An Abstract Study of Completeness in Infinitary Logics

Abstraktní Studium Úplnosti pro Infinitární Logiky

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Aims of the thesis

The goal of this dissertation is to contribute to the theory of abstract algebraic logic with a focus on infinitary logics. Abstract algebraic logic studies non-classical propositional logics, seen as structural consequence relations, via their connections to algebraic semantics. It is a usual practice in the field to consider mostly logics that can be axiomatized by means of rules with finitely many premises (thus every proof in these logics is finite). A logic with this property is called *finitary*; otherwise it is called *infinitary*.[1] Finitarity seems to be a well-motivated restriction especially regarding the argument that reasoning is a process performed by human minds or other finite machines thus, in principle, it should not be infinite. On the other hand, at least in mathematical practice, we deal with implicitly infinite arguments all the time. The best example lies behind the ω-rule: to prove that all natural numbers have some property $\varphi(n)$, that is to prove $\forall n \varphi(n)$, the rule asserts that we need to prove all the instances $\varphi(0), \varphi(1), \varphi(2), \ldots$ What makes this rules feasible in everyday mathematical practice is the fact that we do not have to produce all the particular instances. Indeed, it suffices to have a proof of $\varphi(0)$ and know that this proof can be rewritten to a proof of $\varphi(n)$ for every natural number n. In this sense the ω-rule reduces to the well-known induction principle. Thus, arguably, infinitary rules are, from a philosophical point of view important concepts, and, in many cases, they can be used in actual reasoning, because they can be represented by finite means.

Another motivation for a systematic study of infinitary logics is that they can be used to understand some important algebraic structures such as algebras defined over the unit interval [0,1] of reals that play a prominent role in the field of mathematical fuzzy logic. The main examples that we consider are the standard Łukasiewicz, product, or Gödel algebras, where the first induces the infinitary Łukasiewicz logic $Ł_\infty$, the second one the infinitary product logic Π_∞, while the last one actually induces a finitary logic.

In the thesis we intend to remedy the lack of a systematic study of infinitary propositional logics. Along the way we deal with various, more or less natural, examples of infinitary logics which illustrate the general theory. Some of these examples have a rather unique properties: e.g. we consider a logic semantically defined by an algebra which contains a part of the absolutely free algebra, or a logic presented in a language with only finitely many variables, which ensures some strong properties. However, the main exam-

[1] This notion should not be mistaken with with the one regarding logics with infinitary languages. In our setting all connectives have finite arities.

ples of infinitary logics are the above mentioned Łukasiewicz and product logic, which are intensively studied throughout the text.

In Part I we focus on the most basic logical property: completeness. It is well known that every logic has a semantics based on logical matrices and that this semantics in the usual cases corresponds to the expected algebraic semantics: e.g. the classical logic is complete w.r.t. matrices based on Boolean algebras, and intuitionistic logic w.r.t. matrices based on Heyting algebras. However, we are often interested in completeness with a more refined semantics: in classical logic that is completeness w.r.t. the two-element Boolean algebra, or in case of fuzzy (semilinear) logics it is completeness w.r.t. chains—abstractly, in both cases we are looking for completeness w.r.t. *relatively (finitely) subdirectly irreducible algebras*. There is a key component to obtain this completeness result, the so called *Lindenbaum lemma*: for example in classical logic, using its finitarity, we can show that for every set of formulas, if a formula is not provable from the set, then it can be separated by a maximally consistent theory. The latter then induces an ultrafilter on the free algebra and thus its quotient is the two-element Boolean algebra, as desired.

In non-classical logics, the role of maximally consistent theories is played by other (weaker) notions of theories. In particular, there are two abstract notions: since the collection of all theories is a closure system (it is closed under intersections and the set of all formulas is a theory) we can speak of *(completely) intersection-prime* theories. The abstract Lindenbaum lemma reads as follows, this time in symbols:

> If $\mathit{\Gamma} \nvdash \varphi$, then there is a (completely) intersection-prime theory T such that $T \nvdash \varphi$ and $\mathit{\Gamma} \subseteq T$,

which is true for every finitary logic. Therefore, every finitary logic is complete w.r.t. its relatively (finitely) subdirectly irreducible models. The main objective of the first part is to understand how far the Lindenbaum lemma and completeness can be extended in the realm of infinitary logics.

On the other hand, some special kinds of theories are omnipresent in the literature on non-classical logics:

- *Prime* theories (either $\varphi \in T$ or $\psi \in T$ whenever $\varphi \vee \psi \in T$) in logics with a *disjunction*.
- *Linear* theories (either $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$) in logics with a *implication*.
- *Maximally consistent* theories which are usually interesting in logics with a *negation* that has some strong properties (see below).

Since the prime and linear theories are intersection-prime and maximally consistent theories are completely intersection-prime, the first part can also be seen as an abstract study on the role of these theories in the proof of completeness. Thus, in Part II we study these theories one by one with the aim of understanding how the presence of certain connectives (that is, of disjunction, implication, and negation) interacts with the general study of Part I. It should be noted that in the literature there are plenty of results regarding completeness for infinitary logics, which can be separated into two groups:

- Results for infinitary modal expansions of classical logic [17, 22, 23, 31, 32], where the proofs of the particular instances of the Lindenbaum lemma exploit the strong properties of classical negation.
- Results for infinitary expansions of prominent fuzzy logics [7, 24, 28, 34], where the Lindenbaum lemma is mostly obtained using properties of a disjunction.

Among others we identify two general conditions that ensure provability of the Lindenbaum lemma in infinitary logics. These results subsume most of the above results. The Lindenbaum lemma is provable for

- countably axiomatizable logics with a *strong disjunction*, and
- countably axiomatizable logics that enjoy some generalized version of the *law of the excluded* middle of classical logic.

The main objective of Part II is to present a general study of the mentioned three kinds of theories (linear, prime, and maximally consistent) and their corresponding connectives (implication, disjunction, and negation).

Outline

In the preliminary chapter (Chapter 2), we first present the basic notions and notations of universal algebra and abstract algebraic logic. While the first five sections contain no new material and are in general well known (at least to algebraic logicians), the remaining ones are more or less new. For that reason we decided to call the chapter "Basic concepts" rather than more usual "Preliminaries". In these sections we are mostly occupied with concepts related to infinitarity. In particular, we speak about *compactness* and its tranferability to all algebras. Moreover, we introduce a new notion of an *antitheorem* (a set of formulas that cannot be jointly designated in nontrivial matrix) and study its properties (both compactness and antitheorems are studied in a yet unpublished manuscript [27]). Finally, in the last section we introduce a dual notion to natural extensions, the *natural expansions* (presented in [25]), which can be useful to prove preservation of properties under expansions.

Part I: Hierarchy of infinitary logics

First, in Chapter 3, we present the main classes of logics investigated in the dissertation. In particular, two classes are defined in terms of completeness properties:

- RSI-*complete* logics, that is logics complete w.r.t. relatively subdirectly irreducible models.
- RFSI-*complete* logics, that is logics complete w.r.t. relatively **finitely** subdirectly irreducible models.

Furthermore, two other classes are defined in terms of the two variants of the abstract Lindenbaum lemma (which we call *extension properties*, following [9]):

- the CIPEP class of logics with the ***completely*** *intersection-prime extension property*, and
- the IPEP class of logics with the *intersection-prime extension property*.

and, finally, two classes based on the transferred (semantical) counterparts of these properties:

- the τ-CIPEP class, and
- the τ-IPEP class.

We then explain their basic relations (as described above)—see Figure 0.1.

$$\begin{tikzcd}
\text{Finitary} \arrow[r] \arrow[dr] & \tau\text{-IPEP} \arrow[r] & \text{IPEP} \arrow[r] & \text{RFSI-complete} \\
 & \tau\text{-CIPEP} \arrow[r] \arrow[u] & \text{CIPEP} \arrow[r] \arrow[u] & \text{RSI-complete} \arrow[u]
\end{tikzcd}$$

Figure 0.1: The classes and their basic relations

For protoalgebraic logics, we provide semantical characterizations of the extension properties (C)IPEP by means of *surjective completeness*. Further, we define a notion of *(finitely) subdirectly representable* logic, that is, a logic where the class of all models coincides with class of models subdirectly representable by the relatively (finitely) subdirectly irreducible ones. We show that this property precisely corresponds to the fact that a logic is protoalgebraic and has the transferred extension property (τ-(C)IPEP). The last section of the chapter deals with preservation of these properties under extension and expansions.

In Chapter 4, we answer the questions postulated above: all of them negatively. That is, we present separating examples witnessing that all the above mentioned classes of logics are pairwise different. In particular, there are infinitary logics that do not enjoy the extension properties (the Lindenbaum lemma is not provable) and moreover some of these logics are still RFSI-complete. Therefore, Figure 0.1 in fact describes a new hierarchy of infinitary logics.

Main results of Part I

Most of the results presented in this part is a joint work with Carles Noguera published in [25, 26],[2] we present a short list of the main ones:

- A new hierarchy of infinitary logics is proposed.
- τ-(C)IPEP + protoalgebraicity $\iff$ (finite) subdirect representation.
- The best behaved natural example of an infinitary logic (regarding the position in the hierarchy) is the infinitary Łukasiewicz logic $Ł_\infty$. It has the τ-CIPEP and, in particular, it is subdirectly representable.
- We present a topological result that postulates bounds on the cardinality of logic given by a class of matrices, thus generalizing the well known result saying that strongly finite logics, i.e. those that are complete with respect to finite class of finite matrices, are finite.
- Birkhoff's subdirect representation theorem does not extend beyond quasi-varieties: this is consequence of the fact that the infinitary product logic Π_∞ does not enjoy the τ-IPEP (although it is still in the CIPEP class). In particular, it is not (finitely) subdirectly representable (it is a fuzzy logic not representable by chains). The same is true about its equivalent algebraic semantics. Moreover, this class of algebras is natural and almost a quasi-variety.

[2] We mention that the least trivial separating example was already introduced in the author's master thesis, although in a completely different form.

- We tend to think about the extension properties (C)IPEP as a natural generalizations of finitary (for example it substitutes the role of finitarity in the completeness proofs). However, these properties are not in general preserved under extensions by finitary rules. Though they are preserved under axiomatic extensions and in many cases also to axiomatic expansions.

Part II: Theories and connectives

In the second part we investigate the role of connectives regarding the general theory presented in Part I. Such connectives can be either primitive symbols of the language or definable by sets of formulas (possibly infinite with parameters). The basic connection is established via meta-rules that are typical for each connective. These meta-rules generalize some well-known properties of classical logic. The *semilinearity property* for implication

$$\mathrm{SLP} \qquad \Gamma, \varphi \to \psi \vdash_{\mathrm{CL}} \chi \quad \text{and} \quad \Gamma, \psi \to \varphi \vdash_{\mathrm{CL}} \chi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \chi.$$

The *proof by cases property* for disjunction:

$$\mathrm{PCP} \qquad \Gamma, \varphi \vdash_{\mathrm{CL}} \chi \quad \text{and} \quad \Gamma, \psi \vdash_{\mathrm{CL}} \chi \quad \Longrightarrow \quad \Gamma, \varphi \vee \psi \vdash_{\mathrm{CL}} \chi.$$

And finally, the *law of excluded middle* for negation:

$$\mathrm{LEM} \qquad \Gamma, \varphi \vdash_{\mathrm{CL}} \psi \quad \text{and} \quad \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \psi \quad \Longrightarrow \quad \Gamma \vdash_{\mathrm{CL}} \psi.$$

The validity of the general forms of these meta-rules is equivalent to the fact that the (completely) intersection-prime theories coincide with the particular kinds of theories corresponding to each connective, that is linear, prime, and maximally consistent. Similarly for each connective we define a corresponding notion of extension property (Lindenbaum lemma for linear, prime, and maximally consistent theories), which are again equivalent to the abstract ones in presence of the corresponding meta-rules.

In Chapter 5, we start with *semilinear* logics and linear theories. Semilinear logics were introduced [8] as a mathematical definition of the informal notion of fuzzy logic. They were also studied in the follow-up papers [10, 11]. We contribute to theory of semilinear logics mainly by studying particular examples of infinitary semilinear logics, most notably the infinitary Łukasiewicz and product logic, but we consider many others: for example, we define and study an infinitary version of truth degree preserving logics with truth constants based on (i) Łukasiewicz and (ii) Gödel logic. Regarding our theory, these logics always have the linear extension property (that is, they enjoy some form of the Lindebaum lemma) and, in fact, we see that the IPEP class is the smallest one in the hierarchy which contains all of them.

Then, we focus on logics with a disjunction. Disjunction connectives, of course, were the subjects of many contributions in abstract algebraic logic (e.g. [12, 15, 18, 19, 33]) or more recently also from the perspective of infinitary logics [9]. Most importantly, we prove that countably axiomatizable logics with disjunction enjoy the prime extension property and we demonstrate the applicability of this result by presenting (simple) proofs of completeness for some infinitary logics.

In Chapter 6 we investigate maximally consistent consistent theories which we decided to call *simple*. This is motivated by the notion of simplicity from universal algebra. Unlike in the previous cases (of linear and prime theories), simple theories as far as we know were not systematically studied in the literature.

The starting point of the new theory that we present in this chapter is the recent paper by Raftery [30], where he introduces the notion of the so called *inconsistency lemma*. This is again a generalization of a well known property of classical logic:

$$\Gamma \cup \{\varphi\} \text{ is inconsistent} \quad \Longleftrightarrow \quad \Gamma \vdash \neg\varphi.$$

Observe that the inconsistency lemma in some sense can be seen as a restriction of the classical deduction-detachment theorem:

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \varphi \to \psi.$$

The study of deduction-detachment theorems and their rich hierarchy (including global, local, and parametrized local versions) is one of the important parts of the field of abstract algebraic logic. It generalizes the deduction-detachment theorem of intuitionistic and classical logic to e.g. substructural and modal logics. To compare: the initial paper [30] on inconsistency lemmas was about global version of this property. We aim to continue where the paper left off and extend the theory, analogously to deduction-detachment theorems, to local and parametrized local versions.

Arguably, the right framework to study deduction-detachment theorems is the class of finitary protoalgebraic logics. Indeed, deduction-detachment theorems imply protoalgebraicity and their properties (characterizations, algebraic equivalents, transferability etc.) are always proved for logics in this class. In comparison, even though in [30] inconsistency lemmas were studied in the same class (finitary protoalgebraic logics), we claim it is not the best possible framework. For this purpose we generalize the class of protoalgebraic logics to a richer class of logics that we call *protonegational* (they bear a stronger connection with negation rather than implication). Interestingly enough, in the case of inconsistency lemmas even the assumption of finitarity can weakened to compactness. Schematically on one side we have

- deduction-detachment theorems, protoalgebraic logics, implication, finitarity, and rules,

while on the other side we have

- inconsistency lemmas, protonegational logics, negation, compactness, and inconsistent sets.

We thus start this chapter presenting the protonegational logics, which in essence are precisely logics enjoying the same properties as protoalgebraic logics but restricted to simple theories. In the second section, we deal with the hierarchy of inconsistency lemmas. We also investigate a natural dual notion of this property; in classical logic:

$$\Gamma \cup \{\neg\varphi\} \text{ is inconsistent} \quad \iff \quad \Gamma \vdash \varphi,$$

which is nothing else than the law of excluded middle in disguise. Interestingly enough, the most general form of this property, that is the dual parametrized local inconsistency lemma is a syntactical counterpart of *semisimplicity*. Moreover, countably axiomatizable logics with this property enjoy the simple extension property (which is again a version of the Lindenbaum lemma in infinitary setting). Then, in the next section we define yet another closely related notion, *antistructural completion*, which is, dually to structural completion, the strongest extension with the same simple theories (resp. the same inconsistent sets of formulas/antitheorems). In the remaining three sections, we only briefly suggest possible directions for further research.

- We explain the utility of inconsistency lemmas and antistructural completions in the study of Glivenko-like theorems [1, 3, 4, 5, 6, 20, 21].
- We generalize suitably the standard form of deduction-detachment theorems [13, 14, 16, 29] and inconsistency lemmas [30] to remedy the following shortcoming of the definition in case of infinitary logics. Namely, the idea behind deduction-detachment theorems is to turn rules into theorems, but with the standard definition this cannot in general be achieved for infinitary logics (because we can move only finitely many premises to right-hand side of the turnstile). In fact, it turns out, rather surprisingly, that every logic with some deduction-detachment theorem always enjoys at least the parametrized local one in the stronger setting (every rule corresponds to the provability of some theorems).
- Finally, we provide a possible alternative presentation of protonegational logics. This time generalizing the notion of protoalgebraicity by splitting its defining properties into pair of logics (protoalgebraic pairs).

Main results of Part II

Again, we briefly summarize the main results of this part. The part about disjunction is a joint work with Marta Bílková and Petr Cintula published in [2]. The part about simple theories is a joint work with Adam Přenosil contained in an unpublished manuscript [27].

- The Lindenbaum lemma is proved for logics with a disjunction and countable axiomatic system.
- We demonstrate the applicability of the previous abstract result to prove completeness for some infinitary logics and summarize the already well known (but nowhere published) axiomatizations of these logics.
- We investigate various cut properties and relate them to the pair extension lemma.
- We introduce and study a new hierarchy of inconsistency lemmas. We show that in some setting the full pair extension lemma is equivalent to finitarity.
- A new class of protonegational logics is defined as a framework to study these properties.
- We provide a syntactical characterization for logics such that their simple models are closed under submatrices.
- The dual inconsistency lemma are proved to be a syntactical characterization of semisimplicity.
- Antistructural completions are introduced and studied.
- We describe the local deduction-detachment theorem of the infinitary Łukasiewicz logic, which is so far the only one known in the literature which necessarily uses a family of deduction-detachment terms consisting of infinite sets.
- We fully characterize substructural logics which are Glivenko equivalent to classical logic using only syntactical means (inconsistency lemmas and antistructural completions are used).

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