It is an open question whether the implications are proper or not. We will see that logics with some of these properties will have a particularly nicely behaving antistructural completions. First, let us describe the connection between simply antiadmissible rules and the simple extension property (SEP).

Proposition 6.80. *For every logic* L *with the* MCP, *the following are equivalent:*

(i) $\Gamma \vdash_{\mathrm{L}} \varphi$ *if and only if* $\Gamma \rhd \varphi$ *is simply antiadmissible.*
(ii) L *enjoys the* SEP.

Proof. (i)→(ii): If $\Gamma \nvdash \varphi$ then there is $\Delta \subseteq Fm_{\mathcal{L}}$ such that $\varphi, \Delta \vdash \emptyset$ and $\Gamma, \Delta \nvdash \emptyset$. Use MCP to obtain simple theory T extending $\Gamma \cup \Delta$, then clearly T extends Γ and $\varphi \notin T$.

(ii)→(i): It is easy to prove that for every logic the left-to-right direction of (i) holds. Conversely, assume $\Gamma \nvdash \varphi$ then there is simple theory T such that $\Gamma \subseteq T \nvdash \varphi$. Consequently, $T, \varphi \vdash \emptyset$ but $\Gamma, T \nvdash \emptyset$ (because $\Gamma \subseteq T$)—thus $\Gamma \rhd \varphi$ is not simply antiadmissible. □

We are now ready to summarize the main properties of antistructural completions based on the properties of L:

Theorem 6.81. *Suppose that* L *has the* MCP *and an antitheorem. Moreover, assume that its simply antiadmissible rules are closed under substitutions. Then,* αL *has the following properties:*

(i) *It has the* SEP*.*
(ii) $\varGamma \vdash_{\alpha \mathrm{L}} \varphi \iff T \vdash_{\mathrm{L}} \varphi$ *for every simple* L*-theory* $T \supseteq \varGamma$*.*

Moreover, if L *is protonegational then:*

(iii) *It has the* dPLIL*.*
(iv) *It is protoalgebraic.*

Finally, if additionally L *is compact, has simple theories definable by* $\varPsi$*, and has a small type, then:*

(v) *It has the* dLIL *w.r.t.* $\varPsi$*.*
(vi) *It has* τ-SEP *and hence is semisimple.*
(vii) L *and* αL *have the same simple models, i.e.* $\mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L} = \mathbf{Mod}_{\mathrm{Max}}\,\alpha\mathrm{L}$*.*
(viii) $\alpha\mathrm{L} = \models_{\mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}} = \models_{\mathbf{Mod}^{*}_{\mathrm{Max}}\,\mathrm{L}}$*.*

Proof. (i): Since αL is the logic of antiadmissible rules (Proposition 6.77) and these, by the assumption, coincide with simple antiadmissible ones, we obtain the result by Proposition 6.80.

(ii): The direction from left to right follows straight from the definition of αL (consider the identity substitution). For the other direction use point (i) and the fact that the two logics have the same simple theories (Proposition 6.71).

(iii): Theorem 6.58.

(iv): Let $\Rightarrow(p, q, \bar{r})$ be a parametrized protonegation of L. Then, clearly, $\vdash_{\alpha\mathrm{L}} p \Rightarrow p$ and, by the point (ii), $p, p \Rightarrow q \vdash_{\alpha\mathrm{L}} q$. Therefore, we can obtain a protoimplication for αL from $\Rightarrow$ simply by substituting q for the variables $\bar{r}$.

Or, alternatively, we could use point (i) to show that $\boldsymbol{\varOmega}$ is monotone: suppose $T \subseteq S$, then, by (i), $S = \bigcap S_i$, where all S_i are simple. Protonegationality implies $\boldsymbol{\varOmega}T \subseteq \boldsymbol{\varOmega}S_i$. Hence $\boldsymbol{\varOmega}T \subseteq \bigcap \boldsymbol{\varOmega}S_i \subseteq \boldsymbol{\varOmega}\bigcap S_i = \boldsymbol{\varOmega}S$.

(v): Firstly, if $\varGamma \vdash_{\mathrm{L}} \varphi$, then $\varGamma \cup \{I(\varphi)\}$ is an antitheorem (an easy consequence of the definability and the MCP). Conversely, if $\varGamma \nvdash_{\mathrm{L}} \varphi$, then by the SEP there is a simple theory T extending $\varGamma$ and not containing φ. By the definability there is $I \in \varPsi$ such that $I(\varphi) \subseteq T$. In particular, $\varGamma \cup \{I(\varphi)\}$ is not an antitheorem, which is what we wanted.

(vi): Theorem 6.61.

(vii): Suppose we are given $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}$, since αL has fewer models (it is an extension), if F is a αL-filter it must be simple. So let us check it is indeed an αL-filter: suppose $\varGamma \vdash_{\alpha\mathrm{L}} \varphi$ and $h[\varGamma] \subseteq F$ for some $\boldsymbol{A}$-evaluation h. Since L has definable simple theories, $T \subseteq Fm_{\boldsymbol{\mathcal{L}}}$, the preimage of F over

h, is a simple L-theory extending Γ (see Theorem 6.40). Therefore by point (ii) we obtain $T \vdash_{\mathrm{L}} \varphi$ and since $h[T] \subseteq F$ we conclude $h(\varphi) \in F$.

On the other hand, if $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{Max}}\, \alpha\mathrm{L}$, then obviously F is an L-filter and it must be simple: otherwise it can be extended to a simple one (consequence of the τ-SEP) and, by the already proved direction, it would also be a simple filter of $\alpha\mathrm{L}$ extending F—contradiction.

(viii): If $\Gamma \nvdash_{\alpha\mathrm{L}} \varphi$, then $\Gamma \not\models_{\mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}} \varphi$ by the definition of $\alpha\mathrm{L}$. Conversely, there is $\langle \boldsymbol{A}, F\rangle \in \mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}$ and an evaluation h such that $h[\Gamma] \subseteq F$ and $h(\varphi) \notin F$. Then, as in the previous point, $h^{-1}[F]$ is a simple theory extending Γ and not containing φ. Therefore point (ii) concludes the proof.

Finally, since reductions of simple models are simple and since they define the same logic, we have $\models_{\mathbf{Mod}_{\mathrm{Max}}\,\mathrm{L}} = \models_{\mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{L}}$. □

Example 6.82. The antistructural completion of the basic fuzzy BL is the infinitary Łukasiewicz logic $Ł_\infty$: note that BL clearly satisfies the assumptions of the previous theorem (it is compact by Proposition 2.14). Thus, we can argue using point (viii) of the theorem: indeed, since every simple BL-algebra is up to isomorphism a subalgebra of the standard MV-algebra $[0,1]_Ł$ (see e.g. [11]) we obtain

$$\mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{BL} = \{\langle \boldsymbol{A}, \{1\}\rangle \mid \boldsymbol{A} \text{ is embeddable into } [0,1]_Ł\}.$$

Clearly, the logic of this class is $Ł_\infty$. Also, since $\mathrm{BL} \leq Ł \leq Ł_\infty$, we obtain that $\alpha Ł = Ł_\infty$. Consequently, using points (vi) and (vii) of the theorem, we can easily prove:

$$\mathbf{Alg}^* Ł_\infty = \{\boldsymbol{A} \mid \boldsymbol{A} \text{ is a semisimple BL-algebra (resp. MV-algebra)}\} \quad (6.7)$$

Observe that the previous example shows that the antistructural completion of a finitary logic need not be finitary. On the other hand, there is a condition which ensures that the logic we obtain will be finitary.

Proposition 6.83. *Suppose* L *has the* MCP *and the* LIL *with an inconsistency sequence* $\{\Psi_n\}$*, where every* Ψ_n *is a finite family of finite sets. Then,* $\alpha\mathrm{L}$ *is finitary whenever* L *is.*

Proof. Suppose $\Gamma \vdash_{\alpha\mathrm{L}} \varphi$. Then, for every $I \in \Psi_1$, we obtain $\Gamma, I(\varphi) \vdash_{\alpha\mathrm{L}} \emptyset$ (point (iii) of Theorem 6.81) and, since both logics have the same antitheorems, also $\Gamma, I(\varphi) \vdash_{\mathrm{L}} \emptyset$. Then, by the LIL, there is some $J \in \Psi_{|I|}$ such that $\Gamma \vdash_{\mathrm{L}} J(\bar{\psi})$ where $\bar{\psi} = I(\varphi)$. The collection of all $J(\bar{\psi})$ we obtained this way is, by the assumption, finite. Thus, by finitarity, we can find finite subset Γ' of Γ that proves all of them and, since we can go the same way back, we obtain that $\Gamma' \vdash_{\alpha\mathrm{L}} \varphi$. □

Finally, we shall now give a particular answer to the question whether protonegational logics always has a protonegation without parameters: at least we know that under some natural conditions it is always the case.

Theorem 6.84. *Suppose* L *is protonegational and* $\Rightarrow(p,q,\bar{r})$ *is a parametrized protonegation. If* $\alpha\mathrm{L}$ *enjoys the* SEP *and* σ *is a substitution fixing* p, q*, then also*

$$\sigma[p \Rightarrow q] = \{\sigma(\varphi(p,q,\bar{\chi})) \mid \varphi(p,q,\bar{r}) \in \Rightarrow \textit{ and } \bar{\chi} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}\}$$

is a parametrized protonegation in L*. In particular,* L *has a protonegation.*

Proof. Denote $\Rightarrow'(p,q,\bar{r}) = \sigma[p \Rightarrow q]$. Structurality implies it is reflexive. For *modus ponens* we can use the characterization point (ii) of Theorem 6.81: the proof is the same as (i)→(ii). Thus, it is enough to verify that $\varphi, \varphi \Rightarrow' \psi \vdash_{\alpha\mathrm{L}} \psi$. Let σ' be substitution sending p, q to φ, ψ and every other variable to itself. Then, again by (ii) we know that $p, p \Rightarrow q \vdash_{\alpha\mathrm{L}} q$, hence by structurality $\varphi, \sigma' \circ \sigma[p \Rightarrow q] \vdash_{\alpha\mathrm{L}} \psi$, but clearly $\sigma' \circ \sigma[p \Rightarrow q] \subseteq p \Rightarrow' q$. □

Corollary 6.85. *If* L *is protonegational with an antitheorem and the* MCP*, then any of the following properties ensures that it has a protonegation:*

- *Simply antiadmissible rules of* L *are closed under substitutions.*
- *Simple theories are definable in* L*.*
- L *enjoys the* LIL *or the* dLIL*.*

Proof. The first point follows from Theorem 6.81 and Theorem 6.84. Each of the remaining properties implies the first one (cf. Proposition 6.79). □

Theorem 6.81 shows that $\alpha\mathrm{L}$ often has rather strong properties. In the next example we show the preconditions on the theorem are necessary. To prove this claim, we utilize the previous theorem.

Example 6.86. First, we show that $\alpha\mathrm{FL_w}$ does not enjoy the SEP. Recall the inconsistency sequence of $\mathrm{FL_w}$ defined in Example 6.31. We will denote the elements of Ψ_1 as $\mu_{\bar{r}}(p)$. We start by showing that

$$\Rightarrow(p,q,\bar{r}) = \{\neg[\mu_{\bar{r}}(p) \mathbin{\&} \neg\mu_{\bar{r}}(q)] \mid \mu_{\bar{r}}(p) \in \Psi_1\}$$

is a parametrized protonegation in $\mathrm{FL_w}$. It is easy to see that $\Rightarrow$ is reflexive. For *modus ponens* assume that $T \vdash \varphi, \varphi \Rightarrow \psi$, where T is simple. Then, if $T \nvdash \psi$, we obtain (by the PLIL) that $T \vdash \neg\mu_{\bar{\delta}}(\psi)$ for some $\mu_{\bar{r}} \in \Psi_1$ and some $\bar{\delta} \in Fm_{\mathcal{L}}^{Var_{\mathcal{L}}}$. However, since $T \vdash \varphi$, we obtain that $T \vdash \mu_{\bar{\delta}}(\varphi)$. Hence $T \vdash \mu_{\bar{\delta}}(\psi) \mathbin{\&} \neg\mu_{\bar{\delta}}(\psi)$—contradiction with the fact that T is consistent and $T \vdash p \Rightarrow q$.

Let σ be the substitution sending q to q and all the remaining variables to p. We show that $\Rightarrow' = \sigma[p \Rightarrow q]$ is not a protonegation, which, by the previous theorem, implies that $\alpha\mathrm{FL_w}$ does not enjoy the SEP. Let e be a surjective evaluation on the $\mathrm{FL_w}$-algebra $\boldsymbol{A}$ from Example 6.41 such that $e(p) = 1$ and $e(q) = a$. Let us show that $1 \Rightarrow'^{\boldsymbol{A}} a \subseteq \{1\}$: observe that

$$
\begin{aligned}
1 \Rightarrow'^{\boldsymbol{A}} a =& \{\neg[\mu_{\bar{1}}(1) \,\&\, \neg\mu_{\bar{1}}(a)] \mid \mu_{\bar{r}}(p) \in \Psi_1\} \\
=& \{\neg\neg\mu_{\bar{1}}(a)] \mid \mu_{\bar{r}}(p) \in \Psi_1\}
\end{aligned}
$$

It is easy to see that $\mu_{\bar{1}}(a)$ is equal to either 1 or a: note that the iterated conjugates in μ has as parameters formulas in one variable p, which is evaluated to 1, hence every parameter has either value 1 or 0. Moreover, $\{1, a, 0\}$ is a Heyting algebra which simplifies the computation and implies that $\neg\neg\mu_{\bar{1}}(a) = 1$. Finally, since $\{1\} \in \mathrm{Max}\mathcal{F}i_{\mathrm{FL_w}}\, \boldsymbol{A}$ and σ is surjective, we obtain $T = \sigma^{-1}[\{1\}] \in \mathrm{MaxTh}\,\mathrm{FL_w}$ and $T \vdash p, p \Rightarrow' q$ but $T \nvdash q$. Thus, $\Rightarrow'$ is not a protonegation in $\mathrm{FL_w}$.

Consequently, $\alpha\mathrm{FL_w}$ is not semisimple: because, if it was, then by Proposition 6.25 it would have the SEP. Note that every protoalgebraic logic is fully protonegational and that compactness is preserved under antistructural completions and it implies the τ-MCP (Corollary 6.14).

Also, as a consequence of Theorem 6.58, we obtain that $\alpha\mathrm{FL_w}$ does not enjoy the dPLIL.

Finally, we are ready to conclude the general presentation of antistructurally complete logics by providing a characterization result:

Theorem 6.87. *For every compact protonegational logic* L *with definable simple theories and a small type, the following are equivalent:*

(i) L *is antistructurally complete.*
(ii) *Simple antiadmissible rules are provable in* L.
(iii) L *enjoys the* dPLIL *or, equivalently, the* LEM.
(iv) L *enjoys the* dLIL.
(v) L *is semisimple.*
(vi) L *is complete w.r.t. its simple models:* $\mathrm{L} = \models_{\mathbf{Mod}^*_{\mathrm{Max}}\mathrm{L}}$.
(vii) L *is complete w.r.t. a subclass of simple models, i.e. there is* $\mathbb{K} \subseteq \mathbf{Mod}^*_{\mathrm{Max}}\,\mathrm{L}$ *such that* $\mathrm{L} = \models_{\mathbb{K}}$.

Proof. (i)→(ii): Antistructurally complete logics are characterized by means of antiadmissible rules (Proposition 6.77) and in L these are precisely the simply antiadmissible ones (Proposition 6.79).

(ii)→(iii): By Proposition 6.80, we know that L enjoys the SEP. Consequently, it also enjoys the dPLIL and the LEM (Proposition 6.58).

(iii)→(iv): Suppose that $\Psi(p)$ is a family that defines simple theories. We show that Ψ witnesses also the dLIL for L. If $\Gamma \nvdash \varphi$, then there is a simple theory T not containing φ and extending Γ (note that L enjoys the SEP by Proposition 6.58). Thus, by the definability, $I(\varphi) \subseteq T$ for some $I \in \Psi$. In particular, $\Gamma \cup I(\varphi)$ is not an antitheorem.

The converse direction follows from the fact that $\varphi \cup I(\varphi)$ is an antitheorem (consequence of the MCP).

(iv)→(v): Theorem 6.61.

(v)→(vi): Soundness is clear. Thus, assume that $\Gamma \nvdash \varphi$. Then, there is $\langle \boldsymbol{A}, F \rangle \in \mathbf{Mod}^{*}\,\mathrm{L}$ and an $\boldsymbol{A}$-evaluation v such that $v[\Gamma] \subseteq F$ and $v(\varphi) \notin F$. Moreover, by (v) there is a subdirect representation

$$e : \langle \boldsymbol{A}, F \rangle \hookrightarrow_{\mathbf{SD}} \prod \langle \boldsymbol{A}_i, F_i \rangle,$$

where all $\langle \boldsymbol{A}_i, F_i \rangle$ are simple and reduced. Let $\pi_i \colon \prod \langle \boldsymbol{A}_i, F_i \rangle \twoheadrightarrow \boldsymbol{A}_i$ be the ith projection. Clearly, there must be some i such that $\pi_i ev(\varphi) \notin F_i$. However, since also $\pi_i ev[\Gamma] \subseteq F_i$, we are done (the counter model is $\langle \boldsymbol{A}_i, F_i \rangle$ and $\pi_i ev$ is the witnessing $\boldsymbol{A}_i$-evaluation).

(vi)→(vii): Trivial.

(vii)→(i): By Proposition 6.77, it is enough to show that the antiadmissible rules are provable in L. Contrapositively, assume that $\Gamma \nvdash \varphi$. Then, there is $\langle \boldsymbol{A}, F \rangle \in \mathbb{K} \subseteq \mathbf{Mod}^{*}_{\mathrm{Max}}\,\mathrm{L}$ and an evaluation v such that $v[\Gamma] \subseteq F$ and $v(\varphi) \notin F$. Then, by point (v) of Theorem 6.40, we obtain that $T := v^{-1}[F]$ is a simple theory of L. Moreover, $\varphi \notin T$. Hence $\varphi, T \vdash \emptyset$. On the other hand, since $\Gamma \subseteq T$, we get $\Gamma, T \nvdash \emptyset$. That is, $\Gamma \rhd \varphi$ is not antiadmissible in L. □

The characterization via semisimplicity can be used to provide examples of antistructurally complete logics: every finitary algebraizable logic with a semisimple variety as an equivalent algebraic semantics is antistructurally complete (provided the definability condition is met):

- In [64] it was proved that a global modal logic [10] is semisimple if and only if it is *weakly transitive* and *cyclic* (e.g. the global modal logic S5). Moreover, weak transitivity implies the GDDT (and hence the definability of simple theories).
- In [63] it was proved that an axiomatic extension L of FL_{ew} is semisimple if and only if $\varphi \vee \neg\varphi^k$ is a theorem of L for some natural number k (e.g. the k-valued Łukasiewicz logic). Again these logics enjoys the GDDT.

6.4 Glivenko-like theorems

In this section, we will present few results to demonstrate that the notions we investigated in this chapter, namely, the inconsistency lemmas and anti-structural completions, can be useful in the study of Glivenko-like theorems [4, 14, 15, 16, 17, 54, 55].

We say that a logic L is *Glivenko-equivalent* to L′ if for every set of formulas $\Gamma \cup \{\varphi\}$ we have

$$\Gamma \vdash_{\mathrm{L}} \neg\neg\varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}'} \varphi.$$

Next theorem provides a full characterization of when a given subclassical[9] substructural logic (in our sense an extension of SL) is Glivenko-equivalent to classical logic.

Theorem 6.88. *Let* L *be a finitary subclassical extension of* SL*. Then the following are equivalent:*

(i) L *is Glivenko-equivalent to classical logic: i.e. for every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{SL}}$

$$\Gamma \vdash_{\mathrm{L}} \neg\neg\varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \varphi.$$

(ii) L *"almost" has the inconsistency lemma: i.e.*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \overline{0} \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi, \text{[10]} \tag{6.8}$$

and the following rules are provable in L*:*

$$\neg(\varphi \to \psi) \rhd \neg(\neg\neg\varphi \to {\sim}\neg\psi) \tag{A}$$
$$\neg(\varphi \,\&\, \neg\psi) \lhd\rhd \neg(\varphi \wedge \neg\psi). \tag{Conj}$$

Furthermore, if $\overline{0}$ *is an antitheorem in* L *then both properties imply that* $\alpha\mathrm{L} = \mathrm{CL}$*.*

Proof. Recall the properties of substructural logics from Table 2.1 on page 27.

(i)→(ii): To verify (A), (Conj), it clearly suffices to prove that

$$\Gamma \vdash_{\mathrm{L}} \neg\varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \neg\varphi. \tag{6.9}$$

The left-to-right implication of (6.9) is obvious. For the other one, we can argue by the following chain:

$$\begin{aligned} \Gamma \vdash_{\mathrm{CL}} \neg\varphi \quad &\Longrightarrow \quad \Gamma \vdash_{\mathrm{CL}} {\sim}\varphi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\neg{\sim}\varphi \quad \Longrightarrow \\ &\Longrightarrow \quad \Gamma \vdash_{\mathrm{L}} {\sim}\neg{\sim}\varphi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{L}} {\sim}\varphi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi. \end{aligned}$$

[9] We say that a logic is *subclassical* provided it is weaker than classical logic.

[10] Note that this property does not necessarily satisfy the definition of IL: it need not be the case that $\overline{0}$ is an antitheorem.

Indeed, the right-to-left direction of (6.8) follows by *modus ponens*. The other one can be proved as follows:

$$\begin{aligned} \Gamma, \varphi \vdash_{\mathrm{L}} \overline{0} \quad &\Longrightarrow \quad \Gamma, \varphi \vdash_{\mathrm{CL}} \overline{0} \\ &\Longleftrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \neg\varphi \\ &\Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi, \end{aligned}$$

where the first equivalence is the deduction-detachment theorem of CL, and the second one (6.9).

(ii)→(i): Let L_o be the extension of L by the rule $\overline{0} \vdash \bot$. It is easy to see that $\Gamma \vdash_{\mathrm{L}} \overline{0}$ if and only if $\Gamma \vdash_{\mathrm{L_o}} \bot$. Thus, since $\wedge$ is a conjunction, from (6.8) we obtain

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L_o}} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L_o}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n). \tag{6.10}$$

We show that $\alpha\mathrm{L}_o = \mathrm{CL}$. Since L_o is compact (see Proposition 2.14), it has the MCP (Proposition 6.4), and, because it has the GIL, by Proposition 6.79, all the preconditions of Theorem 6.81 are met, thus $\alpha\mathrm{L}_o$ enjoys the dPLIL. Consequently, since $\alpha\mathrm{L}_o$ has global inconsistency lemma in the same form as L_o (Corollary 6.73), we obtain by Proposition 6.45 that $\alpha\mathrm{L}_o$ has dGIL:

$$\Gamma, \neg\varphi \vdash_{\alpha\mathrm{L_o}} \emptyset \quad \Longleftrightarrow \quad \Gamma \vdash_{\alpha\mathrm{L_o}} \varphi, \tag{6.11}$$

and by Theorem 6.53 we obtain GDDT of the form:

$$\Gamma, \varphi \vdash_{\alpha\mathrm{L_o}} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\alpha\mathrm{L_o}} \neg(\varphi \wedge \neg\psi). \tag{6.12}$$

Next we show that

$$\varphi \to \psi \dashv\vdash_{\alpha\mathrm{L_o}} \neg(\varphi \mathbin{\&} \neg\psi). \tag{6.13}$$

The left-to-right direction holds already in SL: $\varphi \to \psi$ and $\neg\psi \to \neg\psi$ implies $\varphi \mathbin{\&} \neg\psi \to \psi \mathbin{\&} \neg\psi$ and $\psi \mathbin{\&} \neg\psi \to \overline{0}$ is provable, thus the result follows by transitivity of $\to$. For the other direction: $\neg(\varphi \mathbin{\&} \neg\psi) \vdash_{\alpha\mathrm{L_o}} \varphi \to \psi$ is by (6.11) equivalent to $\neg(\varphi \mathbin{\&} \neg\psi), \neg(\varphi \to \psi) \vdash_{\alpha\mathrm{L_o}} \emptyset$. Thanks to (A), it is enough to show that $\neg(\varphi \mathbin{\&} \neg\psi), \neg(\neg\neg\varphi \to {\sim}\neg\psi) \vdash_{\alpha\mathrm{L_o}} \emptyset$. This can be proved by the following chain of equivalences :

$$\begin{aligned} \neg(\varphi \mathbin{\&} \neg\psi), \neg(\neg\neg\varphi \to {\sim}\neg\psi) \vdash_{\alpha\mathrm{L_o}} \emptyset \quad &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi) \vdash_{\alpha\mathrm{L_o}} \neg\neg\varphi \to {\sim}\neg\psi \\ &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi) \vdash_{\alpha\mathrm{L_o}} \neg(\neg\neg\varphi \mathbin{\&} \neg\psi) \\ &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi) \vdash_{\alpha\mathrm{L_o}} \neg(\neg\neg\varphi \wedge \neg\psi) \\ &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi), \neg\neg\varphi, \neg\psi \vdash_{\alpha\mathrm{L_o}} \emptyset \\ &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi), \varphi, \neg\psi \vdash_{\alpha\mathrm{L_o}} \emptyset \\ &\Longleftrightarrow \quad \neg(\varphi \mathbin{\&} \neg\psi) \vdash_{\alpha\mathrm{L_o}} \neg(\varphi \wedge \neg\psi). \end{aligned}$$

Note that the formula $\neg\neg\varphi \to \sim\neg\psi$ is a shortcut for $\neg\neg\varphi \to (\neg\psi \rightsquigarrow \overline{0})$, thus the second equivalence is valid already in SL. The rest is due to the inconsistency lemma, its dual, and the rules (Conj).

Combining (6.12), (6.13), and (Conj) we obtain the GDDT of intuitionistic logic:

$$\Gamma, \varphi \vdash_{\alpha\mathrm{L_o}} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\alpha\mathrm{L_o}} \varphi \to \psi.$$

Consequently, we can easily prove

$$\begin{aligned} &\vdash_{\alpha\mathrm{L_o}} \psi \to (\varphi \to \psi) && \text{(weakening)}\\ &\vdash_{\alpha\mathrm{L_o}} (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) && \text{(contraction)}\\ &\vdash_{\alpha\mathrm{L_o}} (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) && \text{(exchange)} \end{aligned}$$

Consequently, $\vdash_{\alpha\mathrm{L_o}} \varphi \mathbin{\&} \psi \leftrightarrow \varphi \wedge \psi$, which implies that $\&$ is associative. Thus, we obtain that $\mathrm{IL} \leq \alpha\mathrm{L}_o$. Moreover, from (6.11) and Proposition 6.47, $\alpha\mathrm{L}_o$ enjoys the law of excluded middle in the expected form:

$$\frac{\Gamma, \varphi \vdash_{\alpha\mathrm{L_o}} \psi \qquad \Gamma, \neg\varphi \vdash_{\mathrm{L_o}} \psi}{\Gamma \vdash_{\alpha\mathrm{L_o}} \psi}\,.$$

Thus, $\varphi \vee \neg\varphi$ is a theorem of $\alpha\mathrm{L}_o$. In other words $\mathrm{CL} \leq \alpha\mathrm{L}_o$. Finally, since $\alpha\mathrm{L}_o$ is a non-trivial logic (from the assumptions it easily follows that L_o has a simple theory), we obtain $\mathrm{CL} = \alpha\mathrm{L}_o$.

Finally, we can easily prove the desired equivalence: if $\Gamma \vdash_{\mathrm{CL}} \varphi$, then $\Gamma, \neg\varphi \vdash_{\mathrm{CL}} \emptyset$, and, since $\mathrm{CL} = \alpha\mathrm{L}_o$, we obtain $\Gamma, \neg\varphi \vdash_{\mathrm{L_o}} \emptyset$ (the two logics share antitheorems; see Corollary 6.72) which is equivalent to $\Gamma, \neg\varphi \vdash_{\mathrm{L}} \overline{0}$. The rest is one application of (6.8). □

From this theorem, we can immediately obtain the best-known results:

- Every superintuitionistic logic is Glivenko-equivalent to classical logic [55].
- Both Π and SBL are Glivenko-equivalent to classical logic (see [15, 16]): we showed that both logics validate the inconsistency lemma in Example 6.29 and it is easy to verify that they satisfy the rules (A) and (Conj). Note that the theorem implies that both logics have classical logic as their antistructural completion. The theorem also applies to infinitary product logic Π_∞.

As an another application of the theorem, we can easily compute what is the weakest extension of an arbitrary subclassical axiomatic extension of SL which is Glivenko-equivalent to classical logic. This was one of the questions handled e.g. in [54]. The general method is quite simple, it is well known, that all of these logics has deduction-detachment theorem in some form (see e.g. [23]), it is then enough to add axioms that allows us to turn this deduction theorem into the appropriate inconsistency lemma. We describe the method on one example:

Example 6.89. The weakest extension of the full lambek calculus with exchange, $\mathrm{FL_e}$, which is Glivenko-equivalent to classical logic can be axiomatized by adding the following axioms to any presentation of $\mathrm{FL_e}$:

(i) $\neg(\varphi \wedge 1) \to \neg\varphi$,
(ii) (A) as an axiom $\neg(\varphi \to \psi) \to \neg(\neg\neg\varphi \to \neg\neg\psi)$, and
(iii) a variant of (Conj) as axioms $\neg(\varphi \& \psi) \to \neg(\varphi \wedge \psi)$, $\neg(\varphi \wedge \psi) \to \neg(\varphi \& \psi)$.

In the proof of the theorem we saw that a logic Glivenko-equivalent to classical logic must have the same negation fragment as classical logic. In particular, the desired weakest extension of $\mathrm{FL_e}$ necessarily has the newly added axioms: indeed by residuation we can see them as negated formulas, e.g. the first axiom can be seen as $\neg(\neg(\varphi \wedge \overline{1}) \mathbin{\&} \varphi)$. Thus, it remains to check the inconsistency lemma:

$$
\begin{aligned}
\Gamma, \varphi \vdash \overline{0} \quad &\iff \quad \Gamma \vdash (\varphi \wedge \overline{1})^k \to \overline{0} \text{ for some } k \in \omega \\
&\iff \quad \Gamma \vdash \neg(\varphi \wedge \overline{1})^k \text{ for some } k \in \omega \\
&\iff \quad \Gamma \vdash \neg(\varphi \wedge \overline{1}) \\
&\iff \quad \Gamma \vdash \neg\varphi.
\end{aligned}
$$

The first equivalence is the deduction theorem, the second is the definition of negation, the last one is due to the first new axiom and the fact that $\vdash_{\mathrm{FL_e}} \neg\varphi \to \neg(\varphi \wedge 1)$. The bottom-up direction of the third equivalence is clear ($p \vdash_{\mathrm{FL_e}} p \mathbin{\&} p$). The converse one: by the last new axiom we obtain $\Gamma \vdash \neg((\varphi \wedge \overline{1})^{k-1} \wedge (\varphi \wedge \overline{1}))$ and, since $\vdash_{\mathrm{FL_e}} (\varphi \wedge \overline{1})^{k-1} \to (\varphi \wedge \overline{1})$, we conclude $\Gamma \vdash \neg(\varphi \wedge \overline{1})^{k-1}$, and, of course, this process can be repeated finitely many times.

Finally, we show that even the Glivenko-equivalence between the basic fuzzy logic BL and Łukasiewicz logic Ł (see [16]) can be recovered from our theory. Moreover, we show that this correspondence extends to infinitary versions of these logics.

Recall that in Example 6.82 we saw that $\alpha\mathrm{BL} = \text{Ł}_\infty$ and, since $\mathrm{BL} \leq \mathrm{BL}_\infty \leq \text{Ł}_\infty$, also $\alpha\mathrm{BL}_\infty = \text{Ł}_\infty$. This, in particular, implies that all three logics have the same antitheorems (see Corollary 6.72). Consequently, also BL_∞ has the local inconsistency lemma of BL (Example 6.28). The Glivenko-equivalence can be obtained by the following chain of equivalences:

$$
\begin{aligned}
\Gamma \vdash_{Ł_\infty} \varphi \quad &\iff \quad \Gamma, \neg\varphi^k \vdash_{Ł_\infty} \emptyset \text{ for every } k \in \omega \\
&\iff \quad \Gamma, \neg\varphi^k \vdash_{\mathrm{BL}_\infty} \emptyset \text{ for every } k \in \omega \\
&\iff \quad \Gamma \vdash_{\mathrm{BL}_\infty} \neg\neg\varphi,
\end{aligned}
$$

where the first equivalence is the dLIL of the infinitary Łukasiewicz logic (Example 6.51). It remains to argue for the last equivalence: we use the fact that every continuous t-norm can be seen as an ordinal sum of Łukasiewicz, Product, and Gödel t-norm. Indeed, $\Gamma, \neg\varphi^k \vdash_{\mathrm{BL}_\infty} \emptyset$ for every $k \in \omega$ if and only if for every evaluation v on some t-norm such that $v[\Gamma] \subseteq \{1\}$, we obtain that $v(\varphi)$ does not belong to the first component of the ordinal sum provided it is Łukasiewicz. This is equivalent to $\Gamma \vdash_{\mathrm{BL}_\infty} \neg\neg\varphi$. As a corollary, we obtain the well-known

$$\Gamma \vdash_{Ł} \varphi \quad \iff \quad \Gamma \vdash_{\mathrm{BL}} \neg\neg\varphi.$$

6.5 Infinitary deduction theorems and inconsistency lemmas

Above we have proved that the infinitary Łukasiewicz logic enjoys the local deduction-detachment theorem (Theorem 6.55). This is probably the first known deduction theorem valid in infinitary logic. However, the motivation behind DDTs is that they allow to turn consequences provable in our logic into theorems. Thus, the standard notion of 'finitary' DDT, which allows only to move finitely many premises to the right side of the turnstile, calls for strengthening in case of infinitary logics. This is the topic of this section. To this end, we propose a natural stronger version of DDTs and ILs: we will see that any protoalgebraic (protonegational) logic in fact admits also the stronger version of PLDDT (PLIL). Finally, we show that the local DDT of infinitary Łukasiewicz logic can be improved to fulfill the desiderata of deduction theorems.

We say that L has a *(restricted) (surjective) substitution swapping* if for every L-theory T, every (finite) $\Delta \cup \{\varphi\}$, and every (surjective) substitution σ

$$T, \sigma[\Delta] \vdash_{\mathrm{L}} \sigma(\varphi) \quad \iff \quad \sigma^{-1}[T], \Delta \vdash_{\mathrm{L}} \varphi.$$

It is known that the restricted surjective version of this property is equivalent to the PLDDT. We also show the non-surjective one corresponds to the LDDT.

Proposition 6.90. *A logic* L *enjoys the (P)LDDT if and only if it enjoys the restricted (surjective) substitution swapping property.*

Proof. We prove it only for the LDDT case; the other one is a straightforward modification. Note that using induction it easily follows that from the DD family Ψ we can build for every natural number n a family of sets of formulas $\Psi_n(p_1,\dots,p_n,q)$ such that

$$\Gamma,\varphi_1,\dots,\varphi_n \vdash_{\mathrm{L}} \psi \quad\Longleftrightarrow\quad \Gamma \vdash_{\mathrm{L}} I(\varphi_1,\dots,\varphi_n,\psi) \text{ for some } I \in \Psi_n$$

Let $n = |\Delta|$. The left-to-right direction goes as follows:

$$\begin{aligned} T,\sigma[\{\varphi_1,\dots,\varphi_n\}] \vdash_{\mathrm{L}} \sigma(\psi) \quad &\Longleftrightarrow \quad T \vdash_{\mathrm{L}} I(\overline{\sigma(\varphi)},\sigma(\psi)) \text{ for some } I \in \Psi_n \\ &\Longleftrightarrow \quad \sigma^{-1}[T] \vdash_{\mathrm{L}} I(\bar{\varphi},\psi) \text{ for some } I \in \Psi_n \\ &\Longleftrightarrow \quad \sigma^{-1}[T],\varphi_1,\dots,\varphi_n \vdash_{\mathrm{L}} \psi. \end{aligned}$$

Right to left: define a family of deduction sets as

$$\Psi_n = \{I \subseteq Fm_{\mathcal{L}}(\{p_1,\dots,p_n,q\}) \mid p_1,\dots,p_n, I(p_1,\dots,p_n,q) \vdash_{\mathrm{L}} q\}.$$

The right-to-left direction from the definition of LDDT clearly holds. For the other one let $\Gamma,\varphi_1,\dots,\varphi_n \vdash_{\mathrm{L}} \psi$ then let σ be the substitution sending every $p_1,\dots,p_n$ to $\varphi_1,\dots,\varphi_n$, q to ψ, and the rest to p_1. Then, by the substitution swapping,

$$\sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)],p_1,\dots,p_n \vdash_{\mathrm{L}} q.$$

Let σ' be the substitution which is identity on $p_1,\dots,p_n,q$ and sends the rest of the variables to p_1. Then, by structurality,

$$I(p_1,\dots,p_n,q) = \sigma'[\sigma^{-1}[\mathrm{Th}_{\mathrm{L}}(\Gamma)]] \in \Psi_n$$

and it is easy to observe that $\sigma[I] = I(\varphi_1,\dots,\varphi_n,\psi) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$, or equivalently $\Gamma \vdash_{\mathrm{L}} I(\varphi_1,\dots,\varphi_n,\psi)$. □

Inspecting the proof of the previous proposition we realize that the result can be analogously proved for arbitrary Δs (that is, for the non-restricted version of the substitution swapping property) provided we extend the definition of DDTs to infinitely many premises. Before doing that, we observe that every protoalgebraic logic in fact enjoys the full surjective substitution swapping property:

Proposition 6.91. *Every protoalgebraic logic* L *enjoys the surjective substitution swapping property.*

Proof. The right-to-left direction follows immediately by structurality. For the other one we argue as follows

$$\begin{aligned}\sigma^{-1}[T], \varDelta \nvdash_{\mathrm{L}} \varphi \quad &\iff \quad \sigma^{-1}[T], \mathrm{Th}_{\mathrm{L}}(\varDelta) \nvdash_{\mathrm{L}} \varphi \\ &\iff \quad T, \sigma[\mathrm{Th}_{\mathrm{L}}(\varDelta)] \nvdash \varphi \\ &\implies \quad T, \sigma[\varDelta] \nvdash \varphi,\end{aligned}$$

where the second equivalence is due to the correspondence theorem of protoalgebraic logics which implies that $\sigma[\mathrm{Th}_{\mathrm{L}}(\varDelta)]$ is an L-theory. □

In the next definition, we propose a natural generalization of "finitary" deduction-detachment theorems. To this end, by $\bar{p}$ we always mean an infinite sequence of variables of length $|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$ such that the cardinality of the set of remaining variables still has cardinality $|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$, as $\bar{r}$ we will denote this remainder minus a variable q, these variables will serve as parameters.

Definition 6.92. We say that a logic L has the *infinitary parametrized local deduction-detachment theorem* if there is a family of sets of formulas $\varPsi(\bar{p}, q, \bar{r})$ such that for every $\varGamma \cup \{\psi\} \subseteq \mathit{Fm}_{\boldsymbol{\mathcal{L}}}$ and every (possibly infinite) sequence of formulas $\bar{\varphi}$

$$\begin{aligned}\varGamma, \bar{\varphi} \vdash_{\mathrm{L}} \psi \quad \iff \quad &\varGamma \vdash_{\mathrm{L}} I(\bar{\varphi}, \psi, \bar{\delta}) \text{ for some } I \in \varPsi(\bar{p}, q, \bar{r}) \\ &\text{and some } \bar{\delta} \in \mathit{Fm}_{\boldsymbol{\mathcal{L}}}{}^{\bar{r}}.\end{aligned}$$

In a standard fashion we define *infinitary local* and *global deduction-detachment theorems*.

Observe that for finitary logics the standard notion of DDT covers our infinitary one. Next result shows, as suggested above, that protoalgebraic logics always have the infinitary PLDDT:

Corollary 6.93. *Every protoalgebraic logic with enough variables enjoys the infinitary* PLDDT*.*

Proof. Since protoalgebraic logics enjoys the surjective substitution swapping (Proposition 6.91), the result follows analogously to Proposition 6.90. □

Example 6.94. We show that the LDDT of the infinitary Łukasiewicz logic $Ł_\infty$ (Theorem 6.55) can be generalized to the following infinitary version:

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \forall n \in \omega\, \exists m, m' \in \omega, \gamma_1, \ldots, \gamma_m \in \Gamma \text{ such that } \vdash_{Ł_\infty} m'[(\gamma_1 \,\&\, \ldots \,\&\, \gamma_m) \to \varphi^n].$$

Formally, our DD family Ψ contains a set of formulas for every pair of functions $f, g : \omega \to \omega$ of the form

$$\{f(n)[(p_1 \,\&\, \ldots \,\&\, p_{g(n)}) \to q^n] \mid n \in \omega\},$$

where we consider the variables $\bar{p}$ in this set to be unique for every n.

Proof. If $\Gamma \vdash_{Ł_\infty} \varphi$ then for every n, $\Gamma, \neg\varphi^n$ is an antitheorem (by the dLIL —Example 6.51). Since the logic is compact there is a finite subset $\{\gamma_1, \ldots, \gamma_m\}$ of Γ such that $\{\gamma_1, \ldots, \gamma_m\}, \neg\varphi^n$ is still an antitheorem. Thus, by LIL (Example 6.30) we obtain that for some $m' \in \omega$

$$\vdash_{Ł_\infty} \neg(\gamma_1 \,\&\, \ldots \,\&\, \gamma_m \,\&\, \neg\varphi^n)^{m'},$$

which is equivalent to our formulation. The other direction is similar.

Recall that in Example 6.66 we saw that $Ł_\infty$ has the filter extension property, FEP; we obtained the result as a consequence of Theorem 6.65. We mentioned that the result could not be obtained by the well-known characterization from [34] because $Ł_\infty$ is infinitary. However, considering our strengthened notion of LDDT we could in fact use the same result (with an analogous proof), but of course in a slightly stronger formulation (for the details of the proof we refer to the paper).

Proposition 6.95. *For every protoalgebraic logic* L *with enough variables, the following are equivalent:*

(i) L *has the* FEP.
(ii) L *has the infinitary* LDDT.

Other characterization results for deduction-detachment theorems, that are usually restricted to finitary logics only, can be analogously generalized.

Similarly, we can generalize inconsistency lemmas to infinitary versions. Recall the property introduced in Theorem 6.9:

Definition 6.96. A logic L has (*restricted*) *(surjective) substitution swapping property for antitheorems*, if for every (surjective) substitution σ, every L-theory T, and every (finite) $\Delta \subseteq Fm_{\mathcal{L}}$

$$T, \sigma[\Delta] \vdash_{\mathrm{L}} \emptyset \iff \sigma^{-1}[T], \Delta \vdash_{\mathrm{L}} \emptyset.$$

Again we show that these properties characterize inconsistency lemmas:

Proposition 6.97. *Any logic has the* (P)LIL *if and only if it has the restricted (surjective) substitution swapping property for antitheorems.*

Proof. Analogous to the proof of Proposition 6.90. □

Definition 6.98. We say that a logic L has the *infinitary parametrized local inconsistency lemma* if there is a family of sets of formulas $\Psi(\bar{p}, \bar{r})$ such that for every $\Gamma \subseteq Fm_{\mathcal{L}}$ and every (possibly infinite) sequence of formulas $\bar{\varphi}$

$$\Gamma, \bar{\varphi} \vdash_{\mathrm{L}} \emptyset \iff \Gamma \vdash_{\mathrm{L}} I(\bar{\varphi}, \bar{\delta}) \text{ for some } I \in \Psi(\bar{p}, \bar{r}) \text{ and some } \bar{\delta} \in Fm_{\mathcal{L}}{}^{\bar{r}}.$$

In a standard way we define *infinitary local* and *global inconsistency lemmas.*

Every protonegational logic has the surjective substitution swapping for antitheorems—see e.g. Theorem 6.9, point (iii). Thus, extending the characterization from Proposition 6.97 we obtain that every protonegational logic with the MCP and *enough variables for antitheorems* ($\mathrm{card}^{-}\,\mathrm{L} \leq |\mathit{Var}_{\mathcal{L}}|^{+}$) has the infinitary PLIL .

As suggested after the main characterization theorem for protonegational logics (Theorem 6.9), we can use the infinitary PLIL to avoid the assumption of compactness in its formulation to obtain:

Theorem 6.99. *For every logic* L *with an antitheorem and the* MCP *and enough variables for antitheorems, the following are equivalent:*

(i) L *is protonegational.*
(ii) L *satisfies the weak form of the correspondence theorem.*
(iii) L *has the surjective substitution swapping for antitheorems.*
(iv) L *has the infinitary* PLIL *.*
(v) L *has a parametrized protonegation.*
(vi) *The Leibniz congruence is formula definable on simple* L*-theories.*

The essential part of the proof of this results resides in the fact the we can use the infinitary PLIL to obtain a parametrized protonegation as in Theorem 6.8 but counting on a weaker assumptions.

Finally, there arises a question whether finitary GDDT (resp. LDDT) implies its infinitary version as in the case PLDDT: we do not have an answer yet.

6.6 Protoalgebraic pairs

In the last section we present a glimpse of an alternative way in which we can view protonegational logics provided they have a protonegation without parameters. Observe that, if L has a protonegation without parameters $\Rightarrow$ (see e.g. Corollary 6.85), we can see it as a protoimplication with the two defining properties split between L and αL; that is, we have reflexivity in L ($\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$) and *modus ponens* in αL ($\varphi, \varphi \Rightarrow \psi \vdash_{\alpha\mathrm{L}} \psi$). This brings us to the notion of a protoalgebraic pairs, this concept can be seen a refinement of the notion of protoalgebraic logic to pairs of logics.

As in the case of protoalgebraic logics there are several equivalent ways to present protoalgebraic pairs. Throughout this section whenever we speak about pair of logics $\langle \mathrm{L}, \mathrm{L}' \rangle$, we shall always mean an arbitrary pair of logics with the same language.

Definition 6.100. A set of formulas in two variables $\Rightarrow(p, q)$ is called a *protoimplication for a pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$, whenever

$$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \qquad \text{and} \qquad \varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}'} \psi.$$

Definition 6.101. The Leibniz operator is *monotone on theories* of $\langle \mathrm{L}, \mathrm{L}' \rangle$ if $T \subseteq S$ implies $\boldsymbol{\Omega} T \subseteq \boldsymbol{\Omega} S$ whenever $T \in \mathrm{Th}\,\mathrm{L}$ and $S \in \mathrm{Th}\,\mathrm{L}'$. Moreover, the Leibniz operator is *monotone* on filters of $\langle \mathrm{L}, \mathrm{L}' \rangle$ if $F \subseteq G$ implies $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$ whenever $F \in \mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A}$ and $G \in \mathcal{F}i_{\mathrm{L}'}\, \boldsymbol{A}$.

Definition 6.102. A set of a *congruence formulas for a pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$ is a set $\Delta(p, q, \bar{r})$ such that for every algebra $\boldsymbol{A}$ and elements $a, b \in A$:

$$\emptyset \vdash_{\mathrm{L}} \Delta\langle p, p \rangle \text{ and } \langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \text{ whenever } \Delta^{\boldsymbol{A}}\langle a, b \rangle \subseteq F \in \mathcal{F}i_{\mathrm{L}'}\, \boldsymbol{A}.$$

Thus, the condition $\Delta^{\boldsymbol{A}}\langle a, b \rangle \subseteq F$ defines the Leibniz congruence of an L'-filter F whenever $\Delta^{\boldsymbol{A}}\langle a, a \rangle \subseteq \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(\emptyset)$, in particular, whenever $\mathrm{L} \leq \mathrm{L}'$.

Definition 6.103. A *deduction-detachment family* for a pair of logics $\langle \mathrm{L}, \mathrm{L}' \rangle$ is a family of sets of formulas $\Psi(p, q, \bar{r})$ such that for every set of formulas $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\boldsymbol{\mathcal{L}}}$ we have

$$\Gamma, \varphi \vdash_{\mathrm{L}'} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} I(\varphi, \psi, \bar{\delta}) \text{ for some } \bar{\delta} \in Fm_{\boldsymbol{\mathcal{L}}}^{Var_{\boldsymbol{\mathcal{L}}}} \text{ and } I \in \Psi.$$

A pair of logics enjoys *the parametrized local deduction-detachment* (PLDDT) if it has a deduction-detachment family.

Definition 6.104. A pair of logics $\langle \mathrm{L}, \mathrm{L}' \rangle$ enjoys the *surjective substitution swapping property*, if we have

$$T, \sigma[\Delta] \vdash_{\mathrm{L}'} \sigma(\varphi) \quad \Longleftrightarrow \quad \sigma^{-1}[T], \Delta \vdash_{\mathrm{L}'} \varphi,$$

whenever T is an L-theory and σ a surjective substitution.

Now, we can prove that all the above defined properties are equivalent. We say that a pair of logics is *protoalgebraic* provided it has any of them. Let us mention, that once we have appropriately split the properties between the two logics, the proof of the characterization result itself is just a straightforward modification of the one for protoalgebraic logics. However, we provide it anyway for the reader's convenience. For the proof recall that the fundamental set of a logic L

$$\Sigma_{\mathrm{L}}(p, q, \bar{r}) = \{\chi(p, q, \bar{r}) \in Fm_{\mathcal{L}} \mid \emptyset \vdash_{\mathrm{L}} \chi(p, p, \bar{r})\}$$

is the smallest L-theory such that $\langle p, q \rangle \in \boldsymbol{\Omega} \Sigma_{\mathrm{L}}$ (see e.g. [46]).

Theorem 6.105. *The following are equivalent for every pair of logics* $\langle \mathrm{L}, \mathrm{L}' \rangle$*:*

(i) *The Leibniz operator is monotone on filters of* $\langle \mathrm{L}, \mathrm{L}' \rangle$.
(ii) *The Leibniz operator is monotone on theories of* $\langle \mathrm{L}, \mathrm{L}' \rangle$.
(iii) *The pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$ *enjoys the surjective substitution swapping property.*
(iv) *The pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$ *enjoys the* PLDDT.
(v) *The pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$ *has a protoimplication.*
(vi) *The pair* $\langle \mathrm{L}, \mathrm{L}' \rangle$ *has a set of congruence formulas with parameters.*

Proof. (i)→(ii): Trivial.

(ii)→(iii): The right-to-left direction follows by structurality. For the other one, clearly, $\sigma \colon \langle \boldsymbol{Fm}_{\mathcal{L}}, \sigma^{-1}[T] \rangle \to \langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle$ is a strict surjective homomorphism. Consequently, by (ii), we obtain that $\mathrm{Ker}\, \sigma \subseteq \boldsymbol{\Omega}\sigma^{-1}[T] \subseteq \boldsymbol{\Omega} S$, where S is the L′- theory generated by $\sigma^{-1}[T] \cup \Delta$. Therefore, $\sigma \colon \langle \boldsymbol{Fm}_{\mathcal{L}}, S \rangle \to \langle \boldsymbol{Fm}_{\mathcal{L}}, \sigma[S] \rangle$ is also strict. Thus, if we have $\sigma^{-1}[T], \Delta \nvdash_{\mathrm{L}'} \varphi$, then also $T, \sigma[\Delta] \nvdash_{\mathrm{L}'} \sigma(\varphi)$.

(iii)→(iv): Analogous to the proof of Proposition 6.91.

(iv)→(v): Since $p \vdash_{\mathrm{L}'} p$ there is $I(p, q, \bar{r}) \in \Psi$ such that $\emptyset \vdash_{\mathrm{L}} I(p, p, \bar{\delta})$ for some formulas $\bar{\delta}$. We define a protoimplication $\Rightarrow(p, q)$ substituting q for all the parameters $\bar{r}$ in $I(p, q, \bar{\delta})$. By structurality $\emptyset \vdash_{\mathrm{L}} p \Rightarrow p$. By the PLDDT, $p, I(p, q, \bar{\delta}) \vdash_{\mathrm{L}'} q$, thus, structurality, completes the proof.

(v)→(i): Suppose $\mathcal{F}i_{\mathrm{L}}\, \boldsymbol{A} \ni F \subseteq G \in \mathcal{F}i_{\mathrm{L}'}\, \boldsymbol{A}$ and $\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Take a formula $\varphi(p, \bar{r})$ such that $\varphi^{\boldsymbol{A}}(a, \bar{c}) \in G$ for some formulas $\bar{c}$, it is enough to show that $\varphi^{\boldsymbol{A}}(b, \bar{c}) \in G$: from reflexivity of $\Rightarrow$ we obtain

$$\varphi(a,\bar{c}) \Rightarrow^{\boldsymbol{A}} \varphi(b,\bar{c}) \subseteq F \subseteq G,$$

thus *modus ponens* finishes the proof.

(ii)→(vi): We show that Σ_{L} is a set of congruence formulas for $\langle \mathrm{L}, \mathrm{L}' \rangle$. Clearly $\emptyset \vdash_{\mathrm{L}} \Sigma_{\mathrm{L}}\langle p,p\rangle$. Suppose $\Sigma_L^{\boldsymbol{A}}\langle a,b\rangle \subseteq F \in \mathcal{F}i_{\mathrm{L}'}\,\boldsymbol{A}$ for some $a,b \in A$. To prove that $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$, it suffices to show for every formula $\varphi(p,\bar{r})$ and tuple of element $\bar{c}$ of A, that $\varphi^{\boldsymbol{A}}(a,\bar{c}) \in F$ implies $\varphi^{\boldsymbol{A}}(b,\bar{c}) \in F$. Since $\langle p,q\rangle \in \boldsymbol{\Omega}\Sigma_{\mathrm{L}}$, we obtain by (i) that

$$\varphi(p,\bar{r}), \Sigma_{\mathrm{L}} \vdash_{\mathrm{L}'} \varphi(q,\bar{r}),$$

therefore, using the fact that $\Sigma_L^{\boldsymbol{A}}\langle a,b\rangle \subseteq F$, we conclude $\varphi^{\boldsymbol{A}}(b,\bar{c}) \in F$.

(vi)→(ii): If $\langle \varphi,\psi\rangle \in \boldsymbol{\Omega}T$, then $\Delta\langle\varphi,\psi\rangle \subseteq T \subseteq S$ and thus $\langle\varphi,\psi\rangle \in \boldsymbol{\Omega}S$. □

Note that we could have also included the infinitary version of PLDDT analogously to the previous section. Also, similarly to what we have seen so far, we could generalize other properties of the Leibniz hierarchy to pairs of logics, e.g. we could speak about equivalential pairs and so on.

Of course, the prominent example of a protoalgebraic pair is (under some assumptions—cf. Corollary 6.85) the pair $\langle \mathrm{L}, \alpha\mathrm{L}\rangle$. Conversely, since simple theories of L are also theories of αL, we obtain that L is protonegational whenever $\langle \mathrm{L}, \alpha\mathrm{L}\rangle$ is a protoalgebraic pair.

We are now going to see, that protoalgebraic pairs provides another useful insight to the notion of protonegationality. In particular, the equivalent properties of the previous theorem, when applied to $\langle \mathrm{L}, \alpha\mathrm{L}\rangle$, can be seen as a different formulations of the properties equivalent to protonegationality (Theorem 6.9). For example, if the pair enjoys the GDDT (resp. LDDT), then we can construct an inconsistency sequence for the global (resp. local) inconsistency lemma for L by essentially the same construction as in Proposition 6.57, that is, by inserting an antitheorem into the DD family. Conversely, from the global or local inconsistency lemma of L, we can build a DD family for the pair as in Proposition 6.53. Let us describe the second claim in more details: Assume our logic is compact and that it has an antitheorem and the MCP. We construct a DD family out of the inconsistency sequence the same way as in the proposition, that is, we fix

$$\mathcal{F} = \{f : \Psi_1 \to \bigcup_{n\in\omega} \Psi_n \mid f(I) \in \Psi_{|I|+1} \text{ for every } I \in \Psi_1\},$$

and define the family $\Psi \subseteq P(Fm_{\boldsymbol{\mathcal{L}}}(p,q))$ to contain for every $f \in \mathcal{F}$ a set I_f defined as the union of sets

$$f(I)(p, \chi(q)_1, \ldots, \chi(q)_n),$$

where $I = \{\chi(q)_1, \ldots, \chi(q)_n\} \in \Psi_1$ and $f(I) \in \Psi_{n+1}$. The rest of the argument is slightly different: we want to prove that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$

$$\Gamma, \varphi \vdash_{\alpha\mathrm{L}} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} I(\varphi, \psi) \text{ for some } I \in \Psi. \tag{6.14}$$

Recall that the assumptions imply that αL is characterized by means antiadmissible rules (Proposition 6.77). Left to right: by the LIL of L, we know that $\psi, I(\psi)$ is an antitheorem in L for each $I \in \Psi_1$. Thus, from the left-hand side of (6.14), we obtain that $\Gamma, \varphi, I(\psi)$ is an antitheorem in L. Thus, by LIL, we get for every $I \in \Psi_1$ some $J \in \Psi_{|I|+1}$ such that $\Gamma \vdash_{\mathrm{L}} J(\varphi, \overline{I(\psi)})$, which is what we wanted. The other direction goes along the same lines.

Let us show some examples of DDTs for pairs:

Example 6.106. If L is an superintuitionistic logic, then $\langle \mathrm{L}, \mathrm{CL} \rangle$ is a protoalgebraic pair (e.q. $\to$ is a protoimplication for the pair) and it has the obvious global inconsistency lemma (Example 6.27). Consequently the pair has the following DDT: for every set of formulas $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{CL}}$

$$\Gamma, \varphi \vdash_{\mathrm{CL}} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg(\varphi \wedge \neg\psi).$$

Note that this is indeed just a different formulation of the inconsistency lemma of L:

$$\begin{aligned} \Gamma, \varphi \vdash_{\mathrm{L}} \overline{0} \quad &\Longleftrightarrow \quad \Gamma, \varphi \vdash_{\mathrm{CL}} \overline{0} \\ &\Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg(\varphi \wedge \neg\overline{0}) \\ &\Longleftrightarrow \quad \Gamma \vdash_{\mathrm{L}} \neg\varphi. \end{aligned}$$

Example 6.107. Another example is the protoalgebraic pair $\langle \mathrm{BL}, \text{Ł}_\infty \rangle$ of the basic fuzzy logic and the infinitary Łukasiewicz logic. It has a protonegation $\neg(p \,\&\, \neg q)$ and the local DDT in the form (cf. Theorem 6.55):

$$\begin{aligned} \Gamma, \varphi \vdash_{\text{Ł}_\infty} \psi \quad \Longleftrightarrow \quad & \Gamma \vdash_{\mathrm{BL}} \neg(\varphi \,\&\, \neg\psi^n)^{f(n)} \text{ for some } f \colon \omega \to \omega \\ & \text{and every } n \in \omega. \end{aligned}$$

Note that the DDTs of the above pairs resemble the DDT of the antistructurally complete logic (CL and Ł_∞) with the difference that we can use a weaker logic on the right hand side of the equivalence provided we replace $p \to q$ with a different protonegation of the base logic ($\neg(p \wedge \neg q)$ in L and $\neg(p \,\&\, \neg q)$ in BL)—they are equivalent to $\to$ in the antistructural completion.

6.7 Conclusion and remarks

We have presented a new class of protonegational logics and argued that these logics are the right framework to investigate inconsistency lemmas and antistructural completions. We have given a thorough account of these notions and their mutual interplay. However, some important questions are left open:

- Does every protonegational logic have a protonegation without parameters?
- Does protonegationality transfer in general?

Recall that, from Lemma 6.38 and Proposition 6.79, we have obtained the following implications:

$$\begin{aligned} \mathrm{LIL} \quad \text{or} \quad \mathrm{dLIL} \quad &\Longrightarrow \quad \text{definability of simple theories} \\ &\Longrightarrow \quad \text{closure of simply antiadmissible rules} \\ &\qquad\quad \text{under substitutions.} \end{aligned}$$

- Can the previous implications be reversed?

Lastly, we have seen that to argue about protonegational logics (e.g. proving characterizations) we often need to presuppose some conditions like the MCP, having an antitheorem, compactness, etc. This claim, together with the first two questions, makes us wonder whether there could be a better definition of protonegational logics (the notion of protoalgebraic pair is one of the possible candidates).

In the future, regarding the theory of this chapter, among others, we shall attempt to do the following:

- Solve the open problems.
- Translate the notions and results to the algebraic framework.
- Continue the research on Glivenko-like theorems, infinitary deduction-detachment theorems, and protoalgebraic pairs.
- Protonegational logics are presented by restricting the properties of protoalgebraic logics to simple theories. We will consider analogous generalizations with respect to prime and linear theories.

Finally, let us conclude by summarizing all the properties we learned of the infinitary Łukasiewicz logic:

- It is a Rasiowa-implicative logic. In particular, it is algebraizable with equivalent algebraic semantics (see Proposition 3.24 and (6.7) on p. 145):

$$\mathbf{Alg}^*\text{Ł}_\infty = \mathbf{ISP}([0,1]_{\text{Ł}}) = \{\boldsymbol{A} \mid \boldsymbol{A} \text{ is a semisimple MV-algebra}\}.$$

- It is semilinear, $\vee$ is a strong disjunction (Lemma 5.20), and it enjoys the law of excluded middle (Example 6.52). Moreover, all notions of theories (filters) we have investigated (that is, $\vee$-prime, linear, simple, intersection-prime, and completely intersection-prime) coincide in $Ł_\infty$. The proof, that $\vee$-prime theories are simple, is contained in the proof of Proposition 5.21.
- It is semisimple and, consequently, subdirectly representable (see either Theorem 3.22 or comments below Theorem 6.61).
- It enjoys the (infinitary) LDDT and the filter extension property (Theorem 6.55 and Section 6.5).
- It is the antistructural completion of the basic fuzzy logic BL. In particular, it is antistructurally complete (Example 6.82).
- It can be axiomatized relative to Ł by either (see Subsection 5.2.2)

  (i) $\{\varphi \to \psi^n \mid n \in \omega\} \rhd \neg\varphi \vee \psi$, or
  (ii) $\{\neg\varphi \to \varphi^n \mid n \in \omega\} \rhd \varphi$.

- It has continuous connectives w.r.t. the standard unit interval topology [0,1], which allows to prove that its natural extensions are also complete w.r.t. the standard semantics (Proposition 3.21).
- It is Glivenko-equivalent to BL_∞ (see Section 6.4).
- Derivability in $Ł_\infty$ can be reduced to Ł (Proposition 5.23):

$$\mathit{\Gamma} \vdash_{Ł_\infty} \varphi \quad \Longleftrightarrow \quad \mathit{\Gamma} \vdash_{Ł} \neg\varphi \to \varphi^n \text{ for every } n \in \omega.$$

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